MATH 531 Written Homework 1 Solution Radford 10/17/07

1. (**20 points**)

(a) (**3**) First note that $a^2 = 0$ for all $a \in A$ implies $ab = -ba$ for all $a, b \in A$ as

$$0 = (a+b)^2 = a^2 + ab + ba + b^2 = ab + ba.$$

Thus

$$J(a,a,b) = a(ab) + a(ba) + b(aa) = a(ab) + a(-ab) + a(0) = a(ab) - a(ab) + 0 = 0$$

for all $a, b \in A$.

(b) (**4**) Let $a, b, c \in A$. Then rearranging the terms of the first sum

$$a(cb) + c(ba) + b(ac) = b(ac) + a(cb) + c(ba) = c(ba) + b(ac) + a(cb)$$

shows that $J(a,c,b) = J(b,a,c) = J(c,b,a)$. As

$$-J(a,b,c) = -a(bc) - b(ca) - c(ab) = a(-bc) + b(-ca) + c(-ab) = a(cb) + b(ac) + c(ba)$$

all four expressions are equal.

(c) (**3**) You may assume that if $J(a,b,c) = 0$ for all a, b, c in some spanning set then $J = 0$. Thus A is a Lie algebra if and only if $J(a_i, a_j, a_k) = 0$ for all $1 \leq i, j, k \leq n$. Suppose $J(a_i, a_j, a_k) = 0$. By part (b) this equation holds for any rearrangement of the inputs. Thus by part (a) this equation holds if there is duplicates among the inputs. Therefore $J = 0$ if and only if $J(a_i, a_j, a_k) = 0$ holds when $1 \leq i < j < k \leq n$.

(d) (**3**) B is a 2-dimensional algebra over F with basis $\{a, b\}$ and multiplication table

	a	b
a	0	c
b	-c	0

,

where $c \in B$. Once we show that $x^2 = 0$ for all $x \in B$, it follows that B is a Lie algebra since the condition of part (c) is vacuously satisfied. The next lemma applies to parts (d) and (e).

Lemma 1 *Suppose that B is an algebra over F spanned by $\{a_1, \ldots, a_n\}$ which satisfies $a_i^2 = 0$ for all $1 \leq i \leq n$ and $a_i a_j = -a_j a_i$ for all $1 \leq i, j \leq n$. Then $a^2 = 0$ for all $a \in B$.*

Proof: Let $a \in B$. Then $a = \sum_{i=1}^{n} \alpha_i a_i$ where $\alpha_i \in F$. Thus

$$a^2 = \left(\sum_{i=1}^{n} \alpha_i a_i\right)\left(\sum_{j=1}^{n} \alpha_j a_j\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left((\alpha_i a_i)(\sum_{j=1}^{n}\alpha_j a_j)\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i a_i)(\alpha_j a_j)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i\alpha_j)a_i a_j\\
&= \sum_{i=1}^{n}\alpha_i^2 a_i^2 + \sum_{1\le i<j\le n}\alpha_i\alpha_j a_i a_j + \sum_{1\le i>j\le n}\alpha_i\alpha_j a_i a_j\\
&= \sum_{1\le i<j\le n}(\alpha_i\alpha_j)(a_i a_j + a_j a_i)\\
&= 0.
\end{aligned}
$$

□

(e) (**3**) A 3-dimensional algebra B over F with basis $\{x, y, z\}$ and multiplication table

	x	y	z
x	0	cz	by
y	-cz	0	ax
z	-by	-ax	0

,

where $a, b, c \in F$ is a Lie algebra by the lemma and part (c) since

$$J(x,y,z) = x(yz)+y(zx)+z(xy) = x(ax)+y(-by)+z(cz) = ax^2-by^2+cz^2 = a0-b0+c0 = 0.$$

(f) (**4**) $\mathbf{R}^3$ with the cross product is a Lie algebra. [Recall that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \boldsymbol{k} \\ a & b & c \\ a' & b' & c' \end{vmatrix},$$

where

$$\boldsymbol{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.]$$

Using the fact that the determinant function is linear in each row, and has the value 0 when two rows are the same, it follows that $\mathbf{R}^3$ is an algebra with the cross product over $\mathbf{R}$ such that $\mathbf{v}\times\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbf{R}^3$. In particular $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$. (See part (a)). Since

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i},$$

$\mathbf{R}^3$ is a Lie algebra by part (e).

2. (**20 points**) Let $n \geq 2$ and assume that the characteristic of F is not 2.

(a) (**6**) For $1 \leq i < j \leq n$ let $L_{i,j}$ be the span of $x = e_{i\,j}$, $y = e_{j\,i}$, and $h = e_{i\,j} - e_{j\,i}$. Recall that $e_{k\,\ell}e_{r\,s} = \delta_{\ell\,r}e_{k\,s}$. Thus $[x\ y] = h$, $[h\ x] = 2x$, and $[h\ y] = -2y$. Therefore $L_{i,j}$ is a Lie subalgebra of $sl(n, F)$ with multiplication table

	h	x	y
h	0	2x	-2y
x	-2x	0	h
y	2y	-h	0

which is that of $sl(2, F)$ when $n = 2$. We have shown that $L_{i,j} \simeq sl(2, F)$.

Now $sl(n, F)$ has basis consisting of the $e_{i\,j}$'s, $1 \leq i, j \leq n$, where $i \neq j$, and the differences $e_{i\,i} - e_{1\,1}$. (Some detail required.) Thus $sl(n, F) = \sum_{1 \leq i < j \leq n} L_{i,j}$.

(b) (**6**) By part (a) $L = sl(n, F) = L_1 + \cdots + L_r$ where $L_i \simeq sl(2, F)$. Now $[sl(2, F)\ sl(2, F)]$ is spanned by $\{2x, -2y, h\}$ by the table from part (a). Since the characteristic of F is not 2, this set is independent. Therefore $[sl(2, F)\ sl(2, F)] = [sl(2, F)]$; hence $[L_i\ L_i] = L_i$. From this

$$[L\ L] = [L_1 + \cdots + L_r\ L_1 + \cdots + L_r] \supseteq [L_1\ L_1] + \cdots + [L_r\ L_r] = L_1 + \cdots + L_r = L$$

follows and consequently $[L\ L] = L$.

(c) (**8**) First of all, let $D : A \longrightarrow A$ be a derivation of any algebra. For $a, b \in A$ the calculation

$$D^2(ab) = D(D(ab)) = D(D(a)b + aD(b)) = (D^2(a)b + D(a)D(b)) + (D(a)D(b) + aD^2(b))$$

shows that D^2 is a derivation of A if and only if $2D(a)D(b) = 0$ for all $a, b \in A$.

Let $h = e_{1\,1} - e_{2\,2}$, $x = e_{1\,2}$, $y = e_{2\,1}$, and consider the derivation $D = \text{ad}\ h$ of $sl(n, F)$. Then $[D(x)\ D(y)] = [2x\ \ -2y] = -4h$. Since the characteristic of F is not 2, $2[D(x)\ D(y)] = -8h \neq 0$. Thus D^2 is not a derivation of $sl(n, F)$.

3. (**20 points**) Let $n \geq 1$. For $1 \leq r, r', c, c' \leq n$ let $L_{r,r':c,c'}$ be the span of all $e_{i\,j} \in \mathrm{M}(n, F)$ such that $r \leq i \leq r'$ and $c \leq j \leq c'$.

(a) (**5**) Consider $e_{i\,j}, e_{k\,\ell}$ which satisfy $r \leq i, k \leq r'$ and $c \leq j, \ell \leq c'$. Since $e_{i\,j}e_{k\,\ell} = \delta_{j,k}e_{i\,\ell}$ it follows that $L_{r,r':c,c'}$ is closed under matrix multiplication. Thus $L_{r,r':c,c'}$ is a Lie subalgebra of $gl(n, F)$.

$L = L_{1,1:1,n}$ and $a_i = e_{1\,i}$ for all $1 \leq i \leq n$. Then $\{a_1, \ldots, a_n\}$ is a basis for L.

(b) (**5**) $[a_i\,a_j] = e_{1\,i}e_{1\,j} - e_{1\,j}e_{1\,i} = \delta_{i,1}e_{1\,j} - \delta_{j,1}e_{1\,i}$; thus

$$[a_i\,a_j] = \delta_{i,1}a_j - \delta_{j,1}a_i.$$

In particular

$$[a_1\ a_j] = a_j \text{ for } 1 < j \leq n \text{ and } [a_i\ a_j] = 0 \text{ for all } 1 < i \leq j \leq n. \qquad (1)$$

(c) (**5**) Note $\mathrm{Z}(L) \subseteq \mathrm{C}_L(Fa_1) \subseteq \mathrm{N}_L(Fa_1)$. Let $a = \alpha_1 a_1 + \cdots + \alpha_n a_n \in L$. Then

$$[a_1\, a] = \sum_{j=1}^{n} \alpha_j [a_1\, a_j] = \sum_{j=2}^{n} \alpha_j a_j$$

which means $[a_1\, a] \in Fa_1$ if and only if $\alpha_2 = \cdots = \alpha_n = 0$. Therefore $\mathrm{N}_L(Fa_1) = Fa_1$ which means $\mathrm{C}_L(Fa_1) = Fa_1$ as well since $a_1 \in \mathrm{C}_L(Fa_1) \subseteq \mathrm{N}_L(Fa_1) = Fa_1$.

If $n = 1$ then $\mathrm{Z}(L) = \mathrm{C}_L(Fa_1) = Fa_1$ since $L = Fa_1$ is abelian. Now $\mathrm{Z}(L) \subseteq \mathrm{C}_L(Fa_1) = Fa_1$; so when $n > 1$ the calculation $[a_1\; a_2] = a_2$ means that $\mathrm{Z}(L) = (0)$.

(d) (**5**) If $n = 1$ then $L^1 = L^{(1)} = (0)$. Suppose $n > 1$. By (1) $L^1 = L^{(1)}$ is the span of $\{a_2, \ldots, a_n\}$. By (1) we conclude that $L^{(2)} = [L^{(1)}\; L^{(1)}] = (0)$ and $L^2 = [L\; L^1] = L^1$. Therefore $L^2 = L^3 = \ldots$.

4. (**20 points**) Suppose that V is a finite-dimensional vector space over F. Suppose that $\{v_i\}_{1 \le i \le n}$ is a basis for V. Define $\{E_{i\,j}\}_{1 \le i,j \le n} \in \mathrm{End}(V)$ by $E_{i\,j}(v_k) = \delta_{j,k} v_i$. Then for $1 \le i, j, k, \ell, m \le n$ the calculation

$$E_{i\,j} \circ E_{k\,\ell}(v_m) = E_{i\,j}(E_{k\,\ell}(v_m)) = E_{i\,j}(\delta_{\ell,m}(v_k)) = \delta_{\ell,m}\delta_{j,k} v_i = \delta_{j,k} E_{i,\ell}(v_m)$$

shows that $E_{i\,j} \circ E_{k\,\ell} = \delta_{j,k} E_{i,\ell}$.

To show that $\{E_{i\,j}\}_{1 \le i,j \le n}$ is a basis for $\mathrm{End}(V)$ we need only establish independence. Suppose that $\sum_{i,j=1}^{n} \alpha_{i\,j} E_{i\,j} = 0$, where $\alpha_{i,j} \in F$. For fixed $1 \le m \le n$, evaluation of both sides of the equation at v_m yields $\sum_{i=1}^{n} \alpha_{i,m} v_i = 0$. Therefore $\alpha_{i,m} = 0$ for all $1 \le i \le n$. (**5**)

Now suppose that $\{v_1, \ldots, v_n\}$ is a basis of eigenvectors for a. Then there are $\lambda_1, \ldots, \lambda_n \in F$ with $a(v_i) = \lambda_i v_i$ for all $1 \le i \le n$. Since

$$\left(\sum_{i=1}^{n} \lambda_i E_{i\,i} \right)(v_m) = \sum_{i=1}^{n} \lambda_i E_{i\,i}(v_m) = \lambda_m v_m = a(v_m);$$

that is the sum of operators and a agree on a basis, $a = \sum_{i=1}^{n} \lambda_i E_{i\,i}$. (**10**) Now

$$\mathrm{ad}\, a(E_{k\,\ell}) = [a\, E_{k\,\ell}] = \sum_{i=1}^{n} \lambda_i (E_{i\,i} \circ E_{k\,\ell} - E_{k\,\ell} \circ E_{i\,i}) = (\lambda_k - \lambda_\ell) E_{k\,\ell}$$

shows that $E_{k\,\ell}$ is an eigenvector for $\mathrm{ad}\, a$. (**5**)

5. (**20 points**) Let $s \in \mathrm{M}(n, F)$ and set $\mathcal{L}_s = \{x \in \mathrm{M}(n, F) \,|\, x^t s = -sx\}$.

(a) (**5**) Note $0 \in \mathcal{L}_s$; thus $\mathcal{L}_s \neq \emptyset$. Let $x, y, \in \mathcal{L}_s$ and $\alpha \in F$. The calculation

$$(x + \alpha y)^t s = (x^t + \alpha y^t) s = x^t s + \alpha y^t s = -sx + \alpha(-sy) = -s(x + \alpha y)$$

shows that $\mathcal{L}_s$ is a subspace of $\mathrm{M}(n, F)$ and the calculation

$$\begin{aligned}[x\,y]^t s &= (xy - yx)^t s \\ &= (y^t x^t - x^t y^t) s \\ &= y^t(x^t s) - x^t(y^t s) \\ &= y^t(-sx) - x^t(-sy) \\ &= -(y^t s)x + (x^t s)y \\ &= -(-sy)x + (-sx)y \\ &= -s(xy - xy) \\ &= -s[x\,y]\end{aligned}$$

shows that $[x\,y] \in \mathcal{L}_s$. Therefore $\mathcal{L}_s$ is a Lie subalgebra of $gl(n, F)$.

(b) (**5**) Suppose that the characteristic of F is not 2 and s is invertible. Let $x \in \mathcal{L}_s$. Then $x^t s = -sx$ or equivalently $x^t = -sxs^{-1}$. Thus

$$\mathrm{Tr}(x) = \mathrm{Tr}(x^t) = -\mathrm{Tr}(sxs^{-1}) = -\mathrm{Tr}(s^{-1}sx) = -\mathrm{Tr}(x)$$

which shows that $2\mathrm{Tr}(x) = 0$. Since 2 is a unit of F it follows that $\mathrm{Tr}(x) = 0$. Therefore $x \in sl(n, F)$. We have shown $\mathcal{L}_s \subseteq sl(n, F)$.

(c) (**5**) Suppose that $u \in \mathrm{M}(n, F)$ is invertible and $u^{-1} = u^t$. Generally if $v \in \mathrm{M}(n, F)$ is invertible, f_v is an algebra automorphism of $\mathrm{M}(n, F)$, where $f_v(x) = vxv^{-1}$ for all $x \in \mathrm{M}(n, F)$. Note that $f_v^{-1} = f_{v^{-1}}$. (Details needed.) Thus f_v is a Lie algebra automorphism of $gl(n, F)$. Observe that

$$f(x)^t = (uxu^{-1})^t = (u^{-1})^t x^t u^t = (u^t)^{-1} x^t u^t = (u^{-1})^{-1} x^t u^t = ux^t u^{-1} = f(x^t)$$

for all $x \in M(n, F)$.

Let $x \in gl(n, F)$. Then $x \in \mathcal{L}_s$ if and only if $x^t s = -sx$ if and only if $f_u(x^t s) = -f_u(sx)$ if and only if $f_u(x)^t f_u(s) = -f_u(s) f_u(x)$ if and only if $f_u(x) \in \mathcal{L}_{f_u(s)} = \mathcal{L}_{usu^{-1}}$. Since f_u is bijective, we have shown that the restriction $f_u|\mathcal{L}_s : \mathcal{L}_s \longrightarrow \mathcal{L}_{usu^{-1}}$ is a Lie algebra isomorphism.

(d) (**5**) If $n = 1$ then $(0) = sl(1, F) = \mathcal{L}_{(1)}$. Assume $n \geq 2$. Write $s = \sum_{i\,j=1}^n s_{i\,j} e_{i\,j}$ and let $1 \leq k, \ell \leq n$ be distinct. Assume $sl(n, F) = \mathcal{L}_s$. Since $e_{k\,\ell} \in sl(n, F)$ we have $e_{k\,\ell}^t s = -se_{k\,\ell}$, that is $e_{\ell\,k} s = -se_{k\,\ell}$, which is equivalent to

$$\sum_{j=1}^n s_{k\,j} e_{\ell\,j} = -\sum_{i=1}^n s_{i\,k} e_{i\,\ell}.$$

Now if $j \neq \ell$ there is no term on the right-hand side involving $e_{\ell\,j}$. Therefore $s_{k\,j} = 0$.

Suppose $n \geq 3$. Then for all $1 \leq k, j \leq n$ there is an $1 \leq \ell \leq n$ such that $j, k \neq \ell$. Therefore $s_{k\,j} = 0$ which means $s = 0$. As $\mathcal{L}_0 = gl(n, F) \neq sl(n, F)$, we have a contradiction.

Therefore $n = 2$. Since $\{k, \ell\} = \{1, 2\}$, the preceding equation is

$$s_{k\,k} e_{\ell\,k} + s_{k\,\ell} e_{\ell\,\ell} = -(s_{k\,k} e_{k\,\ell} + s_{\ell\,k} e_{\ell\,\ell}),$$

or equivalently $s_{k\,k} = s_{\ell\,\ell} = 0$ and $s_{k\,\ell} = -s_{\ell\,k}$. Since $s \neq 0$ we may assume

$$S \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{1\,2} - e_{2\,1}.$$

It is an easy exercise to check that $sl(2, F) = \mathcal{L}_s$.

Comment: We have shown that $sl(2, F) = C_1$; see page 2 of Humphries.