MATH 531 **Written Homework 5 Solution** **Radford** **11/20/07**

We follow the notation of the text and that used in class. You may use results from the course materials on the class homepage and the text. **This version replaces the previous one**.

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1. (**35 points**) $L_{\alpha_{k,\ell}} = \boldsymbol{R}e_{k\,\ell}$ and $L_{-\alpha_{k,\ell}} = L_{\alpha_{\ell\,k}} = \boldsymbol{R}e_{\ell,k}$.

(a) (**5**) Let $\alpha_{k,\ell} \in \boldsymbol{\Phi}$. Then $k \neq \ell$. The calculation

$$e_{k\,k} - e_{\ell\,\ell} = [e_{k\,\ell}\; e_{\ell\,k}] = \kappa(e_{k\,\ell}, e_{\ell,k})t_{k,\ell} = (2n\delta_{k,k}\delta_{\ell,\ell} - 2\delta_{k,\ell}\delta_{\ell,k})t_{k,\ell} = 2nt_{k,\ell}$$

establishes $t_{k,\ell} = \dfrac{1}{2n}\,(e_{k\,k} - e_{\ell\,\ell})$. See §8.3 Proposition (c) and WH4 Exercise 1(c).

(b) (**5**) Using part (a) observe that

$$\begin{aligned}
(\alpha_{k,\ell},\ \alpha_{r,s}) &= \alpha_{k,\ell}(t_{r,s}) \\
&= \alpha_{k,\ell}(\frac{1}{2n}\,(e_{r\,r} - e_{s,s})) \\
&= \frac{1}{2n}\alpha_{k,\ell}(\sum_{i=1}^{n}(\delta_{i,r} - \delta_{i,s})e_{i\,i}) \\
&= \frac{1}{2n}(\delta_{k,r} - \delta_{k,s} - (\delta_{\ell,r} - \delta_{\ell,s})) \\
&= \frac{1}{2n}\,(\delta_{k,\,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,\,r})\,.
\end{aligned}$$

(c) (**5**) Using part (b) we note

$$||\alpha_{k,\ell}||^2 = (\alpha_{k\,\ell}, \alpha_{k,\ell}) = \frac{1}{2n}(\delta_{k,k} + \delta_{\ell,\ell} - \delta_{k,\ell} - \delta_{\ell,k}) = \frac{1}{n}.$$

Thus $||\alpha_{k,\ell}|| = \sqrt{n}$. Thus

$$\cos\theta = \frac{(\alpha_{k,\ell},\ \alpha_{r,s})}{||\alpha_{k,\ell}||\,||\alpha_{r,s}||} = \frac{\frac{1}{2n}\,(\delta_{k,\,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,\,r})}{\left(\frac{1}{\sqrt{n}}\right)\left(\frac{1}{\sqrt{n}}\right)} = \frac{1}{2}\,(\delta_{k,r} + \delta_{\ell,s} - \delta_{k,s} - \delta_{\ell,r})\,.$$

(d) (**5**) Let $1 \leq i \leq j < k \leq n$ and $\sum_{u=1}^{n} \lambda_u e_{u\,u} \in H$. Then

$$(\alpha_{i\,j} + \alpha_{j\,k})(\sum_{u=1}^{n} \lambda_u e_{u\,u}) = (\lambda_i - \lambda_j) - (\lambda_j - \lambda_k) = \lambda_i - \lambda_k = \alpha_{i\,k}(\sum_{u=1}^{n} \lambda_u e_{u\,u})$$

which shows that $\alpha_{i,j} + \alpha_{j,k} = \alpha_{i,k}$. Now suppose that $1 \leq k < \ell \leq n$. By induction $\alpha_{k\,\ell} = \sum_{i=k}^{\ell-1} \alpha_i$. Thus $\alpha_{\ell,k} = -\alpha_{k,\ell} = \sum_{i=k}^{\ell-1} -\alpha_i$.

Since $\mathbf{\Phi}$ spans H^* we have just shown that $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ spans H^*. Since $\mathrm{Dim}\, H^* = \mathrm{Dim}\, H = n-1$ it follows that Δ is a basis for H^*. Thus Δ is a base for $\mathbf{\Phi}$.

Suppose $1 \leq i, j \leq n$ are distinct. By part (c)

$$\cos\theta = \frac{1}{2}\left(\delta_{i,j} + \delta_{i+1,j+1} - \delta_{i,j+1} - \delta_{i+1,j}\right) = \frac{1}{2}\left(-\delta_{i,j+1} - \delta_{i+1,j}\right)$$

which has the value 0 or $-\dfrac{1}{2}$. Thus angle between two different $\alpha_i, \alpha_j \in \mathbf{\Phi}$ is either $\pi/2$ or $2\pi/3$.

(e) (**5**) We use the calculations of part (c) to deduce

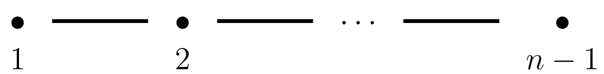
$$<\alpha_i, \alpha_j> = 2\frac{(\alpha_{i,i+1},\ \alpha_{j,j+1})}{||\alpha_{j,j+1}||\,||\alpha_{j,j+1}||} = 2\frac{(\alpha_{i,i+1},\ \alpha_{j,j+1})}{||\alpha_{i,i+1}||\,||\alpha_{j,j+1}||} = 2(\frac{1}{2}\left(\delta_{i,i} + \delta_{i+1,j+1} - \delta_{i,j+1} - \delta_{j,i+1}\right))$$

from which $<\alpha_i, \alpha_j> = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j}$ for all $1 \leq i, j \leq n-1$ follows.

(f) (**5**) From part (e) observe that $<\alpha_i, \alpha_i> = 2$, $<\alpha_i, \alpha_{i+1}> = -1 = <\alpha_{i+1}, \alpha_i>$ for all $1 \leq i < n-1$ and $<\alpha_i, \alpha_j> = 0$ otherwise. Thus the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

(g) (**5**) suppose that $1 \leq i, j \leq n$ are distinct. Then $<\alpha_i, \alpha_j> = 0$ unless $j = i+1$ or $i = j+1$, in which case $<\alpha_i, \alpha_j> \neq 0$, by part (f). Thus the Dynkin diagram of L is



where i represents α_i.

2. (**25 points**)

$$u_i = \begin{pmatrix} \cos\left(\frac{\pi}{m} i\right) \\ \\ \sin\left(\frac{\pi}{m} i\right) \end{pmatrix} \tag{1}$$

for all $0 \leq i < 2m$.

(a) (**7**) $\tau_i(u_j) = u_{2i-j}$ for all $i, j \in \boldsymbol{Z}$ if and only if $2(u_j, u_i)u_i - u_j = u_{2i-j}$ for all $i, j \in \boldsymbol{Z}$. Fix $i, j \in \boldsymbol{Z}$ and set $a = \dfrac{\pi}{m}i$ and $b = \dfrac{\pi}{m}j$. Part (a) comes down to establishing

$$2\left(\begin{pmatrix} \cos b \\ \sin b \end{pmatrix}, \begin{pmatrix} \cos a \\ \sin a \end{pmatrix}\right)\begin{pmatrix} \cos a \\ \sin a \end{pmatrix} - \begin{pmatrix} \cos b \\ \sin b \end{pmatrix} = \begin{pmatrix} \cos(2a-b) \\ \sin(2a-b) \end{pmatrix},$$

or equivalently

$$2(\cos a \cos b + \sin a \sin b)\begin{pmatrix} \cos a \\ \sin a \end{pmatrix} - \begin{pmatrix} \cos b \\ \sin b \end{pmatrix} = \begin{pmatrix} \cos(2a-b) \\ \sin(2a-b) \end{pmatrix},$$

or equivalently

$$2\cos(a-b)\cos a - \cos b = \cos(2a-b) \tag{2}$$

and

$$2\cos(a-b)\sin a - \sin b = \sin(2a-b). \tag{3}$$

As for (2) observe that

$$\begin{aligned} \cos(2a-b) &= \cos((a-b)+a) \\ &= \cos(a-b)\cos a - \sin(a-b)\sin a \\ &= \cos(a-b)\cos a - (\sin a \cos b - \cos a \sin b)\sin a \\ &= \cos(a-b)\cos a - (1-\cos^2 a)\cos b + \cos a \sin b \sin a \\ &= \cos(a-b)\cos a - \cos b + (\cos a \cos b + \sin b \sin a)\cos a \\ &= 2\cos(a-b)\cos a - \cos b. \end{aligned}$$

Replacing a with $\pi/2 - a$ and b with $\pi/2 - b$ in (2) gives (3).

(b) (**5**) $\sigma_i(u_j) = -\tau_i(u_j) = -u_{2i-j} = u_{m+(2i-j)}$.

(c) (**8**) Let $a_i = ||\alpha_i||$ for all $0 \leq i < 2m$. Then $\alpha_i = a_i u_i$ for all $0 \leq i < 2m$ and (3) is satisfied. We extend the definition of a_i to all $i \in \boldsymbol{Z}$ by passing from representatives of the cosets of $2m\boldsymbol{Z}$ in $\boldsymbol{Z}$ to $\boldsymbol{Z}$ by setting $a_{i+2m\ell} = a_i$ for all $\ell \in \boldsymbol{Z}$. Set $\alpha_i = a_i u_i$ for all $i \in \boldsymbol{Z}$. Then the α_i's constitute $\boldsymbol{\Phi}_{\mathsf{n}}$ and $\alpha_{i+2m} = \alpha_i$ for all $i \in \boldsymbol{Z}$.

Note that σ_i is length preserving. Thus $\sigma_i(u_j) = \dfrac{1}{||\alpha_j||}\sigma_i(\alpha_j) = \dfrac{1}{||\sigma_i(\alpha_j)||}\sigma_i(\alpha_j) \in \boldsymbol{\Phi}_{\mathsf{n}}$ for all $i, j \in \boldsymbol{Z}$ which shows that $\sigma_i(\boldsymbol{\Phi}_{\mathsf{n}}) = \boldsymbol{\Phi}_{\mathsf{n}}$ for all $i \in \boldsymbol{Z}$.

To show (1), suppose $i, j \in \boldsymbol{Z}$. Then $a_j u_{m+2i-j} = \sigma_i(\alpha_j) \in \boldsymbol{\Phi}$. Therefore $\sigma_i(\alpha_j)$ and α_{m+2i-j} are scalar multiples which means that $a_j u_{m+2i-j} = \pm a_{m+2i-j} u_{m+2i-j}$. Since the a_ℓ's are positive, $a_j = a_{m+2i-j}$, or equivalently $a_j = a_{m+2(i-j)+j}$ for all $i, j \in \boldsymbol{Z}$. This establishes (1) and (2).

(d) (**5**) Let $\alpha'_i = a_i u_i$ for all $i \in \boldsymbol{Z}$, where the a_i's satisfy (1)–(2) above, and let $\boldsymbol{\Phi}'$ be the set of the α'_i's. Note that $\alpha'_i = \alpha'_{i+2m}$ for all $i \in \boldsymbol{Z}$. Since $\boldsymbol{\Phi}$ spans E, so does $\boldsymbol{\Phi}_{\mathsf{n}}$ and therefore so does $\boldsymbol{\Phi}'$. Since $0 \notin \boldsymbol{\Phi}_{\mathsf{n}}$ and the a_i's are not zero, $0 \notin \boldsymbol{\Phi}'$. Thus (R1) is satisfied.

(R2). Let $i, j \in \boldsymbol{Z}$. Then $-\alpha'_i = a_i(-u_i) = a_{m+i}u_{m+i} \in \boldsymbol{\Phi}'$.

Suppose that $\alpha'_i = c\alpha'_j$ for some $c \in \boldsymbol{R}$. Then u_i and u_j are scalar multiples. Therefore $i = j + km$ for some $k \in \boldsymbol{Z}$. If k is even then $u_i = u_j$ and if k is odd then $u_i = -u_j$. Now $a_i = a_j$ in either case. Therefore $\alpha'_i = \pm\alpha'_j$.

(R3). Let $i, j \in \boldsymbol{Z}$. First of all note that $\sigma_i = \sigma_{\alpha'_i}$. Now $\sigma_i(\alpha'_j) = a_j u_{m+2i-j} = a_{m+2(i-j)+j}u_{m+2i-j} = a_{m+2i-j}u_{m+2i-j} \in \boldsymbol{\Phi}'$.

3. (**20 points**) In light of Exercise 3 and the table on page 45 of Humphreys $m = 2, 3, 4,$ or 6 and, since $\alpha_0 = u_0$, the problem is to find a positive $b \in \boldsymbol{R}$ (which will be the length of α_1) such that with

$$a_{2\ell} = 1 \quad \text{and} \quad a_{2\ell+1} = b \tag{4}$$

for all $\ell \in \boldsymbol{Z}$ the conditions

$$a_{m+\ell} = a_\ell \quad \text{and} \quad 2\left(\frac{a_j}{a_i}\right)(u_i, u_j) = <a_j u_j, a_i u_i> \in \boldsymbol{Z} \tag{5}$$

for all $\ell, i, j \in \boldsymbol{Z}$. *Without loss of generality we may assume* $b \geq 1$ (that is α_0 is a root of minimal length).

$$(u_i, u_j) = \cos\left(\frac{\pi}{m}(i-j)\right)$$

for all $i, j \in \boldsymbol{Z}$. Suppose (4) holds. Observe that $a_{m+\ell} = a_\ell$ for all $\ell \in \boldsymbol{Z}$ holds automatically when m is even and holds if and only if $b = 1$ when $m = 3$. By considering whether or not $i - j$ is even of odd we see that (5) is equivalent to

$$b = 1 \text{ if } m = 3 \text{ and } 2\cos\left(\frac{\pi}{m}(2\ell)\right),\ 2b\cos\left(\frac{\pi}{m}(2\ell+1)\right),\ 2b^{-1}\cos\left(\frac{\pi}{m}(2\ell+1)\right) \in \boldsymbol{Z} \tag{6}$$

for all $\ell \in \boldsymbol{Z}$.

Case 1: $m = 2$. Here $\cos\left(\frac{\pi}{m}(2\ell)\right) = \cos(\pi\ell) \in \{-1, 1\}$ and $\cos\left(\frac{\pi}{m}(2\ell+1)\right) = 0$ for all $\ell \in \boldsymbol{Z}$. Any $b \geq 1$ works.

Case 2: $m = 3$. Since $\cos\left(\frac{n\pi}{3}\right) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ for all $n \in \boldsymbol{Z}$ condition (6) is met.

Case 3: $m = 4$. Here $\cos\left(\frac{\pi}{m}(2\ell)\right) = \cos\left(\frac{\pi}{2}(\ell)\right) \in \{-1, 0, 1\}$ for all $\ell \in \boldsymbol{Z}$. Since $\cos\left(\frac{\pi}{m}(2\ell+1)\right) = \cos\left(\frac{\pi}{4}(2\ell+1)\right) \in \{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$, the remainder of (6) is satisfied if and only if $\sqrt{2}b, \sqrt{2}b^{-1} \in \boldsymbol{Z}$. Writing $b = \frac{m}{\sqrt{2}}, b^{-1} = \frac{n}{\sqrt{2}}$ for some $m, n \in \boldsymbol{Z}$, it is easy to see that the remainder of (6) holds if and only if $b = \sqrt{2}$.

Case 4: $m = 6$. Here $\cos\left(\frac{\pi}{6}(2\ell)\right) = \cos\left(\frac{\pi\ell}{3}\right) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ for all $\ell \in \boldsymbol{Z}$. Since

$$\cos\left(\frac{\pi}{m}(2\ell+1)\right) = \cos\left(\frac{\pi}{6}(2\ell+1)\right) \in \{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\},$$

the remainder of (6) is satisfied if and only if $\sqrt{3}b, \sqrt{3}b^{-1} \in \boldsymbol{Z}$. Mimicking the argument in the last part of Case 3 we see that the remainder of (6) holds if and only if $b = \sqrt{3}$.

Note: The diagrams on page 44 of Humhreys are exactly those which arise in the cases above.

4. (**20 points**) (a) (**10**) Let $i \in \boldsymbol{Z}$. Since σ_i is an isometry of E and $\sigma_i(\boldsymbol{\Phi}) = \boldsymbol{\Phi}$ it follows that $\sigma_i(\boldsymbol{\Phi}_{\mathsf{n}}) = \boldsymbol{\Phi}_{\mathsf{n}}$. Therefore $\mathcal{W}_m$ is a subgroup of $G_{\boldsymbol{\Phi}_{\mathsf{n}}}$. Now $\boldsymbol{\Phi}_{\mathsf{n}}$ spans E since $\boldsymbol{\Phi}$ does. Thus the restriction map $G_{\boldsymbol{\Phi}_{\mathsf{n}}} \longrightarrow \mathrm{Sym}(\boldsymbol{\Phi}_{\mathsf{n}})$ given by $\sigma \mapsto \boldsymbol{\sigma}$ is an injective group homomorphism. Note $\boldsymbol{\sigma}_i(u_j) = u_{m+2i-j}$ for all $j \in \boldsymbol{Z}$ by part (b) of Exercise 2.

(b) (**10**) Let $\boldsymbol{\tau} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_0$. The calculation $\boldsymbol{\tau}(u_j) = \boldsymbol{\sigma}_1(\boldsymbol{\sigma}_0(u_j)) = \boldsymbol{\sigma}_1(u_{m-j}) = u_{m+2-(m-j)} = u_{2+j}$ shows that $\boldsymbol{\tau}(u_j) = u_{2+j}$ for all $j \in \boldsymbol{Z}$. Thus by induction $\boldsymbol{\tau}^\ell(u_j) = u_{2\ell+j}$ for all $0 \leq \ell$ and $j \in \boldsymbol{Z}$. Since $u_k = u_\ell$ if and only if $k \equiv \ell \pmod{2m}$, we conclude that $\boldsymbol{\tau}^\ell = \mathrm{Id}_{\boldsymbol{\Phi}_{\mathsf{n}}}$ if and only if $m|\ell$. Therefore $\boldsymbol{\tau}$ has order m. For $0 \leq \ell$ the calculation

$$\boldsymbol{\tau}^\ell\boldsymbol{\sigma}_0(u_j) = \boldsymbol{\tau}^\ell(\boldsymbol{\sigma}_0(u_j)) = \boldsymbol{\tau}^\ell(u_{m-j}) = u_{2\ell+m-j} = \boldsymbol{\sigma}_\ell(u_j)$$

for all $j \in \boldsymbol{Z}$ shows that $\boldsymbol{\sigma}_\ell = \boldsymbol{\tau}^\ell\boldsymbol{\sigma}_0$. In particular $\mathcal{W}_m$ is generated by $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}_0$.

Let $N = <\boldsymbol{\tau}>$ and $H = <\boldsymbol{\sigma}_0>$. Since $\boldsymbol{\sigma}_i$ has order 2 for all $i \in \boldsymbol{Z}$ we conclude that $\boldsymbol{\tau}^\ell\boldsymbol{\sigma}_0 = (\boldsymbol{\tau}^\ell\boldsymbol{\sigma}_0)^{-1} = \boldsymbol{\sigma}_0^{-1}\boldsymbol{\tau}^{-\ell} = \boldsymbol{\sigma}_0\boldsymbol{\tau}^{-\ell}$, thus

$$\boldsymbol{\tau}^\ell\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_0\boldsymbol{\tau}^{-\ell}, \tag{7}$$

for all $0 \leq \ell$. Therefore $NH = HN$ which means HN is a subgroup of $\mathcal{W}_m$. Since N and H generate $\mathcal{W}_m$, $\mathcal{W}_m = HN$ and N is a normal subgroup of $\mathcal{W}_m$.

Note that $\boldsymbol{\sigma}_0 \in N$ if and only if $\boldsymbol{\sigma}_0 = \boldsymbol{\tau}^\ell$ for some $0 \leq \ell$ if and only if $u_{m-j} = u_{2\ell+j}$ for all $j \in \boldsymbol{Z}$ if and only if $m - 2(j+\ell) \equiv 0 \pmod{2m}$ for all $j \in \boldsymbol{Z}$. The latter implies $m \equiv 0 \pmod{2m}$, a contradiction. Therefore $\boldsymbol{\sigma}_0 \notin N$.

We have shown that $|\mathcal{W}_m| = 2m$. From (7) we conclude $\boldsymbol{\sigma}_0^{-1}\boldsymbol{\tau}\boldsymbol{\sigma}_0 = \boldsymbol{\tau}^{-1}$. Since $\boldsymbol{\sigma}_0, \boldsymbol{\tau}$ generate $\mathcal{W}_m$ and $\boldsymbol{\sigma}_0^2 = \mathrm{Id}_{\boldsymbol{\Phi}_{\mathsf{n}}} = \boldsymbol{\tau}^m$, $\mathcal{W}_m \simeq D_{2m}$.