MATH 531 Written Homework 3 Solution Radford 11/05/07

In the following exercises F is a field, *algebraically closed and of characteristic zero.* We follow the notation of the text and that used in class.

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1. (**25 points**) You may assume that $L = L_1 \oplus \cdots \oplus L_r$ is an algebra. Let $x = \ell_1 \oplus \cdots \oplus \ell_r, y = \ell'_1 \oplus \cdots \oplus \ell'_r, z \in L$. We first show that $\pi_i : L \longrightarrow L_i$ is an algebra map for all $1 \leq i \leq r$ and that $x = y$ if and only if $\pi_i(x) = \pi_i(y)$ for all $1 \leq i \leq r$.

Let $a \in F$. The calculation

$$\begin{aligned}
\pi_i(x+ay) &= \pi_i(\ell_1 \oplus \cdots \oplus \ell_r + a(\ell'_1 \oplus \cdots \oplus \ell'_r)) \\
&= \pi_i(\ell_1 \oplus \cdots \oplus \ell_r + a\ell'_1 \oplus \cdots \oplus a\ell'_r)) \\
&= \pi_i((\ell_1 + a\ell'_1) \oplus \cdots \oplus (\ell_r + a\ell'_r)) \\
&= \ell_i + a\ell'_i \\
&= \pi_i(\ell_1 \oplus \cdots \oplus \ell_r) + a\pi_i(\ell'_1 \oplus \cdots \oplus \ell'_r) \\
&= \pi_i(x) + a\pi_i(y)
\end{aligned}$$

shows that π_i is linear and the calculation

$$\begin{aligned}
\pi_i([x\,y]) &= \pi_i([\ell_1 \oplus \cdots \oplus \ell_r \;\; \ell'_1 \oplus \cdots \oplus \ell'_r]) \\
&= \pi_i([\ell_1 \;\; \ell'_1] \oplus \cdots \oplus [\ell_r \;\; \ell'_r]) \\
&= [\ell_i\,\ell'_i] \\
&= [\pi_i(\ell_1 \oplus \cdots \oplus \ell_r) \;\; \pi_i(\ell'_1 \oplus \cdots \oplus \ell'_r)] \\
&= [\pi_i(x) \;\; \pi_i(y)]
\end{aligned}$$

shows that π_i is multiplicative. Since $\pi_i(x) = \ell_i$ for all $1 \leq i \leq r$ it follows that

$$x = \pi_1(x) \oplus \cdots \oplus \pi_r(x); \tag{1}$$

thus $x = y$ if and only if $\pi_i(x) = \pi_i(y)$ for all $1 \leq i \leq r$.

Since $\pi_i : L \longrightarrow L_i$ is an algebra map, part (b) (**5**) follows once part (a) (**8**) is established.

Let $1 \leq i \leq r$. Since π_i is an algebra map and L_i is a Lie algebra

$$\pi_i([x\ x]) = [\pi_i(x)\ \pi_i(x)] = 0$$

and

$$\begin{aligned}
&\pi_i([x\ [y\ z]] + [y\ [z\ x]] + [z\ [x\ y]]) \\
&= [\pi_i(x)\ [\pi_i(y)\ \pi_i(z)]] + [\pi_i(y)\ [\pi_i(z)\ \pi_i(x)]] + [\pi_i(z)\ [\pi_i(x)\ \pi_i(y)]] \\
&= 0.
\end{aligned}$$

Thus $[x\ x] = 0$ and $[x\ [y\ z]] + [y\ [z\ x]] + [z\ [x\ y]] = 0$ follow by (1).

Part (c). (**12**) Let $\ell' \in L'$ and suppose that $\pi : L' \longrightarrow L$ satisfies $\pi_i \circ \pi = \pi'_i$ for all $1 \leq i \leq r$. Then $\pi(\ell') = \pi_1(\pi(\ell')) \oplus \cdots \oplus \pi_r(\pi(\ell')) = \pi'_1(\ell') \oplus \cdots \oplus \pi'_r(\ell')$; the first equation follows by (1). Thus π is unique. A small argument shows that π is linear. Since each π'_i is a Lie algebra map

$$\begin{aligned}\pi([\ell'\ \ell'']) &= \pi'_1([\ell'\ \ell'']) \oplus \cdots \oplus \pi'_r([\ell'\ \ell'']) \\ &= [\pi'_1(\ell')\ \pi'_1(\ell'')] \oplus \cdots \oplus [\pi'_r(\ell')\ \pi'_r(\ell'')] \\ &= [\pi(\ell')\ \pi(\ell'')]\end{aligned}$$

for all $\ell', \ell'' \in L'$.

2. (**25 points**) (a) (**10**) Since $[x\ y] = y$ and κ is associative

$$\kappa(x, y) = \kappa(x, [x\ y]) = \kappa([x\ x], y) = \kappa(0, y) = 0$$

and

$$\kappa(y, y) = \kappa([x\ y], y) = \kappa(x, [y\ y]) = \kappa(x, 0) = 0.$$

Now

$$(\operatorname{ad} x \circ \operatorname{ad} x)(x) = [x\,[x\,x]] = [x\,0] = 0 \ \text{ and } \ (\operatorname{ad} x \circ \operatorname{ad} x)(y) = [x\,[x\,y]] = [x\,y] = y.$$

Therefore $\kappa(x, x) = \operatorname{Tr}(\operatorname{ad} x \circ \operatorname{ad} x) = \operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1.$

Since κ is symmetric,

$$\begin{pmatrix} \kappa(x,x) & \kappa(x,y) \\ \kappa(y,x) & \kappa(y,y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

In any event $\operatorname{Rad} \kappa \subseteq \operatorname{Rad} L$. Let $\ell = ax + by \in L$. Then $\ell \in \operatorname{Rad} \kappa$ if and only if $\kappa(x, \ell) = 0 = \kappa(y, \ell)$ if and only if $\ell = by$. Thus $Fy = \operatorname{Rad} \kappa \subseteq \operatorname{Rad} L$ is a solvable ideal. (This is trivial since Fy is abelian.) Since L/Fy is one-dimensional, it is abelian and thus solvable. Therefore L is solvable which means $\operatorname{Rad} L = L$.

Part (b). (**15**) There are basically two calculations. Suppose $\{u, v, w\} = \{x, y, z\}$; that is u, v, w are x, y, z in some order. Then $[u\ v] = \alpha_w w$, $[v\ w] = \alpha_u u$, and $[w\ u] = \alpha_v v$ for some $\alpha_u, \alpha_v, \alpha_w \in F$. The calculations

$$(\operatorname{ad} u \circ \operatorname{ad} v)(u) = [u\ [v\ u]] = [u\ (-\alpha_w)w] = (-\alpha_w)(-\alpha_v)v = \alpha_v \alpha_w v;$$

$$(\operatorname{ad} u \circ \operatorname{ad} v)(v) = [u\ [v\ v]] = [u\ 0] = 0;$$

and

$$(\operatorname{ad} u \circ \operatorname{ad} v)(w) = [u\ [v\ w]] = [u\ \alpha_u u] = 0$$

show that $\kappa(u,v) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ \alpha_v\alpha_w & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$. The calculations

$$(\mathrm{ad}\, u \circ \mathrm{ad}\, u)(u) = [u\ [u\ u]] = [u\ 0] = 0;$$

$$(\mathrm{ad}\, u \circ \mathrm{ad}\, u)(v) = [u\ [u\ v]] = [u\ \alpha_w w] = -\alpha_w\alpha_v v;$$

and

$$(\mathrm{ad}\, u \circ \mathrm{ad}\, u)(w) = [u\ [u\ w]] = [u\ (-\alpha_v)v] = -\alpha_v\alpha_w w.$$

show that $\kappa(u,u) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha_v\alpha_w & 0 \\ 0 & 0 & -\alpha_v\alpha_w \end{pmatrix} = -2\alpha_v\alpha_w = -2\widehat{\alpha_u}\alpha_v\alpha_w$. Thus

$$\begin{pmatrix} \kappa(x,x) & \kappa(x,y) & \kappa(x,z) \\ \kappa(y,x) & \kappa(y,y) & \kappa(y,z) \\ \kappa(z,x) & \kappa(z,y) & \kappa(z,z) \end{pmatrix} = \begin{pmatrix} -2bc & 0 & 0 \\ 0 & -2ac & 0 \\ 0 & 0 & -2ab \end{pmatrix}.$$

There are three natural case to consider.

Case 1: None of a, b, c is zero. Then κ is non-singular. Therefore L is semisimple and hence $\mathrm{Rad}\, L = (0)$.

Case 2: Two or three of a, b, c are zero. Then $\kappa = 0$ which means that $\mathrm{Rad}\, \kappa = L$. Since $\mathrm{Rad}\, \kappa \subseteq \mathrm{Rad}\, L$ in any event, $\mathrm{Rad}\, L = L$.

Case 3: Exactly one of a, b, c is zero. Then κ has exactly one non-zero entry (which is diagonal). It is easy to see that $\mathrm{Dim}\,\mathrm{Rad}\, \kappa = 2$ in this case. Lie algebras of dimension two or one are solvable. Therefore $\mathrm{Rad}\, \kappa$ is a solvable ideal of L and $L/\mathrm{Rad}\, \kappa$ is solvable, whence L is solvable and thus $\mathrm{Rad}\, L = L$ again.

3. (**25 points**) $V = gl(n, F) = \bigoplus_{1 \le i,j \le n} Fe_{ij}$ as a vector space. We discover the simple $sl(2, F)$-submodules of submodules V by seeing what submodule each e_{ij} generates, using the rule $e_{ij}e_{k\ell} = \delta_{j,k}e_{i\ell}$ for all $1 \le i, j, k, \ell \le n$. In the problem $n \ge 2$. Note that $e_{ij}e_{k\ell} = 0$, and thus $[e_{ij}\ e_{k\ell}] = 0$, if $\{i,j\} \cap \{k,\ell\} = \emptyset$.

Part (a) Such a decomposition is

$$V = sl(2,F) \oplus F(e_{11} + e_{22}) \bigoplus_{j=3}^{n} (Fe_{1j} \oplus Fe_{2j}) \bigoplus_{j=3}^{n} (Fe_{j2} \oplus Fe_{j1}) \bigoplus_{3 \le i,j \le n} Fe_{ij}.$$

Part (b) We tabulate the results. Note that in each case the dimension of the weight spaces is one, and the weights 0 and 1 can not both appear. Therefore the module is is simple by 7.2 Corollary of the text.

Module	$sl(2,F)$		
Weight spaces	Fe_{21}	$F(e_{11} - e_{22})$	Fe_{12}
Weights	-2	0	2
Maximal vector			e_{12}

Module	$F(e_{11}+e_{22})$
Weight spaces	$F(e_{11}+e_{22})$
Weights	0
Maximal vector	$e_{11}+e_{22}$

Module	$Fe_{2j} \oplus Fe_{1j}$, where $j > 2$.	
Weight spaces	Fe_{2j}	Fe_{1j}
Weights	-1	1
Maximal vector		e_{1j}

Module	$Fe_{j1} \oplus Fe_{j2}$, where $j > 2$.	
Weight spaces	Fe_{j1}	Fe_{j2}
Weights	-1	1
Maximal vector		e_{j2}

Module	Fe_{ij}, where $i, j > 2$.
Weight spaces	Fe_{ij}
Weights	0
Maximal vector	e_{ij}

(**5**) for each type.

4. (**25 points**) You may assume that partial differentiation is a derivation.

(a) (**7**) This follows from: Suppose that $D : A \longrightarrow A$ is a derivation of a commutative associative algebras A over F. For $a \in A$ the endomorphism $D' = \ell_a \circ D$ is a derivation of A.

To prove this we calculate

$$D'(xy) = aD(xy) = a(D(x)y + xD(y)) = (aD(x))y + x(aD(y)) = D'(x)y + xD'(y)$$

for all $x, y \in A$. Since D' is the composite of linear maps it is linear.

(b) (**8**) Recall

$$\boldsymbol{x} = \ell_x \circ \frac{\partial}{\partial y}, \qquad \boldsymbol{y} = \ell_y \circ \frac{\partial}{\partial x}, \qquad \text{and} \qquad \boldsymbol{z} = [\boldsymbol{x}, \boldsymbol{y}].$$

We will use:

Lemma 1 *Let $D, D' : A \longrightarrow A$ be derivations of an algebra over F and suppose $S \subseteq A$ is a subset which generates A as an algebra. Then $D = D'$ if $D(s) = D'(s)$ for all $s \in S$.*

Proof: We need only show that $B = \{a \in A \mid D(a) = D'(a)\}$ is a subalgebra of A. Since $B = \mathrm{Ker}\,(D - D')$, and $D - D'$ is linear, B is a subspace of A. Suppose $a, a' \in B$. Then $D(aa') = D(a)a' + aD(a') = D'(a)a' + aD'(a') = D'(aa')$ which means $aa' \in B$. □

Observe that

$$\mathbf{x}(x) = 0 \qquad \text{and} \qquad \mathbf{x}(y) = x,$$
$$\mathbf{y}(x) = y \qquad \text{and} \qquad \mathbf{y}(y) = 0;$$

therefore

$$\mathbf{z}(x) = \mathbf{x}(\mathbf{y}(x)) - \mathbf{y}(\mathbf{x}(x)) = x \qquad \text{and} \qquad \mathbf{z}(y) = \mathbf{x}(\mathbf{y}(y)) - \mathbf{y}(\mathbf{x}(y)) = -y.$$

In particular $V_1 = Fx \oplus Fy$ is invariant under $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

Let $\mathrm{End}_{V_1}(A)$ be the subspace of endomorphisms T of A such that $T(V_1) \subseteq V_1$. Then $\mathrm{End}_{V_1}(A)$ is a subalgebra of $\mathrm{End}\,(A)$ and the composite π of the restriction map followed by the identification of endomorphisms with matrices with respect to the basis $\mathcal{B} = \{x, y\}$

$$\mathrm{End}_{V_1}(A) \longrightarrow \mathrm{End}\,(V_1) \simeq \mathrm{M}(2, F) \qquad T \mapsto [T|_{V_1}]_{\mathcal{B}}$$

is a map of associative algebras, hence a map of Lie algebras under associative bracket. Note that

$$\pi(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(\mathbf{y}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \pi(\mathbf{z}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to see that $\mathrm{Der}_{V_1}(A)$ is a Lie subalgebra of $\mathrm{End}_{V_1}(A)$. The restriction

$$\pi' : \mathrm{Der}_{V_1}(A) \longrightarrow \mathrm{M}(2, F)$$

of π is injective by the preceding lemma. As $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{Der}_{V_1}(A)$ part (b) follows.

Part (c). (**10**) The calculations

$$\mathbf{x}(x^\ell y^{n-\ell}) = (n-\ell)x^{\ell+1}y^{n-\ell-1} \qquad \text{and} \qquad \mathbf{y}(x^\ell y^{n-\ell}) = \ell x^{\ell-1}y^{n-\ell+1}$$

show that $\mathbf{x}(V_n), \mathbf{y}(V_n) \subseteq V_n$; hence $\mathbf{z}(V_n) = (\mathbf{x}\circ\mathbf{y} - \mathbf{y}\circ\mathbf{x})(V_n) \subseteq \mathbf{x}(\mathbf{y}(V_n)) + \mathbf{y}(\mathbf{x}(V_n)) \subseteq V_n$ for all $n \geq 0$. Therefore V_n is a left $sl(2, F)$-module. Now

$$\mathbf{z}(x^\ell y^{n-\ell}) = \mathbf{x}(\mathbf{y}(x^\ell y^{n-\ell})) - \mathbf{y}(\mathbf{x}(x^\ell y^{n-\ell})) = (\ell(n-\ell+1) - (n-\ell)(\ell+1))x^\ell y^{n-\ell} = (2\ell - n)x^\ell y^{n-\ell}.$$

Thus $V_n = \bigoplus_{\ell=0}^{n} F x^\ell y^{n-\ell}$ and the direct sum of weight spaces and $F x^\ell y^{n-\ell}$ has weight $2\ell - n$. Note that not both 0 and 1 occur as weights since two weights differ by an even integer. Thus V_n is simple by 7.2 Corollary of the text. Note that x^n is a maximal vector.

Addendum to Problem 3(b): Let F be *any field of characteristic not* 2. L is the Lie algebra over F with basis $\{x, y, z\}$ and whose structure is determined by

$$[x\ y] = cz, \quad [y\ z] = ax, \quad [z\ x] = by, \tag{2}$$

where $a, b, c \in F$.

First of all assume $a, b, c \neq 0$. Suppose $a', b', c' \in F$ are non-zero as well. We wish to replace x, y, z by non-zero scalar multiples $x' = \alpha_x x, y' = \alpha_y y, z' = \alpha_z z$ such that

$$[x'\ y'] = c'z', \quad [y'\ z'] = a'x', \quad [z'\ x'] = b'y'.$$

This is equivalent to solving

$$\alpha_x \alpha_y c = \alpha_z c', \quad \alpha_y \alpha_z a = \alpha_x a', \quad \alpha_z \alpha_x b = \alpha_y b'$$

which is done by setting $\alpha_z = \alpha_x \alpha_y \left(\frac{c}{c'}\right)$ and solving

$$\alpha_y^2 = \frac{a'c'}{ac} \quad \text{and} \quad \alpha_x^2 = \frac{b'c'}{bc}. \tag{3}$$

If F is algebraically closed there are always (non-zero) solutions α_y, α_x to these equations. In this case L falls into:

Case 1: $[x\ y] = z, \quad [y\ z] = x, \quad [z\ x] = -y.$

Since $\{x - y, x + y, 2z\}$ is a basis for L,

$$[x - y\ x + y] = 2[x\ y] = 2z,$$

$$[2z\ x - y] = 2([z\ x] - [z\ y]) = 2(-y + x) = 2(x - y) \text{ and}$$
$$[2z\ x + y] = 2([z\ x] + [z\ y]) = 2(-y - x) = -2(x + y),$$

it follows that $\boxed{L = sl(2, F)}$ with $\mathbf{x} = x - y$, $\mathbf{y} = x + y$, and $\mathbf{h} = 2z$.

From this point on $F = \mathbf{R}$ *is the field of real numbers.* Returning to (3) we see that there are solutions $\alpha_y, \alpha_x \in \mathbf{R}$ if a and a' have the same sign, b and b' have the same sign, and c and c' have the same sign. By replacing the basis $\{x, y, z\}$ with $\{x', y', z'\}$ we may assume $a, b, c \in \{-1, 1\}$. Replacing $\{x, y, z\}$ with the basis $\{-x, -y, -z\}$ if necessary we may assume that all of a, b, c are positive or exactly one of these is negative. By reordering $\{x, y, z\}$ if necessary the latter is Case 1. The former is:

Case 2: $[x\ y] = z, \quad [y\ z] = x, \quad [z\ x] = y.$

Let $h = h_x x + h_y y + h_z z \in L$, where $h_x, h_y, h_z \in \mathbf{R}$. We compute $\operatorname{ad} h$.

$$\begin{aligned}
\operatorname{ad} h\,(x) &= h_x[x\,x] + h_y[y\,x] + h_z[z\,x] &&= -h_y z + h_z y \\
\operatorname{ad} h\,(y) &= h_x[x\,y] + h_y[y\,y] + h_z[z\,y] &&= h_x z - h_z x \\
\operatorname{ad} h\,(z) &= h_x[x\,z] + h_y[y\,z] + h_z[z\,z] &&= -h_x y + h_y x.
\end{aligned}$$

The characteristic polynomial of $\operatorname{ad} h$ is therefore

$$\begin{aligned} f(X) &= \begin{vmatrix} X-0 & h_z & -h_y \\ -h_z & X-0 & h_x \\ h_y & -h_x & X-0 \end{vmatrix} \\ &= X(X^2+h_x^2) - h_z(-h_zX - h_xh_y) - h_y(h_xh_z - h_yX) \\ &= X(X^2 + (h_x^2+h_y^2+h_z^2)). \end{aligned}$$

Suppose that $h \neq 0$. Then $\operatorname{ad} h \neq 0$ which means that the minimal polynomial $m(X)$ of $\operatorname{ad} h$ in not X. Since $m(X)$ divides $f(X)$ and the quadratic factor of $f(X)$ is irreducible, $m(X) = f(X)$ which does not split onto linear factors over $\mathbf{R}$. $\operatorname{ad} h$ is not diagonalizable and $\operatorname{ad} h$ is not nilpotent. In particular $\boxed{L \not\simeq sl(2, \mathbf{R})}$. Justification: if $\phi : L' \longrightarrow L''$ is an isomorphisms of Lie algebras and $h \in L'$, then $\operatorname{ad} h$ is diagonalizable (respectively nilpotent) if and only if $\operatorname{ad} \phi(h)$ is diagonalizable (respectively nilpotent).

However, L is simple. To see this, we need only show that $L = \mathbf{R}h + [h\ L] = \mathbf{R}h + T(L)$. For this it suffices to show that $\{h, \operatorname{ad} h\,(y), \operatorname{ad} h\,(z)\}$, or $\{h, \operatorname{ad} h\,(x), \operatorname{ad} h\,(z)\}$, or $\{h, \operatorname{ad} h\,(x), \operatorname{ad} h\,(y)\}$ is linearly independent. The calculations

$$\begin{vmatrix} h_x & -h_z & h_y \\ h_y & 0 & -h_x \\ h_z & h_x & 0 \end{vmatrix} = h_x(h_x^2+h_y^2+h_z^2),$$

$$\begin{vmatrix} h_x & 0 & h_y \\ h_y & h_z & -h_x \\ h_z & -h_y & 0 \end{vmatrix} = -h_y(h_x^2+h_y^2+h_z^2),$$

$$\begin{vmatrix} h_x & 0 & -h_z \\ h_y & h_z & 0 \\ h_z & -h_y & h_x \end{vmatrix} = h_z(h_x^2+h_y^2+h_z^2)$$

bear this out.

Now suppose that one of a, b, c is zero. Using the techniques above one can see that there are four cases to consider.

Case 3: $[x\ y] = 0, \quad [y\ z] = 0, \quad [z\ x] = 0.$

Here L is abelian.

Case 4: $[x\ y] = 0, \quad [y\ z] = 0, \quad [z\ x] = y.$

Note $[L\ L] = \mathbf{R}y$ and $[L\ y] = (0)$. Therefore L is nilpotent and $(\operatorname{ad} h)^2 = 0$ for all $h \in L$.

Case 5: $[x\ y] = 0, \quad [y\ z] = x, \quad [z\ x] = -y.$

Note $[L\ L] = \mathbf{R}x + \mathbf{R}y$ is abelian and $[L\ \mathbf{R}x + \mathbf{R}y] = \mathbf{R}x + \mathbf{R}y$. Thus L is solvable but not nilpotent. Also note that $\operatorname{ad} 2z$ is diagonalizable with eigenvalues $-2, 0, 2$; see Case 1.

Case 6: $[x\ y] = 0, \quad [y\ z] = x, \quad [z\ x] = y.$

Note $[L\, L] = \mathbf{R}x + \mathbf{R}y$ is abelian and $[L\, \mathbf{R}x + \mathbf{R}y] = \mathbf{R}x + \mathbf{R}y$. Thus L is solvable but not nilpotent.

We proceed as in Case 2. Let $h = h_x x + h_y y + h_z z$ where $h_x, h_y, h_z \in \mathbf{R}$. Then

$$\begin{array}{rclcl} \operatorname{ad} h\,(x) & = & h_x[x\,x] + h_y[y\,x] + h_z[z\,x] & = & h_z y \\ \operatorname{ad} h\,(y) & = & h_x[x\,y] + h_y[y\,y] + h_z[z\,y] & = & -h_z x \\ \operatorname{ad} h\,(z) & = & h_x[x\,z] + h_y[y\,z] + h_z[z\,z] & = & -h_x y + h_y x. \end{array}$$

The characteristic polynomial of $\operatorname{ad} h$ is therefore

$$\begin{array}{rcl} f(X) & = & \begin{vmatrix} X - 0 & h_z & -h_y \\ -h_z & X - 0 & h_x \\ 0 & 0 & X - 0 \end{vmatrix} \\ & = & X(X^2 + h_z^2). \end{array}$$

Since $X^2 + h_z^2$ is irreducible when $h_z \neq 0$, it follows that $\operatorname{ad} h$ is not diagonalizable unless $h = 0$. In particular the Lie algebras of Cases 5 and 6 are not isomorphic.

There are six isomorphism types of Lie algebras such that (2) is satisfied when $F = \mathbf{R}$, two of which are simple. The reader is encouraged to analyze the Lie algebras satisfying (2) when F is an algebraically closed field of characteristic zero, of characteristic 2, or of characteristic $p > 2$.