Math 165 Dummy Exam II Lowman F10
Solutions

(1.) Derivative Rules:

Power Rule: $\frac{d}{dx} f(x)^n = n \cdot f(x)^{n-1} \cdot f'(x)$

Exponential Rule: $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'x$

Log Rule: $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$

(2.) Integral Rules:

Power: $\int f'(x) \cdot f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$

Exponential: $\int f'(x) \cdot e^{f(x)} \, dx = e^{f(x)} + C$

Log: $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

3. Log Rules (mostly in base e)

$x = e^y, \quad y = \ln x$ definition

$\ln 1 = 0$

$\ln e = 1$

$e^{\ln x} = x$

$\ln e^x = x$

$\ln(A \cdot B) = \ln A + \ln B$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$\ln(x^n) = n \cdot \ln x$

if $\ln A = \ln B$ then $A = B$

$\log_a x = \frac{\log_b x}{\log_b(a)} = \frac{\ln x}{\ln a}$

$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$; $\int \frac{f'(x)}{f(x)} dx = \ln|x| + C$

(10) $$\ln(4e^x) + \underbrace{\cancelto{0}{\log_2(1)}\cdot(x^2+1)}_{0} + \cancelto{1}{\ln(e)}\cdot\ln(2e^{3x}) = \ln 16 + \underbrace{\underbrace{\ln(e^2)}_{2} - 2}_{0}$$

$$\ln(4e^x) + 0 + 1\cdot\ln(2e^{3x}) = \ln 16 + 0$$

$$\ln(4e^x) + \ln(2e^{3x}) = \ln 16$$

$$\ln(4e^x \cdot 2e^{3x}) = \ln 16$$

$$\ln(8e^x \cdot e^{3x}) = \ln(16)$$

$$\ln(8e^{x+3x}) = \ln(16)$$

$$\ln(8e^{4x}) = \ln 16$$

$$\frac{\cancelto{1}{8}e^{4x}}{\cancel{8}} = \frac{\cancelto{2}{16}}{\cancel{8}}$$

$$e^{4x} = 2$$

$$\ln e^{4x} = \ln 2$$

$$4x = \ln 2$$

$$\boxed{x = \frac{\ln(2)}{4}}$$ exact answer

(check answer in orig. eq.)

8.

Year	t	$t=0$ 1990	$t=10$ 2000	$t=20$ 2010
GDP ($\times 10^9$)	y	100	200	?

Step 1: Use data to find constants in exponential growth function.

$$y = A e^{kt}$$

$t=0$: $100 = A e^{k \cdot 0}$

$$100 = A \cancel{e^0} \to 1$$

$$100 = A$$

Gives $y = 100 e^{kt}$, find k

$t=10$: $2\cancel{00} = 1\cancel{00} e^{k \cdot 10}$

$$2 = e^{10k}$$

$$\ln 2 = \ln\left(e^{10k}\right)$$

$$\ln 2 = 10k$$

$$\boxed{k = \frac{\ln(2)}{10}}$$

$$\boxed{y = 100\, e^{\frac{\ln(2)}{10}\cdot t}}$$

exponential growth function

at $t = 20$ (ie. year 2010)

$$y = 100\, e^{\frac{\ln(2)}{\cancel{10}}\cdot \cancel{20}^{\,2}}$$

$$y = 100\, e^{2\ln 2}$$

$$= 100\, e^{\ln(2^2)}$$

$$= 100\, e^{\ln(4)}$$

$$= 100 \cdot 4$$

$$\boxed{y = 400\ (\times 10^9)}$$ note: GDP doubles every 10 years

(11.) $A = P\left(1 + \frac{r}{k}\right)^{kt}$, $r = 10\% = .10$

$$1{,}000{,}\cancel{000} = 1\cancel{000}\left(1 + \frac{.10}{4}\right)^{4t}$$ ← solve for t

$\frac{1}{4} = .25$

$\frac{.1}{4} = .025$

$$1\,000 = (1 + .025)^{4t} = (1.025)^{4t}$$

$$1000 = (1.025)^{4t}$$

$$\ln(1000) = \ln(1.025^{4t})$$

$$\ln(1000) = 4t\ \ln(1.025)$$

$$4t = \frac{\ln(1000)}{\ln(1.025)}$$

$$\boxed{t = \frac{1}{4}\,\frac{\ln(1000)}{\ln(1.025)}\ \text{years}}$$

This is the exact answer, the non-calculator answer expected on the exam.

If a calculator was available, you would find $t \approx 69.9$ years.

12. $A = Pe^{rt}$

$$\frac{1{,}000{,}\cancel{000}}{1{,}\cancel{000}} = \frac{1{,}\cancel{000}}{\cancel{1000}}\, e^{.10t}$$

$$1000 = e^{.10t}$$

$$\ln 1000 = \ln(e^{.10t})$$

$$\ln 1000 = (.10)t$$

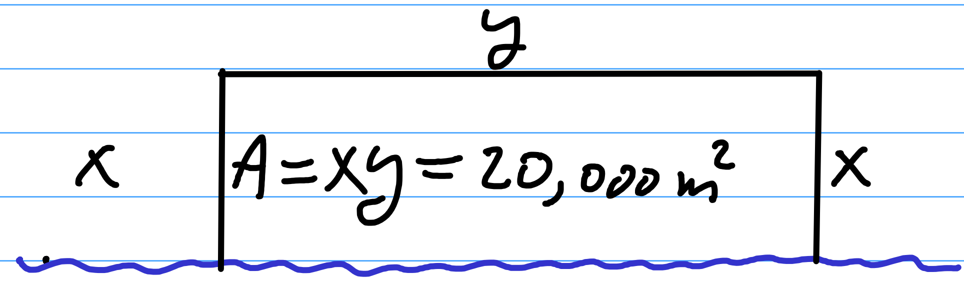
$$t = \frac{\ln(1000)}{(.10)} = \frac{\ln(10^3)}{\left(\frac{1}{10}\right)} = 10 \ln 10^3$$

$$\boxed{t = 30 \ln(10) \text{ years}}$$ exact answer expected on exam.

Using a calculator gives
$y \approx 69.1$ years.

(13.) $\log_2(100.7) = \dfrac{\ln(100.7)}{\ln(2)}$

(7.)



Minimize length: $L = 2x + y$

Subject to constraint: $xy = 20{,}000$

Use constraint to eliminate variable from L.

$$y = \frac{20{,}000}{x}$$

Gives new problem:

Minimize Length:

$$\boxed{L(x) = 2x + \frac{20,000}{x}}$$

$$L' = \frac{d}{dx}\left(2x + 20,000x^{-1}\right)$$

$$= 2 + 20,000 \cdot (-1)x^{-2}$$

$$\boxed{L'(x) = 2 - \frac{20,000}{x^2}}$$

Critical numbers where $L'(x) = 0$

$$2 - \frac{20,000}{x^2} = 0$$

$$2 = \frac{20,000}{x^2}$$

$$x^2 = \frac{20,000}{2}$$

$$x^2 = 10,000$$

$$x^2 = 10^4$$

$$(x^2)^{1/2} = (10^4)^{1/2} \quad \text{(only use + root)}$$

$$x = 10^2 = 100\ m$$

Use 2nd derivative test to check if $x = 100$ gives minimum length

$$L' = 2 - 20{,}000x^{-2}$$

$$L''(x) = 0 - 20{,}000 \cdot (-2)x^{-3}$$

$$L''(x) = +\frac{40{,}000}{x^3}$$

Now 2nd derivative test:

$$L''(100) = \frac{40{,}000}{(100)^3} = \boxed{+} \quad \text{minimum} \checkmark$$

Summary:

$$x = 100\ m$$

$$y = \frac{20{,}000}{x} \quad \text{(from constraint)}$$

$$= \frac{20{,}\cancel{000}}{1\cancel{00}} = 200\ m$$

Total Minimum Length:

$$L_{min} = 2x + y$$

$$= 2(100\,m) + 200\,m$$

$$\boxed{L_{min} = 400\,m.} \quad \boxed{x = 100\,m, \; y = 200\,m}$$

(4.) $$\int_0^1 \underline{(3x^2 + 10x^4)} \cdot (2x^3 + 4x^5 + 1)^9 \, dx$$

try power rule: $\int f'(x) \cdot f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$

need

$$\frac{d}{dx}(2x^3 + 4x^5 + 1) = 2 \cdot 3x^2 + 5 \cdot 4x^4$$

$$= \underline{2(3x^2 + 10x^4)}$$

$$= \frac{1}{2} \cdot \int_0^1 2(3x^2 + 10x^4) \cdot (2x^3 + 4x^5 + 1)^9 \, dx$$

$$= \frac{1}{2} \frac{(2x^3 + 4x^5 + 1)^{10}}{10} \Bigg|_{x=0}^{1}$$

$$= \frac{1}{20}\left[\left(2(1)^3+4(1)^5+1\right)^{10} - \left(2(0)^3+4(0)^5+1\right)^{10}\right]$$

$$= \frac{1}{20}\left[7^{10} - 1^{10}\right] = \frac{1}{20}\left[7^{10} - 1\right]$$

(5.) $$I = \int_1^2 \frac{6x^2+20x^4}{(2x^3+4x^5+1)}\,dx$$

try Log Rule

$$\int \frac{f'}{f}\,dx = \ln|f| + c$$

check

$$\frac{d}{dx}(2x^3+4x^5+1) = 6x^2+20x^4$$

perfect match.

$$I = \Big[\ln|2x^3+4x^5+1|\Big]_{x=1}^{2}$$

$$= \ln|\overbrace{2(2)^3+4(2)^5+1}^{+}| - \ln|\overbrace{2(1)^3+4(1)^5+1}^{+}|$$

$$= \ln(2^4+2^7+1) - \ln(7)$$

ok ans.

$$= \boxed{\ln\left(\frac{2^4+2^7+1}{7}\right)}$$

(6.) $I = \int_2^3 (6x^2 + 20x^4)\, e^{(2x^3+4x^5+1)}\, dx$

try exponential rule: $\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + c$

check $\frac{d}{dx}(2x^3+4x^5+1) = 6x + 20x^4$ perfect match.

$I = \left[e^{2x^3+4x^5+1} \right]_{x=2}^{3}$

$= \left[e^{2(3)^3+4(3)^5+1} \right] - \left[e^{2(2)^3+4(2)^5+1} \right]$

OK answer for exam w/o calculator

(9.) $f(x) = (4+3x)^{2x}$ use logarithmic differentiation to find $f'(x)$

$\ln f(x) = \ln (4+3x)^{2x}$

$\ln f(x) = 2x \cdot \ln(4+3x)$

$\frac{d}{dx}[\ln f(x)] = \frac{d}{dx}[2x \cdot \ln(4+3x)]$ ← use product rule

use log rule

$$\frac{f'(x)}{f(x)} = \cancel{\frac{d}{dx}(2x)} \xrightarrow{} 2 \cdot \ln(4+3x) + 2x \cdot \frac{d}{dx} \ln(4+3x)$$

$$\frac{f'(x)}{f(x)} = 2 \cdot \ln(4+3x) + 2x \, \frac{\cancel{\frac{d}{dx}(4+3x)} \to 3}{4+3x}$$

$$\frac{f'(x)}{f(x)} = 2 \ln(4+3x) + \frac{6x}{4+3x}$$

multiply Both sides by $f(x)$

$$f'(x) = f(x) \left[2 \cdot \ln(4+3x) + \frac{6x}{4+3x} \right]$$

replace $f(x)$ with $f(x) = (4+3x)^{2x}$

$$f'(x) = (4+3x)^{2x} \cdot \left[2 \cdot \ln(4+3x) + \frac{6x}{4+3x} \right]$$

Now evaluate at $x = 1$.

$$f'(1) = (4+3\cdot 1)^{2\cdot 1} \cdot \left[2 \cdot \ln(4+3\cdot 1) + \frac{6 \cdot 1}{4+3\cdot 1} \right]$$

$$= 7^2 \cdot \left[2 \cdot \ln(7) + \frac{6}{7} \right]$$ ok answer

or

$$\boxed{f'(1) = 2 \cdot 7^2 \cdot \ln 7 + 6 \cdot 7}$$