Math 165, Fall '10, Lowman L10 - W4L2 Wed.

Implicite Differentiation

First review with some basic examples.

ex1 $\frac{dx}{dx}=1$, $\frac{dy}{dy}=1$, $\frac{dt}{dt}=1$, $\frac{d(\text{horse})}{d\text{horse}}=1$

$\frac{d}{dx}e^x = e^x$ simple exponential rule

can also use the general exponential rule:

$\frac{d}{dx}e^x = e^x \cdot \cancel{\frac{dx}{dx}}^{\,1} = e^x$

$\frac{d\,e^y}{dy} = e^y$, simple rule

$\frac{d}{dt}e^t = e^t$

Note that in all three examples, the function in the exponent is the same as the variable of differentiation so can use the simple version of the exponential rule.

More examples

$$\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot \frac{du}{dx}$$, general rule

Note that the function in the exponent is not the same as the variable of differentiation so must use the general version of the exponential rule.

$$\frac{d}{dx} e^{y} = e^{y} \cdot y'(x)$$, assume y is a function of x.

$$\frac{d}{dx} e^{x^2+x} = e^{x^2+x} \cdot \frac{d}{dx}(x^2+x)$$

$$= e^{x^2+x} \cdot (2x+1)$$

$$\frac{d}{dy} e^{(y^3+x)} = e^{y^3+x} \cdot \frac{d}{dy}(y^3+x)$$

$$= e^{y^3+x} \cdot (3y^2+1)$$

$$\frac{d}{dx} e^{(y^3+x)} = e^{(y^3+x)} \cdot \frac{d}{dx}(y^3+x)$$

$$= e^{(y^3+x)} \cdot \left(\frac{d}{dx} y^3 + \frac{dx}{dx}\right)$$

$$= e^{(y^3+x)} \cdot \left(3y^2 \cdot \frac{dy}{dx} + 1\right)$$

same

$$= e^{(y^3+x)} \cdot \left(3y^2 \cdot y' + 1\right)$$

Power rule examples:

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

$$\frac{d}{dy} y^n = n \cdot y^{n-1}$$

$$\frac{d}{dt} t^n = n \cdot t^{n-1}$$

use simple version of power rule because the base function is the same as the variable of differentiation.

Note that you can always use the general version of a rule:

$$\frac{d}{dx} x^n = n \cdot x^{n-1} \cdot \cancelto{1}{\frac{dx}{dx}} = n \cdot x^{n-1}$$

$$\frac{d}{dx} (x^2+x)^4 = 4(x^2+x)^3 \cdot \frac{d}{dx}(x^4+x)$$

$$= 4(x^2+x)^3 \cdot (4x^3+1)$$

$$\frac{d}{dy} (y^2+y)^4 = 4(y^2+y)^3 \cdot \frac{d}{dy}(y^2+y)$$

$$= 4(y^2+y)^3 \cdot \left(\frac{d}{dy}y^3 + \frac{d}{dy}y\right)$$

$$= 4(y^2+y)^3 \cdot (3y^2 + 1)$$

$$\frac{d}{dx} y^n = n \cdot y^{n-1} \cdot y'(x),$$ Need general

not same

now.

Implicite Differentiation

Given:

$$2x + 3y = 6x + 7 \quad \boxed{\text{find } y'(x)}$$

This gives y as a function of x using an implicite equation for $y(x)$.

There are two ways to find $y'(x)$

1st method: first solve for y giving $y = y(x)$ as an explicite equation.

2nd method: use implicite differentiation

Here is the 1st method:

(a) Solve for $y = y(x)$

(b) Take derivative of both sides:

$$\frac{d}{dx} y = \frac{d}{dx} y(x) \quad \cdots$$

(a) solve for y:

$$2x + 3y = 6x + 7$$
$$-2x \qquad\qquad -2x$$

$$\cancel{\tfrac{1}{3}}(\cancel{3}y) = \frac{1}{3}(4x + 7)$$

$y = y(x)$

$$y = \frac{4}{3}x + \frac{7}{3}$$

Now take derivative of both sides.

(b)

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{4}{3}x + \frac{7}{3}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{4}{3}x\right) + \cancel{\frac{d}{dx}\left(\frac{7}{3}\right)} \to 0$$

$$y'(x) = \frac{4}{3}\cancel{\frac{dx}{dx}} \to 1 + 0$$

$$\boxed{y'(x) = \frac{4}{3}}$$

Now repeat the problem and find $y(x)$ using implicite differentiation.

$$2x + 3y = 6x + 7 \qquad (y = y(x))$$

$$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(6x + 7)$$

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(6x) + \cancel{\frac{d}{dx}7} \to 0$$

want this

$$2 \cdot \cancel{\frac{dx}{dx}} \to 1 \; + 3 \cdot \boxed{\frac{dy}{dx}} = 6 \cdot \cancel{\frac{dx}{dx}} \to 0 \; + 0$$

$$2 + 3y'(x) = 6$$, now Solve for $y'(x)$

$$-2 \qquad\qquad -2$$

$$\cancel{\tfrac{1}{3}}(\cancel{3}\, y'(x)) = \tfrac{1}{3}(4)$$

$$\boxed{y'(x) = \frac{4}{3}}$$

Same answer

In this case it was easier to not use implicite differention.

Implicite differentiation is useful when it is hard or impossible to 1st solve for $y = y(x)$

Another Example using implicite differentiation

$x\cdot y^3 = x^4 + x^2$, assume $y = y(x)$

$\frac{d}{dx}(x\cdot y^3) = \frac{d}{dx}(x^4 + y^2)$

use product rule

$x\cdot\frac{d}{dx}y^3 + y^3\cdot\cancelto{1}{\frac{dx}{dx}} = \frac{d}{dx}x^4 + \frac{d}{dx}y^2$, where $y=y(x)$

$x\cdot 3y^2\cdot\boxed{y'} + y^3 = 4x^3 + 2y\cdot\boxed{y'}$ now solve for $y'(x)$

$-y^3 \qquad -y^3$

$3xy^2\cdot y' = 4x^3 - y^3 + 2y\cdot y'$

$-2y\cdot y' \qquad -2y\cdot y'$

$3xy^2\cdot y' - 2y\cdot y' = 4x^3 - y^3$

$(3xy^2 - 2y)y' = 4x^3 - y^3$

$$\cancelto{1}{\frac{(3xy^2-2y)}{(3xy^2-2y)}}\,y' = \frac{4x^3-y^3}{(3xy^2-2y)}$$

$$\boxed{y'(x) = \frac{4x^3 - y^3}{3xy^2 - 2y}}$$

In this case, it is to use the original equation for $y(x)$ and substitute.