Math 165 Fall 2010 W3 L1 (wed.)

Definition of Derivative

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

example $f(x) = k$ (a constant)

Use the definition of the derivative to find the slope of the tangent line at x.

$f(x) = k$, $f(x+\Delta x) = k$ $\Rightarrow$

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{k-k}{\Delta x} \right] = \lim_{\Delta x \to 0} \left[\frac{0}{\Delta x} \right] = \lim_{\Delta x \to 0} [0] = 0$$

Gives derivative rule:

$\boxed{\frac{dk}{dx} = 0}$, derivative of a constant is 0. (slope of horozontal line is zero) example:

$f(x) = x$, find slope of $f(x)$ at x.

Use definition of the derivative

$f(x) = (x) \qquad f(x+\Delta x) = (x+\Delta x)$

$$\frac{df}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{(\cancel{x}+\Delta x) - \cancel{x}}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cancel{\Delta x}^{\,1}}{\cancel{\Delta x}}\right]$$, simplify before taking the limit

$$= \lim_{\Delta x \to 0}\left[1\right]$$

$$= 1$$

This gives another derivative rule:

$$\boxed{\frac{dx}{dx} = 1}$$

example: Use the definition of the derivative to find the slope of $f(x) = x^2$ as a function of x.

$f(x) = (x)^2$ $\qquad f(x+\Delta x) = (x+\Delta x)^2 = (x+\Delta x)(x+\Delta x)$

$= x^2 + 2x \cdot \Delta x + (\Delta x)^2$

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{(x+\Delta x)^2 - x^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{[\cancel{x^2} + 2x \cdot \Delta x + (\Delta x)^2] - \cancel{x^2}}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{2x \cdot \Delta x + (\Delta x)^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} \right]$$

$$= \lim_{\Delta x \to 0} [2x + \Delta x]$$

$$= 2x$$

Note: $\Delta x \to 0$, but x does not change.

<u>example</u>: $f(x) = x^3$ find $\frac{df}{dx} = ?$

$$f(x+\Delta x) = (x+\Delta x)^3 = (x+\Delta x)(x+\Delta x)(x+\Delta x)$$

$$= (x+\Delta x)\,(x+\Delta x)^2$$

$$= (x+\Delta x)\left(x^2 + 2x\cdot\Delta x + (\Delta x)^2\right)$$

$$= x^3 + 2x^2\cdot\Delta x + x\cdot(\Delta x)^2$$
$$+ \; x^2\cdot\Delta x + 2x\cdot(\Delta x)^2 + (\Delta x)^3$$

$$= x^3 + 3x^2\cdot\Delta x + 3x\cdot(\Delta x)^2 + (\Delta x)^3$$

$$\frac{df}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{[\cancel{x^3} + 3x^2\cdot\Delta x + 3x\cdot(\Delta x)^2 + (\Delta x)^3] - [\cancel{x^3}]}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cancel{\Delta x}\cdot\left(3x^2 + 3x\cdot\Delta x + (\Delta x)^2\right)}{\cancel{\Delta x}}\right]$$

$$= \lim_{\Delta x \to 0}\left[3x^2 + 3x\cdot\Delta x\right]$$

$$= \boxed{3x^2}$$

<u>note</u>: Δx is going to zero but x is not.

If continue for $f(x) = x^4$, $f(x) = x^6 \dots f(x) = x^n$

you will get:

$$\frac{d x^2}{dx} = 2 \cdot x^1$$

$$\frac{d}{dx}(x^3) = 3 \cdot x^2$$

$$\frac{d}{dx}(x^4) = 4 \cdot x^3$$ (try it!)

$$\frac{d}{dx}(x^5) = 5 \cdot x^4$$

...

$$\boxed{\frac{d}{dx}(x^n) = n \cdot x^{n-1}}$$

Another derivative rule called the <u>Power Rule</u>

Using the definition of the derivative, more rules can be found for finding derivatives of functions.

Here is a partial list.

Derivative Rules

$$\frac{d}{dx}k = 0 \qquad k = \text{constant}$$

$$\frac{dx}{dx} = 1$$

$$\frac{d}{dx}x^n = n \cdot x^{n-1},$$ Simple Power Rule

$$\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x),$$ Constant Times function

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x),$$ Sum Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x)$$ Product Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{df}{dx} - f(x) \cdot \frac{d}{dx}g(x)}{g(x)^2}$$ Quotient Rule

$$\frac{d}{dx}e^x = e^x,$$ Simple Exponential Rule

$$\frac{d}{dx}\ln x = \frac{1}{x},$$ Simple Log Rule

Here are general versions of the Power Rule, Exponential Rule and Log Rule.

$$\frac{d}{dx} u(x)^{n} = n \cdot u(x)^{n-1} \cdot \frac{du}{dx},$$ General Power Rule

$$\frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot \frac{du}{dx},$$ General Exponential Rule

$$\frac{d}{dx} \ln u(x) = \frac{\frac{du}{dx}}{u(x)},$$ General Log Rule

These three rules are very useful.