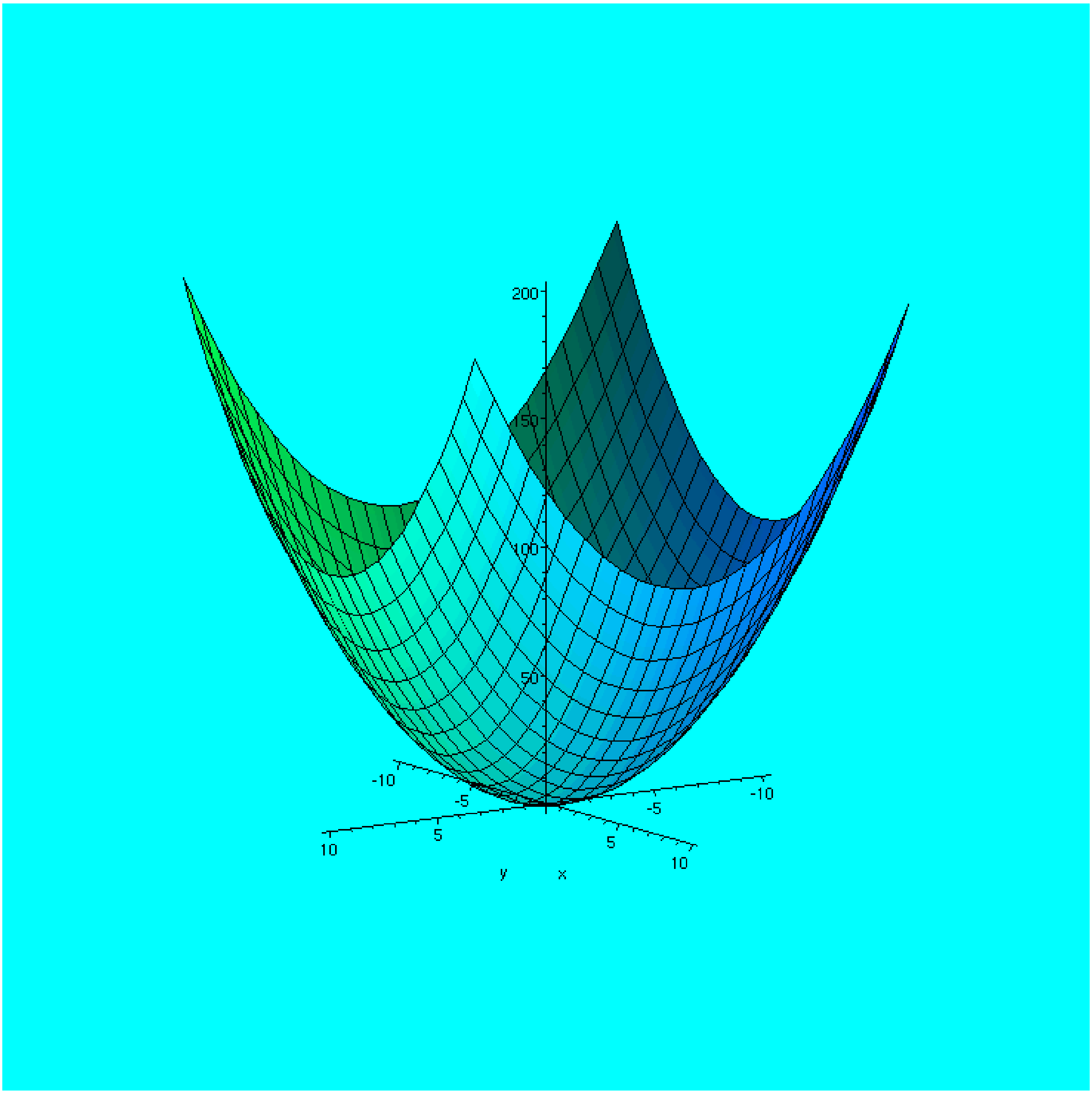
```
> restart;

> # Note, lines beginning with # are comments and not executed by maple

> #Math 165, Spring 2010, Week 14, Friday Lecture Notes

> # Functions of two variables, critical points, partial derivatives.

> # Here is an example of a paraboloid: z = f(x,y) = x^2 + y^2

> # This is how to define the function in maple

> f := (x,y)-> x^2 + y^2;
```

$$f := (x, y) \rightarrow x^2 + y^2$$

```
> # now graph the function

> plot3d(f(x,y), x = -10..10, y=-10..10);
```



> **# The point at (x,y) = (0,0) is a relative minimum. The actual point on the graph is (x,y,z)=(0,0,0)**

> **# Observe that all tangent lines to the point at (0,0) have zero slope.**

> **# To show that the point (x,y,z)=(0,0,0) is a critical point we only need to show that the tangent along**

> **# the x-direction has zero slope and the tangent along the y-direction also has zero-slope.**

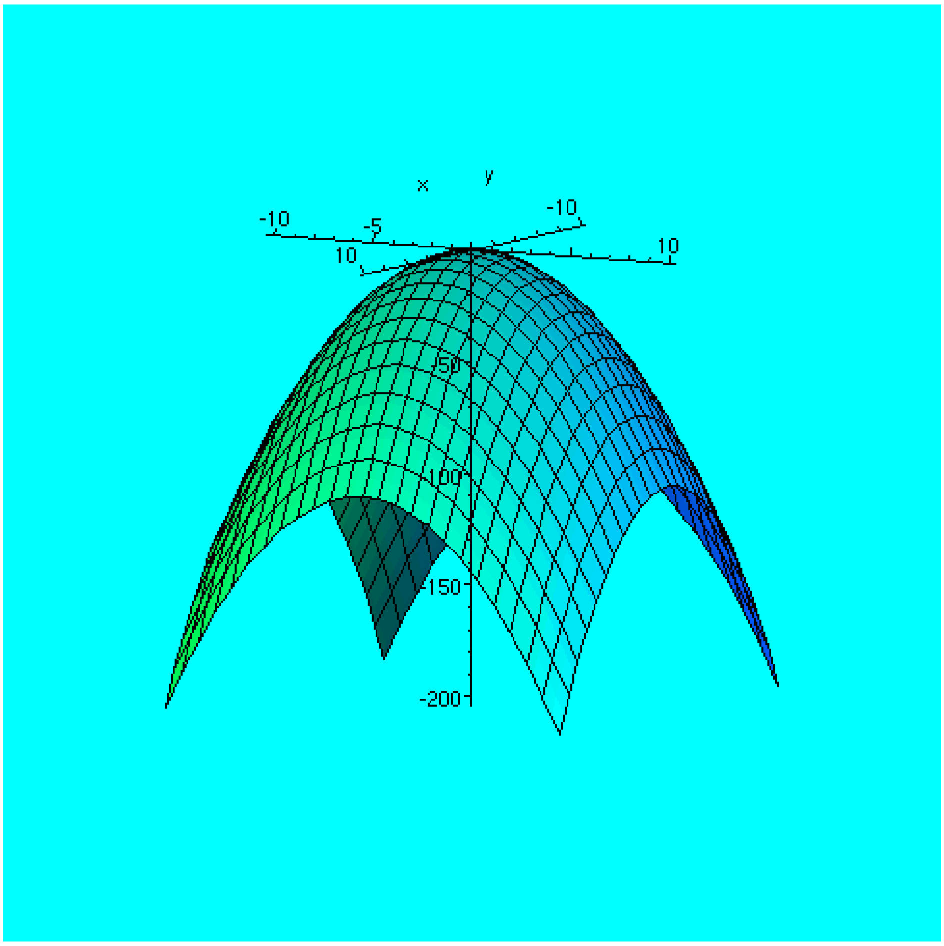
> **#**

> **# The curves on the graph are called contours. There are two kinds of contours.**

```
> # One kind results from keeping y fixed at a single value and graphing function. For example
> # if y=0 then the function z = f(x,y) = x^2+y^2 becomes the contour z = f(x,0)= x^2. This is a parabola
> # that opens up and has zero slope at (x,y)=(0,0). The
> # first derivative fx = 2x and second derivative fxx = 2. Notice that the first derivative is zero at
> # the critical point (0,0) and the second derivative is + (holds water). From this you would conclude
> # that the function is a minimum at (x,y)=(0,0) when only letting x vary.
> #
> # Now repeat in the other direction. Let x=0 and let y change. This gives the contour z = f(0,y) = y^2.
> # This contour has slope (first derivative along y-dir) fy = 2y and second derivative along the y-direction
> # fyy = 2. The first derivative, fy, of the contour at (x,y)=(0,0) is zero at the critical point and the
> # second derivative is positive at the CP. So from this you would conclude that the function is a minimum
> # at (x,y)=(0,0) when only letting only y vary.
> #
> # Putting the information from both directions together: both fx=0 and fy=0 =>(x,y)=(0,0) are coordinates of a
> # critical point and the second derivatives fxx = +, fyy = + tell us that the CP is a Relative Maximum.
> #
> #There is a slightly more complicated version of the Second Derivative Test for functions z=f(x,y). However,
> # it just extends what you already know about the second derivative and the second derivative test for y=f(x).
> #
> # Now look at an inverted paraboloid.
> f := (x,y)-> -x^2 - y^2;
```

$$f := (x, y) \rightarrow -x^2 - y^2$$

> **plot3d(f(x,y), x=-10..10, y=-10..10);**



> **# The critical point occurs where all tangent lines have zero slope. i.e. where both fx=0 and fy=0.**

> **# fx = -2x and fy -2y, set both = 0 and solve for x and y of the critical point. solving the system**

> **# of equations 2x=0 and 2y=0 gives only one solution: (x,y)=(0,0) are the coordinates of the only**

> **# critical point. z = f(0,0) = 0^2+0^2 = 0. This gives the actual critical point (x,y,z)=(0,0,0).**

> **# Looking at the second derivatives at (0,0) fxx = -2 = negative and fyy = -2 = negative. The graph at the**

> **# CP is concave up in both directions so the CP is a Relative Maximum.**

> **#**

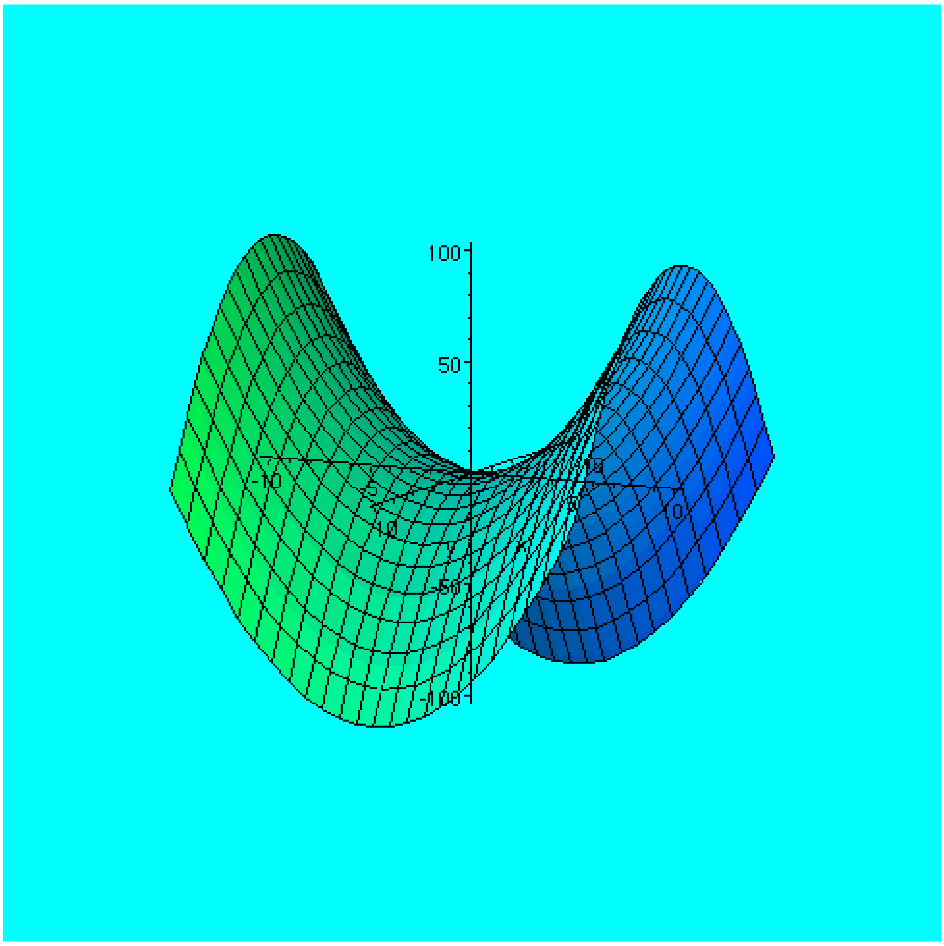
> **# Note, for details on notation and how to find partial derivatives, see the actual lecture notes.**

> **# What is given here is just a reminder of what was actually done in class.**

```
> #

> # Here is an important observation for these two simple examples that will also be true when we consider

> # more general functions.

> # 1. Critical points occur where fx=0 and fy=0 (slopes in both directions are 0 at a single point)

> # 2. When the CP was a minimum: both fxx=(+) and fyy=(+). Concave up in both directions gives the product fxx * fyy = (+)

> # 3. When the CP was a maximum: both fxx=(-) and fyy=(-). Concave down in both directions gives the product

> # fxx * fyy = (+)

> # 4. Observe that for both a maximum and a minimum the product fxx * fyy = (positive). This is because both

> # fxx and fyy have the same sign (both pos for min or both neg for max). This will be a useful memory aid

> # when using the general version of the second derivative test D(x,y) = fxx * fyy - fxy * fyx. If D(x,y)

> # is positive at a critical point then the point is either a relative max or relative min. To find out

> # which one, you must look at the sign of either fxx or fyy (does not matter which one) at the critical point to see if they are both positive (min) or both negative (max).

> #

> # Now look at a simple example of a saddle surface with a saddle point.

> f := (x,y)-> -x^2 + y^2;

>
```

$$f := (x, y) \rightarrow -x^2 + y^2$$

```
> plot3d(f(x,y), x=-10..10, y=-10..10);
```



S

> # you can see that there is a critical point at (x,y)=(0,0). Both fx=0 and fy=0 at this point.

> # Along the x-direction at the CP the contour is concave down (fxx = -) and along the y-direction

> # the contour is concave up (fyy = +)at the CP. These are the chatacteristics of Saddle Pointsl

> #

> # Observe that for a saddle point:

> # 1. fx=0 and fy=0 (it is a critical point)

> # 2. fxx = (-) and fyy = (+)

> # 3. At the saddle point D(x,y) = fxx * fyy - fxy * fyx evaluates to a negative value because the first

> # term in D determine the sign of D and will be (-) because the contours go in opposite directions giving.

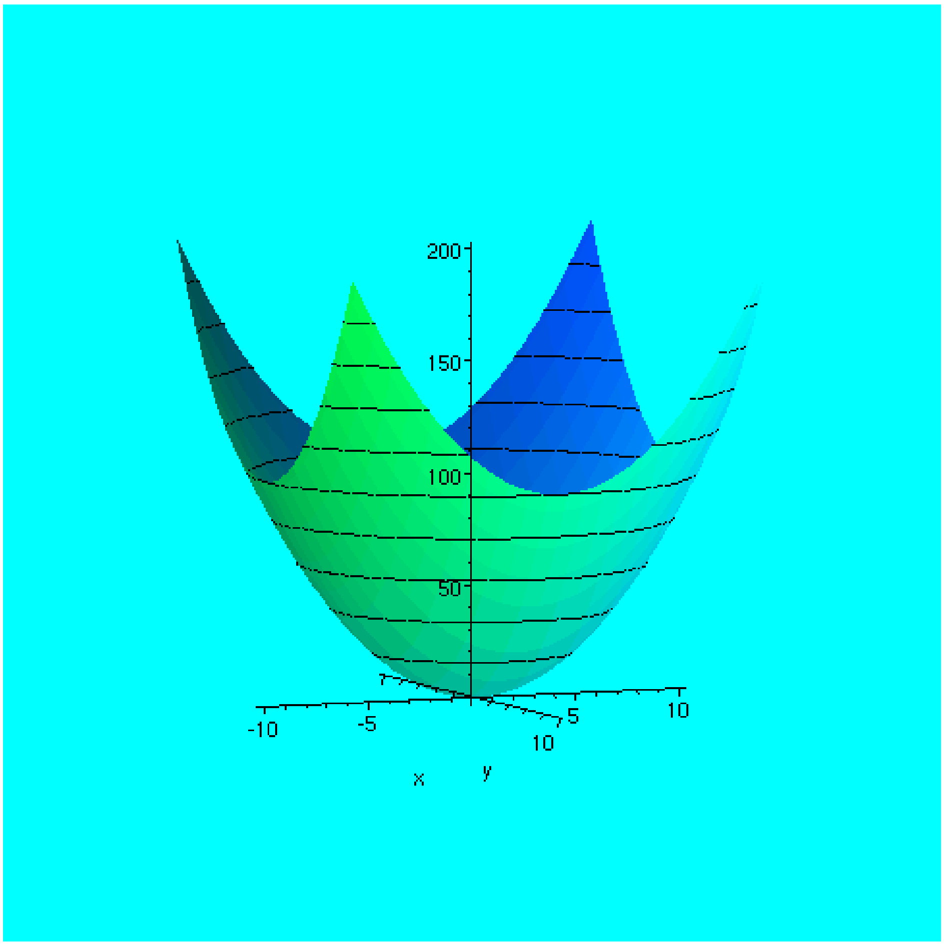
> # opposite signs to fxx and fyy => D(x,y) is negative at a saddle point.

> #

> **# Using D(x,y) in this way to determine what kind of critical points is an extension of the second derivative test for functions of one variable y = f(x).**

```
> #

> # Level Curves (instead of contours) are also useful when graphing the
  above three functions.

> # A Level Curve is the graph that results on the surface of z=f(x,y) when the
  z, the height of the graph, is

> # kept at a constant value. Level curves are useful in applications.

> #

> f := (x,y)->  x^2 + y^2;

> plot3d(f(x,y), x=-10..10, y=-10..10);
```

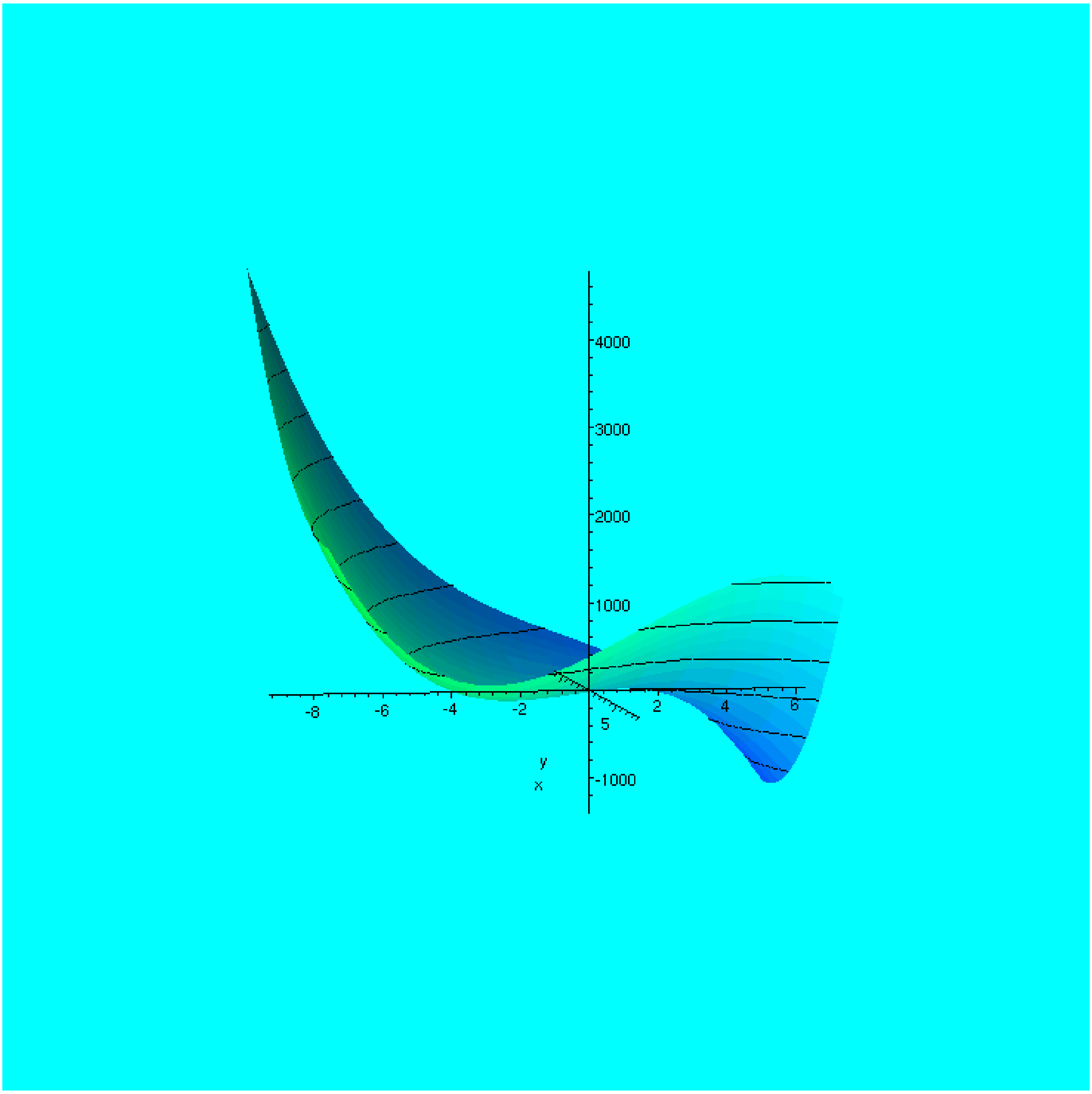
$$f := (x, y) \rightarrow x^2 + y^2$$




```
> # here is the function from problem 2 in special assignment 5

> f := (x,y)-> 12*x^2 - 4*y^3 + 24*x*y + 20;
```

$$f := (x, y) \rightarrow 12\,x^2 - 4\,y^3 + 24\,x\,y + 20$$

> **plot3d(f(x,y), x = -6..9, y = -9..6);**



>