SECOND MIDTERM MA 415,
MARCH 25TH 2008

To receive credit for an exercise your solution must be justified.

1. EXERCISE (25 POINTS)

Find, using the method of least squares, the linear function $y = a_0 + a_1 x$ that best fits the data

x	-1	0	1	2
y	1	1	2	3

SOLUTION

We are trying to find the least possible squares solution to the system of equations

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

So the normal equations are

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix},$$

or

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}.$$

We solve this using row reduction

$$\left(\begin{array}{cc|c} 4 & 2 & 7 \\ 2 & 6 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 0 & -10 & -7 \\ 2 & 6 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 3 & 7/2 \\ 0 & 1 & 7/10 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 7/5 \\ 0 & 1 & 7/10 \end{array}\right)$$

So the least possible squares solution is $y = \frac{7}{5} + \frac{7}{10}x$.

2. Exercise (50 points)

Compute the QR factorisation of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 3 \end{pmatrix}.$$

I.e., write $A = QR$, where Q is orthogonal and R is upper triangular with non-zero diagonal entries.

Solution

We have $r_{11} = \|\text{first column}\| = \sqrt{2}$, so replacing the first column by $\vec{u}_1 = \frac{\text{first column}}{r_{11}}$

$$A \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{\sqrt{2}} & 0 & 3 \end{pmatrix}$$

Now, $r_{12} = \text{second column} \bullet u_1 = 0$ and $r_{13} = \text{third column} \bullet \vec{u}_1 = 2\sqrt{2}$, so subtracting $r_{12}\vec{u}_1$ and $r_{13}\vec{u}_1$ from resp. the second and third column

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{\sqrt{2}} & 0 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & 2 & 4 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}$$

Again, $r_{22} = \|\text{second column}\| = 2$, so replacing the second column by $\vec{u}_2 = \frac{\text{second column}}{r_{22}}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & 2 & 4 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & 1 & 4 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}$$

and $r_{23} = \text{third column} \bullet \vec{u}_2 = 4$, so subtracting $r_{23}\vec{u}_2$ from the third column

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & 1 & 4 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}$$

Finally, $r_{33} = \|\text{third column}\| = \sqrt{2}$, so replacing the third column by $\vec{u}_3 = \frac{\text{third column}}{r_{33}}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -1 \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = Q$$

Therefore,

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 2\sqrt{2} \\ 0 & 2 & 4 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = QR$$

3. Exercise (25 points)

Let

$$Q = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{pmatrix}.$$

(a) Define what it means for a square matrix to be orthogonal.

(b) Determine whether the orthogonal matrix Q is proper or improper.

(c) Let $\vec{u}_1, \vec{u}_2, \vec{u}_3$ be the columns of Q and write $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$, i.e., find the coordinates of $\vec{b}$ with respect to the basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ of $\mathbb{R}^3$.

Solution

(a) Q is orthogonal if and only if $Q^tQ = I$ or equivalently if and only if the columns of Q form an orthonormal set.

(b)

$$\det Q = detQ^t = \frac{1}{\sqrt{6}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \det \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} = \frac{1}{6}(-6) = -1.$$

So Q is improper.

(c) $\vec{b} = \sum_{i=1}^{3} (\vec{b} \bullet \vec{u}_i)\vec{u}_i = \frac{2}{\sqrt{6}}\vec{u}_1 + \sqrt{2}\vec{u}_2 - \frac{1}{\sqrt{3}}\vec{u}_3.$

4. Exercise (20 points)

(a) Compute the LDL^t factorisation of the symmetric matrix

$$K = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

I.e., find a lower triangular matrix L with 1's in the diagonal and a diagonal matrix D such that $LDL^t = K$.

(b) Is K positive definite?

Solution

(a)

$$\begin{aligned} K &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(c) Since D has a non-positive diagonal entry, K is not positive definite.