8.2.1 (a)

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \quad p_A(\lambda) = \det \begin{pmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4$$

$$= (3-\lambda)(-1-\lambda)$$

So A has eigenvalues 3 and -1.

The corresponding eigenspaces are:

$$V_3 := \ker \begin{pmatrix} 1-3 & -2 \\ -2 & 1-3 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$V_{-1} := \ker \begin{pmatrix} 1+1 & -2 \\ -2 & 1+1 \end{pmatrix} = \ker \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

and eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

8.2.1(e)

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}, \quad P_A(\lambda) = (3-\lambda)\big((2-\lambda)(3-\lambda)-1\big)$$
$$+1\big(-1(3-\lambda)\big)+0$$

$$= (3-\lambda)(6-2\lambda-3\lambda+\lambda^2-1) - 3 + \lambda = 15 - 15\lambda + 3\lambda^2 - 5\lambda + 5\lambda^2 - \lambda^3$$
$$-3+\lambda = 12 - 19\lambda + 8\lambda^2 - \lambda^3$$

$$= -\lambda^3 + \text{trace}\, A \cdot \lambda - \Big(\sum_{1\le i<j\le 3}(a_{ii}a_{jj} - a_{ij}a_{ji})\Big)\lambda + \det A$$

(using the formula shown in class)

By inspection, we see that $\lambda = 1$ is a root, so by long division we have

$$\begin{array}{l|l}
-\lambda+1 \;\big)\; -\lambda^3+8\lambda^2-19\lambda+12 \;\big|\; \lambda^2-7\lambda+12 & -\lambda+3 \;\big)\; \lambda^2-7\lambda+12 \;\big|\; -\lambda+4 \\
\quad -\lambda^3+\lambda^2 & \quad \lambda^2-3\lambda \\
\quad\quad 7\lambda^2-19\lambda+12 & \quad\quad -4\lambda+12 \\
\quad\quad 7\lambda^2-7\lambda & \quad\quad -4\lambda+12 \\
\quad\quad\quad -12\lambda+12 & \quad\quad\quad 0 \\
\quad\quad\quad -12\lambda+12 & \\
\quad\quad\quad 0 &
\end{array}$$

i.e., $P_A(\lambda) = (1-\lambda)(\lambda^2-7\lambda+12) = (1-\lambda)(3-\lambda)(4-\lambda)$

Again by inspection or by solving the polynomial by standard methods, we find $\lambda = 3$ to be another root. And $\lambda = 4$ is the last root.

8.2.1 (e) cont'd.

So the eigenvalues of A are $\lambda = 1, 3, 4$.

$$V_1 := \ker \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \ker \begin{pmatrix} 2 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$V_3 = \ker \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$V_4 = \ker \begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

So eigenvectors are $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

8.2.4

(a) To say that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = 0$ for the matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is to ask that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - 0 \cdot I) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ i.e., } \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \ker \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So, eg, $A = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$ works.

8.2.4 cont'd

Again for A 3×3, we want

$\begin{pmatrix}1\\2\\3\end{pmatrix} \in \ker(A+1\cdot I)$, so if $A+1\cdot I = \begin{pmatrix}6 & -3 & 0\\ 6 & 0 & -2\\ 0 & 0 & 0\end{pmatrix}$,

and $A = \begin{pmatrix}5 & -3 & 0\\ 6 & -1 & -2\\ 0 & 0 & -1\end{pmatrix}$ this works.

8.2.5

(a) $A = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ \alpha & \beta & \gamma\end{pmatrix}$ has characteristic equation

$$0 = p_A(\lambda) = -\lambda\left(-\lambda(\gamma-\lambda)-\beta\right) - 1\left(0(\gamma-\lambda)-\alpha 1\right)$$
$$= -\lambda^3 + \gamma\lambda^2 + \beta\lambda + \alpha .$$

(b) So any polynomial of degree ≤ 3 is the characteristic polynomial of some 3×3 matrix.

8.2.7 (a)

$A = \begin{pmatrix}i & 1\\ 0 & -1+i\end{pmatrix}$, $p_A(\lambda) = \lambda^2 - (2i-1)\lambda + (-i-1)$

$$= \lambda^2 + (1-2i)\lambda - 1 - i$$
$$= (i-\lambda)(i-1-\lambda)$$

So A has eigenvalues $\lambda = i, i-1$.

$V_i = \ker\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix} = \mathbb{R}\begin{pmatrix}1\\0\end{pmatrix}$, eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}$.

$V_{i-1} = \ker\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = \mathbb{R}\begin{pmatrix}1\\-1\end{pmatrix}$

8.2.14

$$A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

(a)

$$p_A(\lambda) = -\lambda^3 + \operatorname{trace} A \cdot \lambda^2 - \sum_{1 \le i < j \le 3} \left(a_{ii}a_{jj} - a_{ij}a_{ji}\right)\lambda + \det A$$

$$= -\lambda^3 + 3\lambda^2 - (-1+3-3-12)\lambda + (-3 - 4\cdot 9 + 4\cdot 6)$$

$$= -\lambda^3 + 3\lambda^2 + 13\lambda - 15$$ having root $\lambda = 1$.

Using long division:

$$\begin{array}{r|l} -\lambda+1 & -\lambda^3 + 3\lambda^2 + 13\lambda - 15 \quad \big| \; \lambda^2 - 2\lambda - 15 \\ & -\lambda^3 + \lambda^2 \\ & \quad 2\lambda^2 + 13\lambda - 15 \\ & \quad 2\lambda^2 - 2\lambda \\ & \qquad 15\lambda - 15 \\ & \qquad 15\lambda - 15 \\ \hline & \qquad\qquad 0 \end{array}$$

$$\begin{array}{r|l} & -\lambda + 5 \\ -\lambda-3 & \lambda^2 - 2\lambda - 15 \\ & \lambda^2 + 3\lambda \\ & \quad -5\lambda - 15 \end{array}$$

$$p_A(\lambda) = (1-\lambda)(\lambda^2 - 2\lambda - 15) = (1-\lambda)(-3-\lambda)(5-\lambda)$$

So the three eigenvalues are $\lambda = 1, -3, 5$ with eigenspaces

$$V_1 = \ker \begin{pmatrix} 0 & 4 & 4 \\ 3 & -2 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \ker \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$$

$$V_{-3} = \ker \begin{pmatrix} 4 & 4 & 4 \\ 3 & 2 & 0 \\ 0 & 2 & 6 \end{pmatrix} = \ker \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

$$V_5 = \ker \begin{pmatrix} -4 & 4 & 4 \\ 3 & -6 & 0 \\ 0 & 2 & -2 \end{pmatrix} = \ker \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

So the eigenvectors are $\begin{pmatrix}2\\3\\-3\end{pmatrix}, \begin{pmatrix}-2\\3\\-1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}$.

(b) $\quad \operatorname{trace} A = 3 = 1-3+5$

(c) $\quad \det A = 1(-3) - 4(9) + 4(6) = -3-12 = -15$

$$= 1\cdot(-3)\cdot 5 .$$

8.2.18

$$J_a = \begin{pmatrix} a & 1 & & 0 \\ & a & \ddots & \\ & & \ddots & 1 \\ 0 & & & a \end{pmatrix}$$ having characteristic

polynomial $\quad p_{J_a}(\lambda) = (a-\lambda)^n$ with single root $\lambda = a$.

So the only eigenvalue is $\lambda = a$ and with corresponding eigenspace

$$V_a = \ker(J_a - a\cdot I) = \ker\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = \mathbb{R}\cdot\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \mathbb{R}\cdot\vec{e}_1$$

8.2.51 $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has characteristic polynomial

$$p_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad-cd) ,$$

Now $\quad p_A(A) = A^2 - (a+d)A + (ad-bc)I$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

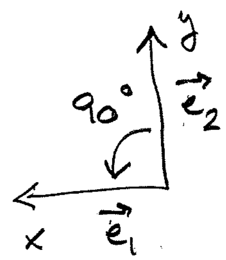
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O .$$

7.1.30 (a)

$$L : \mathbb{R}^3 \to \mathbb{R} \quad , \quad L(x,y,z) = 3x - y + 2z = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

7.1.5

The xy-plane seen in the direction of the positive z-axis looks like



So a rotation L of $90°$ counterclockwise around the z-axis satisfies

$$L(\vec{e}_1) = -\vec{e}_2 \quad , \quad L(\vec{e}_2) = \vec{e}_1 \quad , \quad L(\vec{e}_3) = \vec{e}_3$$

So the matrix reps. of L wrt the standard basis $\mathcal{E}$ is

$$A = \left([L(\vec{e}_1)]_{\mathcal{E}} \, , [L(\vec{e}_2)]_{\mathcal{E}} \, , [L(\vec{e}_3)]_{\mathcal{E}} \right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

7.2.12 (e)

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{matrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 1 \end{pmatrix}$$

$$-R_2 + R_3 \to R_3 \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 0 \end{pmatrix} \; -\tfrac{1}{3} R_3 \to R_3 \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; -2R_3 + R_1 \to R_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; R_2 \leftrightarrow R_3 \; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

7.2.12(e) cont'd.

So $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

So A is the composition of

- a reflection along the line $x=0$, $y=z$
- a shear along the x-axis of magnitude 2 in the coordinate plane xz.
- a stretch of the z-axis of magnitude $-\frac{1}{3}$.
- a shear along the z-axis of magnitude 1 in the coordinate plane yz
- a shear along the z-axis of magnitude 2 in the coordinate plane xz
- a shear along the y-axis of magnitude 2 in the coordinate plane xy.