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Banach spaces without minimal subspaces

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Abstract

We prove three new dichotomies for Banach spaces à la W.T. Gowers' dichotomies. The three dichotomies characterise respectively the spaces having no minimal subspaces, having no subsequentially minimal basic sequences, and having no subspaces crudely finitely representable in all of their subspaces. We subsequently use these results to make progress on Gowers' program of classifying Banach spaces by finding characteristic spaces present in every space. Also, the results are used to embed any partial order of size $\aleph_1$ into the subspaces of any space without a minimal subspace ordered by isomorphic embeddability.

Keywords: Minimal Banach spaces; Dichotomies; Classification of Banach spaces

Contents

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1. Introduction

In the paper [20], W.T. Gowers initiated a celebrated classification theory for Banach spaces. Since the task of classifying all (even separable) Banach spaces up to isomorphism is extremely complicated (just how complicated is made precise in [11]), one may settle for a *loose classification of Banach spaces up to subspaces*, that is, look for a list of classes of Banach spaces such that:

(a) each class is *pure*, in the sense that if a space belongs to a class, then every subspace belongs to the same class, or maybe, in the case when the properties defining the class depend on a basis of the space, every block subspace belongs to the same class,
(b) the classes are *inevitable*, i.e., every Banach space contains a subspace in one of the classes,
(c) any two classes in the list are disjoint,
(d) belonging to one class gives a lot of information about operators that may be defined on the space or on its subspaces.

We shall refer to this list as the *list of inevitable classes of Gowers*. Many classical problems are related to this classification program, as for example the question whether every Banach space contains a copy of c_0 or ℓ_p, solved in the negative by B.S. Tsirelson in 1974 [42], or the unconditional basic sequence problem, also solved negatively by Gowers and B. Maurey in 1993 [21]. Ultimately one would hope to establish such a list so that any classical space appears in one of the classes, and so that belonging to that class would yield most of the properties which are known for that space. For example, any property, which is known for Tsirelson's space, is also true for any of its block subspaces. So Tsirelson's space is a pure space, and, as such, should appear in one of the classes with a reasonable amount of its properties. Also, presumably the nicest among the classes would consist of the spaces isomorphic to c_0 or ℓ_p, $1 \leqslant p < \infty$.

After the discovery by Gowers and Maurey of the existence of *hereditarily indecomposable* (or HI) spaces, i.e., spaces such that no subspace may be written as the direct sum of infinite-dimensional subspaces [21], Gowers proved that every Banach space contains either an HI subspace or a subspace with an unconditional basis [19]. These were the first two examples of inevitable classes. We shall call this dichotomy the *first dichotomy* of Gowers. He then used his famous Ramsey or determinacy theorem [20] to refine the list by proving that any Banach space contains a subspace with a basis such that either no two disjointly supported block subspaces are isomorphic, or such that any two subspaces have further subspaces which are isomorphic. He

called the second property *quasi minimality*. This *second dichotomy* divides the class of spaces with an unconditional basis into two subclasses (up to passing to a subspace). Finally, recall that a space is *minimal* if it embeds into any of its subspaces. A quasi minimal space which does not contain a minimal subspace is called *strictly quasi minimal*, so Gowers again divided the class of quasi minimal spaces into the class of strictly quasi minimal spaces and the class of minimal spaces.

Obviously the division between minimal and strictly quasi minimal spaces is not a real dichotomy, since it does not provide any additional information. The main result of this paper is to provide the missing dichotomy for minimality, which we shall call the *third dichotomy*.

A first step in that direction was obtained by A.M. Pełczar, who showed that any strictly quasi minimal space contains a further subspace with the additional property of not containing any subsymmetric sequence [35]. The first author proved that the same holds if one replaces subsymmetric sequences by embedding homogeneous sequences (any subspace spanned by a subsequence contains an isomorphic copy of the whole space) [10].

A crucial step in the proofs of [35] and [10] is the notion of *asymptoticity*. An asymptotic game of length k in a space E with a basis is a game where I plays integers n_i and II plays block vectors x_i supported after n_i, and where the outcome is the length k sequence (x_i). Asymptotic games have been studied extensively and the gap between finite-dimensional and infinite-dimensional phenomena was usually bridged by fixing a constant and letting the length of the game tend to infinity. For example, a basis is *asymptotic* ℓ_p if there exists C such that for any k, I has a winning strategy in the length k asymptotic game so that the outcome is C-equivalent to the unit vector basis of ℓ_p^k.

In [35] it is necessary to consider asymptotic games of *infinite* length, which are defined in an obvious manner. The outcome is then an infinite block sequence. The proof of the theorem in [35] is based on the obvious fact that if a basic sequence (e_i) is subsymmetric and (x_i) is a block sequence of (e_i), then II has a strategy in the infinite asymptotic game in $E = [e_i]$ to ensure that the outcome is equivalent to (x_i). In [10] a similar fact for embedding homogeneous basic sequences is obtained, but the proof is more involved and a more general notion of asymptoticity must be used. Namely, a *generalised asymptotic game* in a space E with a basis (e_i) is a game where I plays integers n_i and II plays integers m_i and vectors x_i such that $\operatorname{supp}(x_i) \subseteq [n_1, m_1] \cup \cdots \cup [n_i, m_i]$, and the outcome is the sequence (x_i), which may no longer be a block basis.

The second author analysed infinite asymptotic games in [38] (a previous study had also been undertaken by E. Odell and T. Schlumprecht in [34]), showing that the most obvious necessary conditions are, in fact, also sufficient for II to have a strategy to play inside a given set. This was done through relating the existence of winning strategies to a property of subspaces spanned by vectors of the basis with indices in some intervals of integers. Now the methods of [38] extend to the setting of generalised asymptotic games and motivate the following definition. A space Y is *tight* in a basic sequence (e_i) if there is a sequence of successive intervals $I_0 < I_1 < I_2 < \cdots$ of $\mathbb{N}$ such that for all infinite subsets $A \subseteq \mathbb{N}$, we have

$$Y \not\sqsubseteq \left[e_n \;\middle|\; n \notin \bigcup_{i \in A} I_i \right],$$

where $Y \sqsubseteq X$ denotes that Y embeds into X. In other words, any embedding of Y into $[e_i]$ has a "large" image with respect to subsequences of the basis (e_i) and cannot avoid an infinite number of the subspaces $[e_n]_{n \in I_i}$. We then define a *tight basis* as a basis such that every subspace is tight in it and a *tight space* as a space with a tight basis.

As we shall prove in Lemma 3.7, using the techniques of [38], essentially a block subspace $Y = [y_i]$ is not tight in (e_i), when II has a winning strategy in the generalised asymptotic game in $[e_i]$ for producing a sequence equivalent to (y_i). This relates the notion of tight bases to the methods of [10], and by extending these methods we prove the main result of this paper:

Theorem 1.1 *(3rd dichotomy). Let E be a Banach space without minimal subspaces. Then E has a tight subspace.*

Theorem 1.1 extends the theorems of [35,10], since it is clear that a tight space cannot contain a subsymmetric or even embedding homogeneous block sequence. This dichotomy also provides an improvement to the list of Gowers: a strictly quasi minimal space must contain a tight quasi minimal subspace. Example 3.6 shows that this is a non-trivial refinement of the unconditional and strictly quasi minimal class, and Corollary 4.3 states that Tsirelson's space is tight. Theorem 1.1 also refines the class of HI spaces in the list, i.e., every HI space contains a tight subspace, although it is unknown whether the HI property for a space with a basis does not already imply that the basis is tight.

Our actual examples of tight spaces turn out to satisfy one of two stronger forms of tightness. The first is called *tightness with constants*. A basis (e_n) is tight with constants when for every infinite-dimensional space Y, the sequence of successive intervals $I_0 < I_1 < \cdots$ of $\mathbb{N}$ witnessing the tightness of Y in (e_n) may be chosen so that $Y \not\sqsubseteq_K [e_n \mid n \notin I_K]$ for each K. This is the case for Tsirelson's space.

The second kind of tightness is called *tightness by range*. Here the range, range x, of a vector x is the smallest interval of integers containing its support, and the range of a block subspace $[x_n]$ is $\bigcup_n \operatorname{range} x_n$. A basis (e_n) is tight by range when for every block subspace $Y = [y_n]$, the sequence of successive intervals $I_0 < I_1 < \cdots$ of $\mathbb{N}$ witnessing the tightness of Y in (e_n) may be defined by $I_k = \operatorname{range} y_k$ for each k. This is equivalent to no two block subspaces with disjoint ranges being comparable. In a companion paper [14], we show that tightness by range is satisfied by an HI space and also by a space with unconditional basis both constructed by Gowers.

It turns out that there are natural dichotomies between each of these strong forms of tightness and respective weak forms of minimality. For the first notion, we define a space X to be *locally minimal* if for some constant K, X is K-crudely finitely representable in any of its subspaces. Notice that local minimality is easily incompatible with tightness with constants. Using an equivalent form of Gowers' game, as defined by J. Bagaria and J. López-Abad [4], we prove:

Theorem 1.2 *(5th dichotomy). Any Banach space E contains a subspace with a basis that is either tight with constants or is locally minimal.*

The ideas involved in the notion of local minimality also make sense for block representability, which allows us to connect these notions with asymptoticity of basic sequences. Proving a simple dichotomy for when a space contains an asymptotically ℓ_p subspace, we are led to the following dichotomy for when a Banach space contains a copy of either c_0 or ℓ_p.

Theorem 1.3 *(The c_0 and ℓ_p dichotomy). Suppose X is a Banach space not containing a copy of c_0 nor of ℓ_p, $1 \leqslant p < \infty$. Then X has a subspace Y with a basis such that either*

(1) $\forall M\ \exists n\ \forall U_1, \dots, U_{2n} \subseteq Y\ \exists u_i \in \mathcal{S}_{U_i}$

$$\big(u_1 < \cdots < u_{2n}\ \&\ (u_{2i-1})_{i=1}^n \nsim_M (u_{2i})_{i=1}^n\big).$$

(2) *For all block bases* (z_n) *of* $Y = [y_n]$ *there are intervals* $I_1 < I_2 < I_3 < \cdots$ *such that* $(z_n)_{n\in I_K}$ *is not* K*-equivalent to a block sequence of* $(y_n)_{n\notin I_K}$.

Here, of course, the variables range over infinite-dimensional *spaces.*

Property (1) indicates some lack of homogeneity and (2) some lack of minimality. It is interesting to see which conditions the various examples of Banach spaces not containing c_0 or ℓ_p satisfy; obviously, Tsirelson's space and its dual satisfy (2) and indeed (2) is the only option for spaces being asymptotic ℓ_p. On the other hand, Schlumprecht's space S satisfies (1).

There is also a dichotomy concerning tightness by range. This direction for refining the list of inevitable classes of spaces was actually suggested by Gowers in [20]. P. Casazza proved that if a space X has a shrinking basis such that no block sequence is *even–odd* (the odd subsequence is equivalent to the even subsequence), then X is not isomorphic to a proper subspace, see [17]. So any Banach space contains either a subspace, which is not isomorphic to a proper subspace, or is saturated with even–odd block sequences, and, in the second case, we may find a further subspace in which Player II has a winning strategy to produce even–odd sequences in the game of Gowers associated to his Ramsey theorem. This fact was observed by Gowers, but it was unclear to him what to deduce from the property in the second case.

We answer this question by using Gowers' theorem to obtain a dichotomy which on one side contains tightness by range, which is a slightly stronger property than the Casazza property. On the other side, we define a space X with a basis (x_n) to be *subsequentially minimal* if every subspace of X contains an isomorphic copy of a subsequence of (x_n). This last property is satisfied by Tsirelson's space and will also be shown to be incompatible with tightness by range.

Theorem 1.4 *(4th dichotomy). Any Banach space* E *contains a subspace with a basis that is either tight by range or is subsequentially minimal.*

It is easy to check that the second case in Theorem 1.4 may be improved to the following hereditary property of a basis (x_n), that we call *sequential minimality*: every block sequence of $[x_n]$ has a further block sequence (y_n) such that every subspace of $[x_n]$ contains a copy of a subsequence of (y_n).

The five dichotomies and the interdependence of the properties involved can be visualised in the following diagram.

Unconditional basis	∗∗ 1st dichotomy ∗∗	Hereditarily indecomposable
⇑		⇓
Tight by support	∗∗ 2nd dichotomy ∗∗	Quasi minimal
⇓		⇑
Tight by range	∗∗ 4th dichotomy ∗∗	Sequentially minimal
⇓		⇑
Tight	∗∗ 3rd dichotomy ∗∗	Minimal
⇑		⇓
Tight with constants	∗∗ 5th dichotomy ∗∗	Locally minimal

From a different point of view, coming from combinatorics and descriptive set theory, Theorem 1.1 also has important consequences for the isomorphic classification of separable Banach spaces. To explain this, suppose that X is a Banach space and $SB_\infty(X)$ is the class of all infinite-dimensional subspaces of X. Then the relation $\sqsubseteq$ of isomorphic embeddability induces a partial order on the set of biembeddability classes of $SB_\infty(X)$ and we denote this partial order by $\mathbb{P}(X)$. Many questions about the isomorphic structure of X translate directly into questions about the structure of $\mathbb{P}(X)$, e.g., X has a minimal subspace if and only if $\mathbb{P}(X)$ has a minimal element and X is quasi minimal if and only if $\mathbb{P}(X)$ is downwards directed. In some sense, a space can be said to be pure in case the complexity of $\mathbb{P}(X)$ does not change by passing to subspaces and Gowers [20, Problem 7.9], motivated by this, asked for a classification of, or at least strong structural information about, the partial orders P for which there is a Banach space X saturated with subspaces $Y \subseteq X$ such that $P \cong \mathbb{P}(Y)$. A simple diagonalisation easily shows that such P either consist of a single point (corresponding to a minimal space) or are uncountable, and, using methods of descriptive set theory and metamathematics, this was successively improved in [12] and [37] to either $|P| = 1$ or P having a continuum size antichain. Using a strengthening of Theorem 1.1, we are now able to show that such P, for which $|P| > 1$, have an extremely complex structure by embedding any partial order of size at most $\aleph_1$ into them.

For $A, B \subseteq \mathbb{N}$, we write $A \subseteq^* B$ to mean that $A \setminus B$ is finite.

Theorem 1.5. *Given a Banach space X, let $\mathbb{P}(X)$ be the set of all biembeddability classes of infinite-dimensional subspaces of X, partially ordered under isomorphic embeddability. Let P be a poset for which there exists a Banach space X such that X is saturated with subspaces Y such that $\mathbb{P}(Y) \cong P$. Then either $|P| = 1$, or $\subseteq^*$ embeds into P. In the second case it follows that*

(a) *any partial order of size at most $\aleph_1$ embeds into P, and*
(b) *any closed partial order on a Polish space embeds into P.*

From the point of view of descriptive set theory, it is more natural to study another problem, part of which was originally suggested to us by G. Godefroy some time ago. Namely, the space $SB_\infty(X)$, for X separable, can easily be made into a standard Borel space using the Effros–Borel structure. In this way, the relations of isomorphism, $\cong$, and isomorphic embeddability, $\sqsubseteq$, become analytic relations on $SB_\infty(X)$ whose complexities can be measured through the notion of Borel reducibility. We obtain Theorem 1.5 as a consequence of some finer results formulated in this language and that are of independent interest.

In Section 8, we put all the dichotomies together in order to make progress on the loose classification mentioned above. In connection with this, we shall also rely on work by A. Tcaciuc [41], who proved a dichotomy for containing a strongly asymptotic ℓ_p basis, i.e., a basis such that finite families of disjointly supported (but not necessarily successive) normalised blocks supported "far enough" are uniformly equivalent to the basis of ℓ_p^n. Using just the first four dichotomies, in Theorem 8.3 we find 6 classes of inevitable spaces, 4 of which are known to be non-empty, while if we use all 5 dichotomies plus Tcaciuc's, we find 19 classes. Out of these, 8 of them are known to be non-empty, though for 4 of the examples, we will need the results of a companion paper [14] where these are constructed and investigated.

The resulting classification gives fairly detailed knowledge about the various types of inevitable spaces, though much work remains to be done. In particular, the new dichotomies explains some of the structural differences between the wealth of new exotic spaces constructed

in the wake of the seminal paper of Gowers and Maurey [21]. It seems an interesting task to determine which of the remaining 11 of the 19 cases are non-empty.

2. Preliminaries

2.1. Notation, terminology, and conventions

We shall in the following almost exclusively deal with infinite-dimensional Banach spaces, so to avoid repeating this, we will always assume our spaces to be infinite-dimensional. The spaces can also safely be assumed to be separable, but this will play no role and is not assumed. Moreover, all spaces will be assumed to be over the field of real numbers $\mathbb{R}$, though the results hold without modification for complex spaces too.

Suppose E is a Banach space with a normalised Schauder basis (e_n). Then, by a standard Skolem hull construction, there is a countable subfield $\mathbf{F}$ of $\mathbb{R}$ containing the rational numbers $\mathbb{Q}$ such that for any finite linear combination

$$\lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n$$

with $\lambda_i \in \mathbf{F}$, we have $\|\lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n\| \in \mathbf{F}$. This means that any $\mathbf{F}$-linear combination of (e_n) can be normalised, while remaining a $\mathbf{F}$-linear combination. Thus, as the set of $\mathbb{Q}$ and hence also $\mathbf{F}$-linear combinations of (e_n) are dense in E, also the set of $\mathbf{F}$-linear normalised combinations of (e_n) are dense in the unit sphere $\mathcal{S}_E$.

A *block vector* is a normalised finite linear combination $x = \lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n$ where $\lambda_i \in \mathbf{F}$. We insist on blocks being normalised and $\mathbf{F}$-linear and will be explicit on the few occasions that we deal with non-normalised blocks. The restriction to $\mathbf{F}$-linear combinations is no real loss of generality, but instead has the effect that there are only countably many blocks. We denote by $\mathbf{D}$ the set of blocks. The *support*, $\operatorname{supp} x$, of a block $x = \lambda_0 e_0 + \lambda_1 e_1 + \cdots + \lambda_n e_n$ is the set of $i \in \mathbb{N}$ such that $\lambda_i \neq 0$ and the *range*, $\operatorname{range} x$, is the smallest interval $I \subseteq \mathbb{N}$ containing $\operatorname{supp} x$.

A *block* (*sub*)*sequence*, *block basis*, or *blocking* of (e_n) is an infinite sequence (x_n) of blocks such that $\operatorname{supp} x_n < \operatorname{supp} x_{n+1}$ for all n and a *block subspace* is the closed linear span of a block sequence. Notice that if X is a block subspace, then the associated block sequence (x_n) such that $X = [x_n]$ is uniquely defined up to the choice of signs $\pm x_n$. So we shall sometimes confuse block sequences and block subspaces. For two block subspaces $X = [x_n]$ and $Y = [y_n]$, write $Y \leqslant X$ if $Y \subseteq X$, or, equivalently, $y_n \in \operatorname{span}(x_i)$ for all n. Also, let $Y \leqslant^* X$ if there is some N such that $y_n \in \operatorname{span}(x_i)$ for all $n \geqslant N$.

When we work with block subspaces of some basis (e_n), we will assume that we have chosen the same countable subfield $\mathbf{F}$ of $\mathbb{R}$ for all block sequences (x_n) of (e_n), and hence a vector in $[x_n]$ is a block of (x_n) if and only if it is a block of (e_n), so no ambiguity occurs. We consider the set $bb(e_n)$ of block sequences of (e_n) as a closed subset of $\mathbf{D}^{\mathbb{N}}$, where $\mathbf{D}$ is equipped with the discrete topology. In this way, $bb(e_n)$ is a Polish, i.e., separable, completely metrisable space. If $\Delta = (\delta_n)$ is a sequence of positive real numbers, which we denote by $\Delta > 0$, and $\mathbb{A} \subseteq bb(e_n)$, we designate by $\mathbb{A}_\Delta$ the set

$$\mathbb{A}_\Delta = \big\{(y_n) \in bb(e_n) \;\big|\; \exists (x_n) \in bb(e_n)\ \forall n\ \|x_n - y_n\| < \delta_n\big\}.$$

If A is an infinite subset of $\mathbb{N}$, we denote by $[A]$ the space of infinite subsets of A with the topology inherited from 2^A. Also, if $a \subseteq \mathbb{N}$ is finite,

$$[a, A] = \big\{ B \in [\mathbb{N}] \bigm| a \subseteq B \subseteq a \cup \big(A \cap [\max a + 1, \infty[\big)\big\}.$$

We shall sometimes confuse infinite subsets of $\mathbb{N}$ with their increasing enumeration. So if $A \subseteq \mathbb{N}$ is infinite, we denote by A_n the $(n+1)$st element of A in its increasing enumeration (we start counting at 0).

A Banach space X *embeds* into Y if X is isomorphic to a closed subspace of Y. Since we shall work with the embeddability relation as a mathematical object itself, we prefer to use the slightly non-standard notation $X \sqsubseteq Y$ to denote that X embeds into Y.

Given two Banach spaces X and Y, we say that X is *crudely finitely representable* in Y if there is a constant K such that for any finite-dimensional subspace $F \subseteq X$ there is an embedding $T : F \to Y$ with constant K, i.e., $\|T\| \cdot \|T^{-1}\| \leqslant K$.

Also, if $X = [x_n]$ and $Y = [y_n]$ are spaces with bases, we say that X is *crudely block finitely representable* in Y if for some constant K and all k, there are (not necessarily normalised) blocks $z_0 < \cdots < z_k$ of (y_n) such that $(x_0, \ldots, x_k) \sim_K (z_0, \ldots, z_k)$.

Two Banach spaces are said to be *incomparable* if neither one embeds into the other, and *totally incomparable* if no subspace of one is isomorphic to a subspace of the other.

We shall at several occasions use non-trivial facts about the Tsirelson space and its p-convexifications, for which our reference is [8], and also facts from descriptive set theory that can all be found in [24]. For classical facts in Banach space theory we refer to [27].

2.2. Gowers' block sequence game

A major ingredient in several of our proofs will be the following equivalent version of Gowers' game due to J. Bagaria and J. López-Abad [4].

Suppose $E = [e_n]$ is given. Player I and II alternate in choosing blocks $x_0 < x_1 < x_2 < \cdots$ and $y_0 < y_1 < y_2 < \cdots$ as follows: Player I plays in the kth round of the game a block x_k such that $x_{k-1} < x_k$. In response to this, II either chooses to pass, and thus play nothing in the kth round, or plays a block $y_i \in [x_{l+1}, \ldots, x_k]$, where l was the last round in which II played a block.

I	$x_0 \cdots x_{k_0}$		$x_{k_0+1} \cdots x_{k_1}$	
II		$y_0 \in [x_0, \ldots, x_{k_0}]$		$y_1 \in [x_{k_0+1}, \ldots, x_{k_1}]$

We thus see I as constructing a block sequence (x_i), while II chooses a block subsequence (y_i). This block subsequence (y_i) is then called the *outcome* of the game. (Potentially the blocking could be finite, but the winning condition can be made such that II loses unless it is infinite.) We now have the following fundamental theorem of Gowers (though he only proves it for real scalars, it is clear that his proof is valid for the field **F** too).

Theorem 2.1. *(See W.T. Gowers [20].) Suppose (e_n) is a Schauder basis and $\mathbb{A} \subseteq bb(e_i)$ is an analytic set such that any $(x_i) \in bb(e_i)$ has a block subsequence (y_i) belonging to $\mathbb{A}$, then for all $\Delta > 0$, there is a block subsequence $(v_i) \in bb(e_i)$ such that II has a strategy to play in $\mathbb{A}_\Delta$ if I is restricted to play blockings of (v_i).*

2.3. A trick and a lemma

We gather here a couple of facts that will be used repeatedly later on.

We shall at several occasions use coding with inevitable subsets of the unit sphere of a Banach space, as was first done by López-Abad in [28]. So let us recall here the relevant facts and set up a framework for such codings.

Suppose E is an infinite-dimensional Banach space with a basis not containing a copy of c_0. Then by the solution to the distortion problem by Odell and Schlumprecht [31] there is a block subspace $[x_n]$ of E and two closed subsets F_0 and F_1 of the unit sphere of $[x_n]$ such that $\operatorname{dist}(F_0, F_1) = \delta > 0$ and such that for all block bases (y_n) of (x_n) there are block vectors v and u of (y_n) such that $v \in F_0$ and $u \in F_1$. In this case we say that F_0 and F_1 are *positively separated, inevitable, closed subsets of* $\mathcal{S}_{[x_n]}$.

We can now use the sets F_0 and F_1 to code infinite binary sequences, i.e., elements of $2^{\mathbb{N}}$ in the following manner. If (z_n) is a block sequence of (x_n) such that for all n, $z_n \in F_0 \cup F_1$, we let $\varphi((z_n)) = \alpha \in 2^{\mathbb{N}}$ be defined by

$$\alpha_n = \begin{cases} 0, & \text{if } z_n \in F_0; \\ 1, & \text{if } z_n \in F_1. \end{cases}$$

Since the sets F_0 and F_1 are positively separated, this coding is fairly rigid and can be extended to block sequences (v_n) such that $\operatorname{dist}(v_n, F_0 \cup F_1) < \frac{\delta}{2}$ by letting $\varphi((v_n)) = \beta \in 2^{\mathbb{N}}$ be defined by

$$\beta_n = \begin{cases} 0, & \text{if } \operatorname{dist}(v_n, F_0) < \frac{\delta}{2}; \\ 1, & \text{if } \operatorname{dist}(v_n, F_1) < \frac{\delta}{2}. \end{cases}$$

In this way we have that if (z_n) and (v_n) are block sequences with $z_n \in F_0 \cup F_1$ and $\|v_n - z_n\| < \frac{\delta}{2}$ for all n, then $\varphi((z_n)) = \varphi((v_n))$.

One can now use elements of Cantor space $2^{\mathbb{N}}$ to code other objects in various ways. For example, let $\mathbb{H}$ denote the set of finite non-empty sequences $(q_0, q_1, \dots, q_n)$ of rationals with $q_n \neq 0$. Then, as $\mathbb{H}$ is countable, we can enumerate it as $\vec{h}_0, \vec{h}_1, \dots$. If now (y_n) and (v_n) are block sequences with $\varphi((v_n)) = 0^{n_0}10^{n_1}10^{n_2}1\cdots$, then (v_n) codes an infinite sequence $\Psi((v_n), (y_n)) = (u_n)$ of finite linear combinations of (y_n) by the following rule:

$$u_k = q_0 y_0 + q_1 y_1 + \cdots + q_m y_m,$$

where $\vec{h}_{n_k} = (q_0, \dots, q_m)$.

We should then notice three things about this type of coding:

- It is *inevitable*, i.e., for all block sequences (y_n) of (x_n) and $\alpha \in 2^{\mathbb{N}}$, there is a block sequence (v_n) of (y_n) with $\varphi((v_n)) = \alpha$.
- It is *continuous*, i.e., to know an initial segment of $(u_n) = \Psi((v_n), (y_n))$, we only need to know initial segments of (v_n) and of (y_n).
- It is *stable under small perturbations*. I.e., given $\epsilon > 0$, we can find some $\Delta = (\delta_n)$ only depending on ϵ and the basis constant of (x_n) with the following property. Assume that (v_n) and (y_n) are block bases of (x_n) with $v_n \in F_0 \cup F_1$ for all n and such that $\Psi((v_n), (y_n)) = (u_n)$ is a block sequence of (y_n) with $\frac{1}{2} < \|u_n\| < 2$. Then whenever (v'_n) and (y'_n) are other

block sequences of (x_n) with $\|v_n - v'_n\| < \frac{\delta}{2}$ and $\|y_n - y'_n\| < \delta_n$ for all n, the sequence $\Psi((v'_n), (y'_n)) = (u'_n)$ will be a block sequence of (y'_n) that is $1 + \epsilon$-equivalent to (u_n).

One can of course consider codings of other objects than sequences of vectors and, depending on the coding, obtain similar continuity and stability properties.

The inevitability of the coding is often best used in the following form.

– Suppose $\mathbb{B}$ is a set of pairs $((y_n), \alpha)$, where (y_n) is a block sequence of (x_n) and $\alpha \in 2^{\mathbb{N}}$, such that for all block sequences (z_n) of (x_n) there is a further block sequence (y_n) and an α such that $((y_n), \alpha) \in \mathbb{B}$. Then for all block sequences (z_n) of (x_n) there is a further block sequence (y_n) such that for all n, $y_{2n+1} \in F_0 \cup F_1$ and $((y_{2n}), \varphi((y_{2n+1}))) \in \mathbb{B}$.

To see this, let (z_n) be given and notice that by the inevitability of the coding there is a block sequence (w_n) of (z_n) such that $w_{3n+1} \in F_0$ and $w_{3n+2} \in F_1$. Pick now a block sequence (v_n) of (w_{3n}) and an α such that $((v_n), \alpha) \in \mathbb{B}$. Notice now that between v_n and v_{n+1} there are block vectors w_{3i_n+1} and w_{3i_n+2} of (z_n) belonging to F_0, respectively F_1. Thus, if we let $y_{2n} = v_n$ and set

$$y_{2n+1} = \begin{cases} w_{3i_n+1}, & \text{if } \alpha_n = 0; \\ w_{3i_n+2}, & \text{if } \alpha_n = 1, \end{cases}$$

then $((y_{2n}), \varphi((y_{2n+1}))) \in \mathbb{B}$.

Lemma 2.2. *Let $(x_n^0) \geqslant (x_n^1) \geqslant (x_n^2) \geqslant \cdots$ be a decreasing sequence of block bases of a basic sequence (x_n^0). Then there exists a block basis (y_n) of (x_n^0) such that (y_n) is $\sqrt{K}$-equivalent with a block basis of (x_n^K) for every $K \geqslant 1$.*

Proof. Let $c(L)$ be a constant depending on the basis constant of (x_n^0) such that if two block bases differ in at most L terms, then they are $c(L)$-equivalent. Find now a sequence $L_1 \leqslant L_2 \leqslant \cdots$ of non-negative integers tending to $+\infty$ such that $c(L_K) \leqslant \sqrt{K}$. We can now easily construct an infinite block basis (y_n) of (x_n^0) such that for all $K \geqslant 1$ at most the first L_K terms of (y_n) are not blocks of $(x_n^K)_{n=L_K+1}^{\infty}$. Then (y_n) differs from a block basis of (x_n^K) in at most L_K terms and hence is $\sqrt{K}$-equivalent with a block basis of (x_n^K). □

3. Tightness

3.1. Tight bases

The following definition is central to the rest of the paper.

Definition 3.1. Consider a Banach space E with a basis (e_n) and let Y be an arbitrary Banach space. We say that Y is *tight in the basis* (e_n) if there is a sequence of successive non-empty intervals $I_0 < I_1 < I_2 < \cdots$ of $\mathbb{N}$ such that for all infinite subsets $A \subseteq \mathbb{N}$, we have

$$Y \not\subseteq \left[e_n \;\middle|\; n \notin \bigcup_{i \in A} I_i \right].$$

In other words, if Y embeds into $[e_n]_{n\in B}$, then $B \subseteq \mathbb{N}$ intersects all but finitely many intervals I_i.

We say that (e_n) is *tight* if every infinite-dimensional Banach space Y is tight in (e_n).

Finally, an infinite-dimensional Banach space X is *tight* if it has a tight basis.

Also, the following more analytical criterion will prove to be useful. For simplicity, denote by P_I the canonical projection onto $[e_n]_{n\in I}$.

Lemma 3.2. *Let X be a Banach space, (e_n) a basis for a space E, and (I_n) finite intervals such that* $\min I_n \xrightarrow[n\to\infty]{} \infty$ *and for all infinite $A \subseteq \mathbb{N}$,*

$$X \not\subseteq [e_n]_{n\notin\bigcup_{k\in A} I_k}.$$

Then whenever $T : X \to [e_n]$ is an embedding, we have $\liminf_k \|P_{I_k} T\| > 0$.

Proof. Suppose towards a contradiction that $T : X \to E$ is an embedding such that for some infinite $A \subseteq \mathbb{N}$,

$$\lim_{\substack{k\to\infty \\ k\in A}} \|P_{I_k} T\| = 0.$$

Then, by passing to an infinite subset of A, we can suppose that $\sum_{k\in A} \|P_{I_k} T\| < \frac{1}{2}\|T^{-1}\|^{-1}$ and that the intervals $(I_n)_{n\in A}$ are disjoint. Thus, the sequence of operators $(P_{I_k} T)_{k\in A}$ is absolutely summable and therefore the operator $\sum_{k\in A} P_{I_k} T : X \to E$ exists and has norm $< \frac{1}{2}\|T^{-1}\|^{-1}$.

But then for $x \in X$ we have

$$\left\|\sum_{k\in A} P_{I_k} T x\right\| \leqslant \left\|\sum_{k\in A} P_{I_k} T\right\| \cdot \|x\| \leqslant \frac{1}{2\|T^{-1}\|}\|x\| \leqslant \frac{1}{2\|T^{-1}\|}\left\|T^{-1}\right\| \cdot \|Tx\| = \frac{1}{2}\|Tx\|,$$

and hence also

$$\left\|\left(T - \sum_{k\in A} P_{I_k} T\right)x\right\| \geqslant \|Tx\| - \left\|\sum_{k\in A} P_{I_k} T x\right\| \geqslant \|Tx\| - \frac{1}{2}\|Tx\| = \frac{1}{2}\|Tx\|.$$

So $T - \sum_{k\in A} P_{I_k} T$ is still an embedding of X into E. But this is impossible as $T - \sum_{k\in A} P_{I_k} T$ takes values in $[e_n]_{n\notin\bigcup_{k\in A} I_k}$. □

Proposition 3.3. *A tight Banach space contains no minimal subspaces.*

Proof. Suppose (e_n) is a tight basis for a space E and let Y be any subspace of E. Pick a block subspace $X = [x_n]$ of E that embeds into Y. Since Y is tight in (e_n), we can find a sequence of intervals (I_i) such that Y does not embed into $[e_n]_{n\in B}$ whenever $B \subseteq \mathbb{N}$ is disjoint from an infinite number of intervals I_i. By passing to a subsequence (z_n) of (x_n), we obtain a space $Z = [z_n]$ that is a subspace of some $[e_n]_{n\in B}$ where $B \subseteq \mathbb{N}$ is disjoint from an infinite number of intervals I_i, and hence Y does not embed into Z. Since Z embeds into Y, this shows that Y is not minimal. □

The classical example of space without minimal subspaces is Tsirelson's space T and it is not too difficult to show that T is tight. This will be proved later on as a consequence of a more general result.

Any block sequence of a tight basis is easily seen to be tight. And also:

Proposition 3.4. *If E is a tight Banach space, then every shrinking basic sequence in E is tight.*

Proof. Suppose (e_n) is a tight basis for E and (f_n) is a shrinking basic sequence in E. Let Y be an arbitrary space and find intervals $I_0 < I_1 < \cdots$ associated to Y for (e_n), i.e., for all infinite subsets $A \subseteq \mathbb{N}$, we have $Y \not\sqsubseteq [e_n \mid n \notin \bigcup_{i\in A} I_i]$.

We notice that, since (e_n) is a basis, we have for all m

$$\|P_{I_k}|_{[f_i|i\leqslant m]}\| \xrightarrow[k\to\infty]{} 0, \tag{1}$$

and, since (f_n) is shrinking and the P_{I_k} have finite rank, we have for all k

$$\|P_{I_k}|_{[f_i|i>m]}\| \xrightarrow[m\to\infty]{} 0. \tag{2}$$

Using alternately (1) and (2), we can construct integers $k_0 < k_1 < \cdots$ and intervals $J_0 < J_1 < \cdots$ such that

$$\|P_{I_{k_n}}|_{[f_i|i\notin J_n]}\| < \frac{2}{n+1}.$$

To see this, suppose k_{n-1} and J_{n-1} have been defined and find some large $k_n > k_{n-1}$ such that

$$\|P_{I_{k_n}}|_{[f_i|i\leqslant \max J_{n-1}]}\| \leqslant \frac{1}{n+1}.$$

Now, choose m large enough that

$$\|P_{I_{k_n}}|_{[f_i|i>m]}\| \leqslant \frac{1}{n+1},$$

and set $J_n = [\max J_{n-1} + 1, m]$. Then $\|P_{I_{k_n}}|_{[f_i|i\notin J_n]}\| < \frac{2}{n+1}$. It follows that if $A \subseteq \mathbb{N}$ is infinite and $T : Y \to [f_i]_{i\notin\bigcup_{n\in A} J_n}$ is an embedding, then $\lim_{n\in A} \|P_{I_{k_n}} T\| = 0$, which contradicts Lemma 3.2. So (J_n) witnesses that Y is tight in (f_n). □

Corollary 3.5. *If a tight Banach space X is reflexive, then every basic sequence in X is tight.*

Notice that, since c_0 and ℓ_1 are minimal, we have by the classical theorem of James, that if X is a tight Banach space with an unconditional basis, then X is reflexive and so every basic sequence in X is tight.

Example 3.6. The symmetrisation $S(T^{(p)})$ of the p-convexification $T^{(p)}$ of Tsirelson's space, $1 < p < +\infty$, does not contain a minimal subspace, yet it is not tight.

Proof. Since $S(T^{(p)})$ is saturated with isomorphic copies of subspaces of $T^{(p)}$ and $T^{(p)}$ does not contain a minimal subspace, it follows that $S(T^{(p)})$ does not have a minimal subspace. The canonical basis (e_n) of $S(T^{(p)})$ is symmetric, therefore $S(T^{(p)})$ is not tight in (e_n) and so (e_n) is not tight. By reflexivity, no basis of $S(T^{(p)})$ is tight. □

3.2. A generalised asymptotic game

Suppose $X = [x_n]$ and $Y = [y_n]$ are two Banach spaces with bases. We define the game $H_{Y,X}$ with constant $C \geqslant 1$ between two players I and II as follows: I will in each turn play a natural number n_i, while II will play a not necessarily normalised block vector $u_i \in X$ and a natural number m_i such that

$$u_i \in X[n_0, m_0] + \cdots + X[n_i, m_i],$$

where, for ease of notation, we write $X[k, m]$ to denote $[x_n]_{k \leqslant n \leqslant m}$. Diagrammatically,

$$\begin{array}{lccccccc} \mathbf{I} & n_0 & & n_1 & & n_2 & & n_3 \quad \cdots \\ \mathbf{II} & & u_0, m_0 & & u_1, m_1 & & u_2, m_2 & & u_3, m_3 \ \cdots \end{array}$$

We say that the sequence $(u_i)_{i\in\mathbb{N}}$ is the *outcome* of the game and say that II wins the game if $(u_i) \sim_C (y_i)$.

For simplicity of notation, if $X = [x_n]$ is space with a basis, Y a Banach space, $I_0 < I_1 < I_2 < \cdots$ a sequence of non-empty intervals of $\mathbb{N}$ and K is a constant, we write

$$Y \sqsubseteq_K (X, I_i)$$

if there is an infinite set $A \subseteq \mathbb{N}$ containing 0 such that

$$Y \sqsubseteq_K \left[x_n \,\middle|\, n \notin \bigcup_{i\in A} I_i \right],$$

i.e., Y embeds with constant K into the subspace of X spanned by $(x_n)_{n\notin\bigcup_{i\in A} I_i}$. Also, write

$$Y \sqsubseteq (X, I_i)$$

if there is an infinite set $A \subseteq \mathbb{N}$ such that $Y \sqsubseteq [x_n \mid n \notin \bigcup_{i\in A} I_i]$. Notice that in the latter case we can always demand that $0 \in A$ by perturbating the embedding with a finite rank operator.

It is clear that if $Y = [y_n]$ and II has a winning strategy in the game $H_{Y,X}$ with constant K, then for any sequence of intervals (I_i), $Y \sqsubseteq_K (X, I_i)$.

Modulo the determinacy of open games, the next lemma shows that the converse holds up to a perturbation.

Lemma 3.7. *Suppose $X = [x_n]$ is space with a basis and K, ϵ are positive constants such that for all block bases Y of X there is a winning strategy for I in the game $H_{Y,X}$ with constant $K + \epsilon$. Then there is a Borel function $f : bb(X) \to [\mathbb{N}]$ such that for all Y if $I_j = [f(Y)_{2j}, f(Y)_{2j+1}]$, then*

$$Y \not\sqsubseteq_K (X, I_j).$$

Proof. Notice that the game $H_{Y,X}$ is open for player I and, in fact, if $\mathbf{D}_{K+\epsilon}$ denotes the set of blocks u with $\frac{1}{K+\epsilon} \leqslant \|u\| \leqslant K+\epsilon$, then the set

$$\mathbb{A} = \big\{(Y, \vec{p}) \in bb(X) \times (\mathbb{N} \times \mathbf{D}_{K+\epsilon} \times \mathbb{N})^{\mathbb{N}} \;\big|\; \text{either } \vec{p} \text{ is a legal run of the game } H_{Y,X} \text{ with constant } K+\epsilon \text{ in which I wins or } \vec{p} \text{ is not a legal run of the game } H_{Y,X}\big\}$$

is Borel and has open sections $\mathbb{A}_Y = \{\vec{p} \in (\mathbb{N} \times \mathbf{D}_{K+\epsilon} \times \mathbb{N})^{\mathbb{N}} \mid (Y, \vec{p}) \in \mathbb{A}\}$. Also, since there are no rules for the play of I in $H_{Y,X}$, $\mathbb{A}_Y$ really corresponds to the winning plays for I in $H_{Y,X}$ with constant $K+\epsilon$. By assumption, I has a winning strategy to play in $\mathbb{A}_Y$ for all Y, and so by the theorem on strategic uniformisation (see (35.32) in [24]), there is a Borel function $\sigma : Y \mapsto \sigma_Y$ that to each Y associates a winning strategy for I in the game $H_{Y,X}$ with constant $K+\epsilon$.

Now let $\Delta = (\delta_n)$ be a sequence of positive reals such that for all $2KC$-basic sequences of blocks (w_n) of X with $\frac{1}{K} \leqslant \|w_n\| \leqslant K$ (where C is the basis constant of X) and sequences of vectors (u_n), if for all n, $\|w_n - u_n\| < \delta_n$, then $(w_n) \sim_{\sqrt{1+\epsilon/K}} (u_n)$. We also choose sets $\mathbb{D}_n$ of finite (not necessarily normalised) blocks with the following properties:

- for each finite $d \subseteq \mathbb{N}$, the number of vectors $u \in \mathbb{D}_n$ such that $\operatorname{supp} u = d$ is finite,
- for all blocks vectors w with $\frac{1}{K} \leqslant \|w\| \leqslant K$, there is some $u \in \mathbb{D}_n$ with $\operatorname{supp} w = \operatorname{supp} u$ such that $\|w - u\| < \delta_n$.

This is possible since the K-ball in $[x_i]_{i\in d}$ is totally bounded for all finite $d \subseteq \mathbb{N}$. So for all $2KC$-basic sequences (w_n) of blocks with $\frac{1}{K} \leqslant \|w_n\| \leqslant K$, there is some $(u_n) \in \prod_n \mathbb{D}_n$ such that $\operatorname{supp} w_n = \operatorname{supp} u_n$ and $\|w_n - u_n\| < \delta_n$ for all n, whence $(w_n) \sim_{\sqrt{1+\epsilon/K}} (u_n)$.

Suppose now that $Y = [y_n]$ is given. For each $p = (n_0, u_0, m_0, \ldots, n_i, u_i, m_i)$, where $u_j \in \mathbb{D}_j$ for all j and

I	n_0	n_1	$\cdots$	n_i
II	u_0, m_0	u_1, m_1	$\cdots$	u_i, m_i

is a legal position in the game $H_{Y,X}$ in which I has played according to σ_Y, we write $p < k$ if $n_j, u_j, m_j < k$ for all $j \leqslant i$. Notice that for all k there are only finitely many such p with $p < k$, so we can define

$$\alpha(k) = \max\big(k, \max\big\{\sigma_Y(p) \;\big|\; p < k\big\}\big)$$

and set $I_k = [k, \alpha(k)]$. Clearly, the sequence (I_k) can be computed in a Borel fashion from Y. The I_k are not necessarily successive, but their minimal elements tend to ∞, so to prove the lemma it is enough to show that Y does not K-embed into $[x_n]$ avoiding an infinite number of I_k including I_0.

Suppose now for a contradiction that $A \subseteq \mathbb{N}$ is infinite, $0 \in A$ and $y_i \mapsto w_i$ is a K-embedding into $[x_n \mid n \notin \bigcup_{k\notin A} I_k]$. By perturbing the embedding slightly, we can suppose that the w_i are blocks such that $\frac{1}{K} \leqslant \|w_i\| \leqslant K$ and we still have a $K\sqrt{1+\epsilon/K}$-embedding. Using the defining properties of $\mathbb{D}_i$, we find $u_i \in \mathbb{D}_i$ such that $\|w_i - u_i\| < \delta_i$ and $\operatorname{supp} w_i = \operatorname{supp} u_i$ for all i, whereby $(u_i) \sim_{\sqrt{1+\epsilon/K}} (w_i) \sim_{K\sqrt{1+\epsilon/K}} (y_i)$, and therefore $(u_i) \sim_{K+\epsilon} (y_i)$.

We now proceed to define natural numbers n_i, m_i, and $a_i \in A$ such that for $p_i = (n_0, u_0, m_0, \ldots, n_i, u_i, m_i)$, we have

(i) $a_0 = 0$ and $[0, n_0[\subseteq I_{a_0}$,
(ii) $m_i = a_{i+1} - 1$,
(iii) p_i is a legal position in $H_{Y,X}$ in which I has played according to σ_Y,
(iv) $]m_i, n_{i+1}[\subseteq I_{a_{i+1}}$.

Let $a_0 = 0$ and $n_0 = \sigma_Y(\emptyset) = \alpha(0)$, whence $I_{a_0} = [0, \alpha(0)] = [0, n_0]$. Find a_1 such that $n_0, u_0, a_0 < a_1$ and set $m_0 = a_1 - 1$. Then $p_0 = (n_0, u_0, m_0)$ is a legal position in $H_{Y,X}$ in which I has played according to σ_Y, $p_0 < a_1$, so $n_1 = \sigma_Y(n_0, u_0, m_0) \leqslant \alpha(a_1)$, and therefore $]m_0, n_1[\subseteq I_{a_1} = [a_1, \alpha(a_1)]$.

Now suppose by induction that $n_0, \dots, n_i$ and $a_0, \dots, a_i$ have been defined. Since $[0, n_0[\subseteq I_{a_0}$ and $]m_j, n_{j+1}[\subseteq I_{a_{j+1}}$ for all $j < i$, we have

$$u_i \in X[n_0, m_0] + \cdots + X[n_{i-1}, m_{i-1}] + X[n_i, \infty[\,.$$

Find some a_{i+1} greater than all of $n_0, \dots, n_i, u_0, \dots, u_i, a_0, \dots, a_i$ and let $m_i = a_{i+1} - 1$. Then

$$u_i \in X[n_0, m_0] + \cdots + X[n_{i-1}, m_{i-1}] + X[n_i, m_i]$$

and $p_i = (n_0, u_0, m_0, \dots, n_i, u_i, m_i)$ is a legal position played according to σ_Y. Since $p_i < a_{i+1}$ also

$$n_{i+1} = \sigma_Y(n_0, u_0, m_0, \dots, n_i, u_i, m_i) \leqslant \alpha(a_{i+1}).$$

Thus $]m_i, n_{i+1}[\subseteq I_{a_{i+1}} = [a_{i+1}, \alpha(a_{i+1})]$.

Now since $p_0 \subseteq p_1 \subseteq p_2 \subseteq \cdots$, we can let $\vec{p} = \bigcup_i p_i$ and see that $\vec{p}$ is a run of the game in which I followed the strategy σ_Y and II has played (u_i). Since σ_Y is winning for I, this implies that $(u_i) \nsim_{K+\epsilon} (y_i)$ contradicting our assumption. □

Lemma 3.8. *Suppose $X = [x_n]$ is a space with a basis and Y is a space such that for all constants K there are intervals $I_0^{(K)} < I_1^{(K)} < I_2^{(K)} < \cdots$ such that $Y \not\sqsubseteq_K (X, I_j^{(K)})$. Then there are intervals $J_0 < J_1 < J_2 < \cdots$ such that $Y \not\sqsubseteq (X, J_j)$. Moreover, the intervals (J_j) can be computed in a Borel manner from $(I_i^{(K)})_{i,K}$.*

Proof. By induction we can construct intervals $J_0 < J_1 < J_2 < \cdots$ such that J_n contains one interval from each of $(I_i^{(1)}), \dots, (I_i^{(n)})$ and if $M = \min J_n - 1$ and $K = \lceil n \cdot c(M) \rceil$, then $\max J_n > \max I_0^{(K)} + M$, where $c(M)$ is a constant such that if two subsequences of (x_n) differ in at most M terms then they are $c(M)$ equivalent. It then follows that if $A \subseteq \mathbb{N}$ is infinite, then

$$Y \not\sqsubseteq [x_n]_{n \notin \bigcup_{i \in A} J_i}.$$

To see this, suppose towards a contradiction that $A \subseteq \mathbb{N}$ is infinite and that for some integer N,

$$Y \sqsubseteq_N [x_n]_{n \notin \bigcup_{i \in A} J_i}.$$

Pick then $a \in A$ such that $a \geqslant N$ and set $M = \min J_a - 1$ and $K = \lceil a \cdot c(M) \rceil$. Define an isomorphic embedding T from

$$\left[x_n \;\middle|\; n \notin \bigcup_{i \in A} J_i \right]$$

into

$$\left[x_n \;\middle|\; n \notin \bigcup_{i \in A} J_i \;\&\; n > \max J_a \right] + \left[x_n \;\middle|\; \max I_0^{(K)} < n \leqslant \max J_a \right]$$

by setting

$$T(x_n) = \begin{cases} x_n, & \text{if } n > \max J_a; \\ x_{\max I_0^{(K)} + n + 1}, & \text{if } n \leqslant M. \end{cases}$$

This is possible since $\max J_a > \max I_0^{(K)} + M$. Also, since T only changes at most M vectors from (x_n), it is a $c(M)$ embedding. Therefore, by composing with T and using that $N \cdot c(M) \leqslant a \cdot c(M) \leqslant K$, we see that

$$Y \sqsubseteq_K \left[x_n \;\middle|\; n \notin \bigcup_{i \in A} J_i \;\&\; n > \max J_a \right] + \left[x_n \;\middle|\; \max I_0^{(K)} < n \leqslant \max J_a \right].$$

In particular, as almost all J_i contain an interval $I_l^{(K)}$, we can find and infinite set $B \subseteq \mathbb{N}$ containing 0 such that

$$Y \sqsubseteq_K \left[x_n \;\middle|\; n \notin \bigcup_{i \in B} I_i^{(K)} \right],$$

which is a contradiction. □

Lemma 3.9. *Let $E = [e_n]$ be given and suppose that for all block subspaces $Z \leqslant E$ and constants C there is a block subspace $X \leqslant Z$ such that for all block subspaces $Y \leqslant X$, I has a winning strategy in the game $H_{Y,X}$ with constant C. Then there is a block subspace $X \leqslant E$ and a Borel function $f : bb(X) \to [\mathbb{N}]$ such that for all normalised block bases $Y \leqslant X$, if we set $I_j = [f(Y)_{2j}, f(Y)_{2j+1}]$, then*

$$Y \not\sqsubseteq (X, I_j).$$

Proof. Using the hypothesis inductively together with Lemma 3.7, we can construct a sequence $X_0 \geqslant X_1 \geqslant X_2 \geqslant \cdots$ of block subspaces X_K and corresponding Borel functions $f_K : bb(X_K) \to [\mathbb{N}]$ such that for all $V \leqslant X_K$ if $I_j = [f_K(V)_{2j}, f_K(V)_{2j+1}]$, then $V \not\sqsubseteq_{K^2} (X_K, I_j)$.

Pick by Lemma 2.2 some block X_∞ of X_0 that is $\sqrt{K}$-equivalent with a block sequence Z_K of X_K for every $K \geqslant 1$. Then for any block sequence Y of X_∞ and any $K \geqslant 1$ there is some block sequence $V \leqslant Z_K \leqslant X_K$ such that Y is $\sqrt{K}$-equivalent with V. Let (I_j) be the intervals given

by $f_K(V)$ so that $V \not\sqsubseteq_{K^2} (X_K, I_j)$. We can then in a Borel way from (I_j) construct intervals (J_j) such that $V \not\sqsubseteq_{K^2} (Z_K, J_j)$ and therefore also $Y \not\sqsubseteq_K (X_\infty, J_j)$.

This means that there are Borel functions $g_K : bb(X_\infty) \to [\mathbb{N}]$ such that for all $Y \leqslant X_\infty$ if $J_j^K(Y) = [g_K(Y)_{2j}, g_K(Y)_{2j+1}]$, then $Y \not\sqsubseteq_K (X_\infty, J_j^K(Y))$. Using Lemma 3.8 we can now in a Borel manner in Y define intervals $L_0^Y < L_1^Y < \cdots$ such that

$$Y \not\sqsubseteq \big(X_\infty, L_j^Y\big).$$

Letting $f : bb(X_\infty) \to [\mathbb{N}]$ be the Borel function corresponding to $Y \mapsto (L_j^Y)$, we have our result. □

As will be clear in Section 7 it can be useful to have a version of tightness that not only assures us that certain intervals exist, but also tells us how to obtain these. Thus, we call a basis (e_n) *continuously tight* if there is a continuous function $f : bb(e_n) \to [\mathbb{N}]$ such that for all normalised block bases X, if we set $I_j = [f(X)_{2j}, f(X)_{2j+1}]$, then

$$X \not\sqsubseteq \big([e_n], I_j\big),$$

i.e., X does not embed into $[e_n]$ avoiding an infinite number of the intervals I_j.

We shall now improve Lemma 3.9 to conclude continuous tightness from its hypothesis.

Lemma 3.10. *Let $E = [e_n]$ be given and suppose that for all block subspaces $Z \leqslant E$ and constants C there is a block subspace $X \leqslant Z$ such that for all block subspaces $Y \leqslant X$, I has a winning strategy in the game $H_{Y,X}$ with constant C. Then there is a continuously tight block subspace $X \leqslant E$.*

Proof. We observe that E does not contain a copy of c_0. Indeed if Z is a block subspace of E spanned by a block sequence which is C-equivalent to the unit vector basis of c_0, then for any $Y \leqslant X \leqslant Z$, II has a winning strategy in the game $H_{Y,X}$ with constant C^2. We shall then use codings with inevitable subsets. So find first a block subspace Z of E such that there are inevitable, positively separated, closed subsets F_0 and F_1 of $\mathcal{S}_Z$. By Lemma 3.9, we can find a further block subspace V of Z and a Borel function $g : bb(V) \to [\mathbb{N}]$ such that for all $Y \leqslant V$, if $I_j = [g(Y)_{2j}, g(Y)_{2j+1}]$, then $Y \not\sqsubseteq (V, I_j)$. Define the set

$$\mathbb{A} = \big\{(y_n) \in bb(V) \;\big|\; y_{2n} \in F_0 \;\Leftrightarrow\; n \notin g\big((y_{2n+1})\big) \text{ and } y_{2n} \in F_1 \;\Leftrightarrow\; n \in g\big((y_{2n+1})\big)\big\}.$$

Obviously, $\mathbb{A}$ is Borel and, using inevitability, one can check that any block basis of V contains a further block basis in $\mathbb{A}$. Thus, by Gowers' Determinacy Theorem, we have that for all $\Delta > 0$ there is a block sequence X of V such that II has a strategy to play into $\mathbb{A}_\Delta$ when I plays block subspaces of X. Choosing $\Delta > 0$ sufficiently small, this easily implies that for some block basis X of E, there is a continuous function $h : bb(X) \to bb(X) \times [\mathbb{N}]$ that to each $W \leqslant X$ associates a pair $(Y, (I_n))$ consisting of a block sequence Y of W and a sequence of intervals (I_n) such that $Y \not\sqsubseteq (V, I_j)$. Notice now that continuously in the sequence (I_j), we can construct intervals (J_j) such that $Y \not\sqsubseteq (X, J_j)$ and hence also $W \not\sqsubseteq (X, J_j)$. So the continuous function $f : bb(X) \to [\mathbb{N}]$ corresponding to $W \mapsto (J_j)$ witnesses the continuous tightness of X. □

We shall need the following consequence of continuous tightness in Section 7.

Lemma 3.11. *Suppose (e_n) is continuously tight. Then there is a continuous function $f : [\mathbb{N}] \to [\mathbb{N}]$ such that for all $A, B \in [\mathbb{N}]$, if B is disjoint from an infinite number of intervals $[f(A)_{2i}, f(A)_{2i+1}]$, then $[e_n]_{n\in A}$ does not embed into $[e_n]_{n\in B}$.*

Proof. It is enough to notice that the function $h : [\mathbb{N}] \to bb(e_n)$ given by $h(A) = (e_n)_{n\in A}$ is continuous. So when composed with the function witnessing continuous tightness we have the result. □

3.3. A game for minimality

For L and M two block subspaces of E, define the infinite game $G_{L,M}$ with constant $C \geqslant 1$ between two players as follows. In each round I chooses a subspace $E_i \subseteq L$ spanned by a finite block sequence of L, a normalised block vector $u_i \in E_0 + \cdots + E_i$, and an integer m_i. In the first round II plays an integer n_0, and in all subsequent rounds II plays a subspace F_i spanned by a finite block sequence of M, a (not necessarily normalised) block vector $v_i \in F_0 + \cdots + F_i$ and an integer n_{i+1}. Moreover, we demand that $n_i \leqslant E_i$ and $m_i \leqslant F_i$.

Diagrammatically,

I	$n_0 \leqslant E_0 \subseteq L$		$n_1 \leqslant E_1 \subseteq L$	$\cdots$
	$u_0 \in E_0, m_0$		$u_1 \in E_0 + E_1, m_1$	
II	n_0	$m_0 \leqslant F_0 \subseteq M$		$m_1 \leqslant F_1 \subseteq M$ $\cdots$
		$v_0 \in F_0, n_1$		$v_1 \in F_0 + F_1, n_2$

The *outcome* of the game is the pair of infinite sequences (u_i) and (v_i) and we say that II wins the game if $(u_i) \sim_C (v_i)$.

Lemma 3.12. *Suppose that X and Y are block subspaces of E and that player II has a winning strategy in the game $H_{Y,X}$ with constant C. Then II has a winning strategy in the game $G_{Y,X}$ with constant C.*

Proof. We shall in fact prove that II has a winning strategy in a game that is obviously harder for her to win. Namely, we shall suppose that II always plays $n_i = 0$, which obviously puts less restrictions on the play of I. Moreover, we do not require I to play the finite-dimensional spaces E_i, which therefore also puts fewer restrictions on I in subsequent rounds. Therefore, we shall suppress all mention of E_i and n_i and only require that the u_i are block vectors in Y.

While playing the game $G_{Y,X}$, II will keep track of an auxiliary play of the game $H_{Y,X}$ in the following way. In the game $G_{Y,X}$ we have the following play

I	$u_0 \in Y, m_0$	$u_1 \in Y, m_1$	$\cdots$
II	$m_0 \leqslant F_0 \subseteq X$		$m_1 \leqslant F_1 \subseteq X$ $\cdots$
	$v_0 \in F_0$		$v_1 \in F_0 + F_1$

We write each vector $u_i = \sum_{j=0}^{k_i} \lambda_j^i y_j$ and may for simplicity of notation assume that $k_i < k_{i+1}$. The auxiliary run of $H_{Y,X}$ that II will keep track of is as follows, where II plays according to her winning strategy for $H_{Y,X}$.

$$\begin{array}{lccccccccc} \mathbf{I} & m_0 & & \cdots & m_0 & & m_1 & & \cdots & m_1 & & \cdots \\ \mathbf{II} & & w_0, p_0 & \cdots & & w_{k_0}, p_{k_0} & & w_{k_0+1}, p_{k_0+1} & \cdots & & w_{k_1}, p_{k_1} & \cdots \end{array}$$

To compute the v_i and F_i in the game $G_{Y,X}$, II will refer to the play of $H_{Y,X}$ and set

$$v_i = \sum_{j=0}^{k_i} \lambda_j^i w_j,$$

and let

$$F_i = X\big[m_i, \max\{p_{k_{i-1}+1}, \ldots, p_{k_i}\}\big].$$

It is not difficult to see that $m_i \leqslant F_i \subseteq X$, $v_i \in F_0 + \cdots + F_i$, and that the F_i and v_i only depends on $u_0, \ldots, u_i$ and $m_0, \ldots, m_i$. Thus this describes a strategy for II in $G_{Y,X}$ and it suffices to verify that it is a winning strategy.

But since II follows her strategy in $H_{Y,X}$, we know that $(w_i) \sim_C (y_i)$ and therefore, since u_i and v_i are defined by the same coefficients over respectively (y_i) and (w_i), we have that $(v_i) \sim_C (u_i)$. □

3.4. A dichotomy for minimality

We are now in condition to prove the central result of this paper.

Theorem 3.13 *(3rd dichotomy). Let E be a Banach space with a basis (e_i). Then either E contains a minimal block subspace or a continuously tight block subspace.*

Proof. Suppose that E has no continuously tight block basic sequence. By Lemma 3.10, we can, modulo passing to a block subspace, suppose that for some constant C and for all block subspaces $X \leqslant E$ there is a further block subspace $Y \leqslant X$ such that I has no winning strategy in the game $H_{Y,X}$ with constant C. By the determinacy of open games, this implies that for all block subspaces $X \leqslant E$ there is a further block subspace $Y \leqslant X$ such that II has a winning strategy in the game $H_{Y,X}$ with constant C.

A *state* is a pair (a, b) with $a, b \in (\mathbf{D}' \times \mathbb{F})^{<\omega}$, where $\mathbb{F}$ is the set of subspaces spanned by finite block sequences and $\mathbf{D}'$ the set of not necessarily normalised blocks, such that $|a| = |b|$ or $|a| = |b| + 1$. The set S of states is countable, and corresponds to the possible positions of a game $G_{L,M}$ after a finite number of moves were made, restricted to elements that affect the outcome of the game from that position (i.e., m_i's and n_i's are forgotten).

For each state $s = (a, b)$ we will define the game $G_{L,M}(s)$ in a manner similar to the game $G_{L,M}$ depending on whether $|a| = |b|$ or $|a| = |b| + 1$. To avoid excessive notation we do this via two examples:

If $a = (a_0, A_0, a_1, A_1)$, $b = (b_0, B_0, b_1, B_1)$, the game $G_{L,M}(s)$ will start with II playing some integer n_2, then I playing (u_2, E_2, m_2) with $n_2 \leqslant E_2 \subseteq L$ and $u_2 \in A_0 + A_1 + E_2$, II playing (v_2, F_2, n_3) with $m_2 \leqslant F_2 \subseteq M$ and $v_2 \in B_0 + B_1 + F_2$, etc., and the outcome of the game will be the pair of infinite sequences $(a_0, a_1, u_2, \ldots)$ and $(b_0, b_1, v_2, \ldots)$.

If $a = (a_0, A_0, a_1, A_1)$, $b = (b_0, B_0)$, the game $G_{L,M}(s)$ will start with I playing some integer m_1, then II playing (v_1, F_1, n_2) with $m_1 \leqslant F_1 \subseteq M$ and $v_1 \in B_0 + F_1$, I playing (u_2, E_2, m_2)

with $n_2 \leqslant E_2 \subseteq L$ and $u_2 \in A_0 + A_1 + E_2$, etc., and the outcome of the game will be the pair of infinite sequences $(a_0, a_1, u_2, \ldots)$ and $(b_0, v_1, v_2, \ldots)$.

The following lemma is well known and easily proved by a simple diagonalisation.

Lemma 3.14. *Let N be a countable set and let $\mu : bb(E) \to \mathcal{P}(N)$ satisfy either*

$$V \leqslant^* W \quad \Rightarrow \quad \mu(V) \subseteq \mu(W)$$

or

$$V \leqslant^* W \quad \Rightarrow \quad \mu(V) \supseteq \mu(W).$$

Then there exists a stabilising block subspace $V_0 \leqslant E$, i.e., such that $\mu(V) = \mu(V_0)$ for any $V \leqslant^* V_0$.

Let now $\tau : bb(E) \to \mathcal{P}(S)$ be defined by

$$s \in \tau(M) \quad \Leftrightarrow \quad \exists L \leqslant M \text{ such that player II has a winning strategy in } G_{L,M}(s).$$

By the asymptotic nature of the game we see that $M' \leqslant^* M \Rightarrow \tau(M') \subseteq \tau(M)$, and therefore there exists $M_0 \leqslant E$ which is stabilising for τ. We then define a map $\rho : bb(E) \to \mathcal{P}(S)$ by setting

$$s \in \rho(L) \quad \Leftrightarrow \quad \text{player II has a winning strategy in } G_{L,M_0}(s).$$

Again $L' \leqslant^* L \Rightarrow \rho(L') \supseteq \rho(L)$ and therefore there exists $L_0 \leqslant M_0$ which is stabilising for ρ. Finally, the reader will easily check that $\rho(L_0) = \tau(L_0) = \tau(M_0)$, see, e.g., [35] or [10].

Lemma 3.15. *For every $M \leqslant L_0$, II has a winning strategy for the game $G_{L_0,M}$.*

Proof. Fix M a block subspace of L_0. We begin by showing that $(\emptyset, \emptyset) \in \tau(L_0)$. To see this, we notice that as $L_0 \leqslant E$, there is a $Y \leqslant L_0$ such that II has a winning strategy for H_{Y,L_0} and thus, by Lemma 3.12, also a winning strategy in G_{Y,L_0} with constant C. So $(\emptyset, \emptyset) \in \tau(L_0)$.

We will show that for all states

$$\big((u_0, E_0, \ldots, u_i, E_i), (v_0, F_0, \ldots, v_i, F_i)\big) \in \tau(L_0),$$

there is an n such that for all $n \leqslant E \subseteq L_0$ and $u \in E_0 + \cdots + E_i + E$, we have

$$\big((u_0, E_0, \ldots, u_i, E_i, u, E), (v_0, F_0, \ldots, v_i, F_i)\big) \in \tau(L_0).$$

Similarly, we show that for all states

$$\big((u_0, E_0, \ldots, u_{i+1}, E_{i+1}), (v_0, F_0, \ldots, v_i, F_i)\big) \in \tau(L_0)$$

and for all m there are $m \leqslant F \subseteq M$ and $v \in F_0 + \cdots + F_i + F$ such that

$$\big((u_0, E_0, \ldots, u_{i+1}, E_{i+1}), (v_0, F_0, \ldots, v_i, F_i, v, F)\big) \in \tau(L_0).$$

Since the winning condition of $G_{L_0,M}$ is closed, this clearly shows that II has a winning strategy in $G_{L_0,M}$ (except for the integers m and n, $\tau(L_0)$ is a winning quasi strategy for II).

So suppose that

$$s = \big((u_0, E_0, \dots, u_i, E_i), (v_0, F_0, \dots, v_i, F_i)\big) \in \tau(L_0) = \rho(L_0),$$

then II has a winning strategy in $G_{L_0,M_0}(s)$ and hence there is an n such that for all $n \leqslant E \subseteq L_0$ and $u \in E_0 + \cdots + E_i + E$, II has a winning strategy in $G_{L_0,M_0}(s')$, where

$$s' = \big((u_0, E_0, \dots, u_i, E_i, u, E), (v_0, F_0, \dots, v_i, F_i)\big).$$

So $s' \in \rho(L_0) = \tau(L_0)$.

Similarly, if

$$s = \big((u_0, E_0, \dots, u_{i+1}, E_{i+1}), (v_0, F_0, \dots, v_i, F_i)\big) \in \tau(L_0) = \tau(M)$$

and m is given, then as II has a winning strategy for $G_{L,M}(s)$ for some $L \leqslant M$, there are $m \leqslant F \subseteq M$ and $v \in F_0 + \cdots + F_i + F$ such that II has a winning strategy in $G_{L,M}(s')$, where

$$s' = \big((u_0, E_0, \dots, u_{i+1}, E_{i+1}), (v_0, F_0, \dots, v_i, F_i, v, F)\big).$$

So $s' \in \tau(M) = \tau(L_0)$. □

Choose now $Y = [y_i] \leqslant L_0$ such that II has a winning strategy in H_{Y,L_0}. We shall show that any block subspace M of L_0 contains a C^2-isomorphic copy of Y, which implies that Y is $C^2 + \epsilon$-minimal for any $\epsilon > 0$.

To see this, notice that, since II has a winning strategy in H_{Y,L_0}, player I has a strategy in the game $G_{L_0,M}$ to produce a sequence (u_i) that is C-equivalent with the basis (y_i). Moreover, we can ask that I plays $m_i = 0$. Using her winning strategy for $G_{L_0,M}$, II can then respond by producing a sequence (v_i) in M such that $(v_i) \sim_C (u_i)$. So $(v_i) \sim_{C^2} (y_i)$ and $Y \sqsubseteq_{C^2} M$. □

Finally we observe that by modifying the notion of embedding in the definition of a tight basis, we obtain variations of our dichotomy theorem with a weaker form of tightness on one side and a stronger form of minimality on the other.

Theorem 3.16. *Every Banach space with a basis contains a block subspace $E = [e_n]$ which satisfies one of the two following properties*:

(1) *For any $[y_i] \leqslant E$, there exists a sequence (I_i) of successive intervals such that for any infinite subset A of $\mathbb{N}$, the basis (y_i) does not embed into $[e_n]_{n \notin \bigcup_{i \in A} I_i}$ as a sequence of disjointly supported blocks , resp. as a permutation of a block sequence, resp. as a block sequence.*
(2) *For any $[y_i] \leqslant E$, (e_n) is equivalent to a sequence of disjointly supported blocks of $[y_i]$, resp. (e_n) is permutatively equivalent to a block sequence of $[y_i]$, resp. (e_n) is equivalent to a block sequence of $[y_i]$.*

The case of block sequences immediately implies the theorem of Pełczar [35].

The fact that the canonical basis of T^* is strongly asymptotically ℓ_∞ implies easily that it is tight for "embedding as a sequence of disjointly supported blocks" although T^* is minimal in

the usual sense. We do not know of other examples of spaces combining one form of minimality with another form of tightness in the above list.

4. Tightness with constants and crude stabilisation of local structure

We shall now consider a stronger notion of tightness, which is essentially local in nature. Let E be a space with a basis (e_n). There is a particularly simple case when a sequence (I_i) of intervals associated to a subspace Y characterises the tightness of Y in (e_n). This is when for all integer constants K, $Y \not\sqsubseteq_K [e_n]_{n\notin I_K}$. This property has the following useful reformulations.

Proposition 4.1. *Let E be a space with a basis (e_n). The following are equivalent*:

(1) *For any block sequence (y_n) there are intervals $I_0 < I_1 < I_2 < \cdots$ such that for all K,*

$$[y_n]_{n\in I_K} \not\sqsubseteq_K [e_n]_{n\notin I_K}.$$

(2) *For any space Y, there are intervals $I_0 < I_1 < I_2 < \cdots$ such that for all K,*

$$Y \not\sqsubseteq_K [e_n]_{n\notin I_K}.$$

(3) *No space embeds uniformly into the tail subspaces of E.*
(4) *There is no K and no subspace of E which is K-crudely finitely representable in any tail subspace of E.*

A basis satisfying properties (1), (2), (3), (4), as well as the space it generates, will be said to be *tight with constants*.

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

To prove (3) $\Rightarrow$ (4) assume some subspace Y of E is K-crudely finitely representable in any tail subspace of E. Without loss of generality, we may assume that $Y = [y_n]$ is a block subspace of E. We pick a subsequence (z_n) of (y_n) in the following manner. Let $z_0 = y_0$, and if $z_0, \dots, z_{k-1}$ have been chosen, we choose z_k supported far enough on the basis (e_n), so that $[z_0, \dots, z_{k-1}]$ has a $2K$-isomorphic copy in $[e_n \mid k \leqslant n < \min(\operatorname{supp} z_k)]$. It follows that for any k, $Z = [z_n]$ has an M-isomorphic copy in the tail subspace $[e_n \mid n \geqslant k]$ for some M depending only on K and the constant of the basis (e_n).

To prove (4) $\Rightarrow$ (1), let $c(L)$ be a constant such that if two block sequences differ in at most L terms, then they are $c(L)$-equivalent. Now assume (4) holds and let (y_n) be a block sequence of (e_n). Suppose also that $I_0 < \cdots < I_{K-1}$ have been chosen. By (4) applied to $Y = [y_n]_{n=\max I_{K-1}+1}^{\infty}$, we can then find m and $l > \max I_{K-1}$ such that $[y_n]_{n=\max I_{K-1}+1}^{l}$ does not $K \cdot c(\max I_{K-1} + 1)$-embed into $[e_n]_{n=m}^{\infty}$. Let now

$$I_K = [\max I_{K-1} + 1, l + m]$$

and notice that, as $[y_n]_{n=\max I_{K-1}+1}^{l} \subseteq [y_n]_{n\in I_K}$, we have that $[y_n]_{n\in I_K}$ does not $K \cdot c(\max I_{K-1} + 1)$-embed into $[e_n]_{n=m}^{\infty}$. Also, since $(e_n)_{n=m}^{\infty}$ and

$$(e_n)_{n=0}^{\max I_{K-1}} {}^\frown (e_n)_{n=\max I_{K-1}+1+m}^{\infty}$$

only differ in $\max I_{K-1}+1$ many terms, $[y_n]_{n\in I_K}$ does not K-embed into

$$[e_n]_{n=0}^{\max I_{K-1}} + [e_n]_{n=\max I_{K-1}+1+m}^{\infty},$$

and thus not into the subspace $[e_n]_{n\notin I_K}$ either. □

It is worth noticing that a basis (e_n), tight with constants, is necessarily continuously tight. For a simple argument shows that in order to find the intervals I_K satisfying (1) above, one only needs to know a beginning of the block sequence (y_n) and hence the intervals can be found continuously in (y_n). From Proposition 4.1 we also deduce that any block basis or shrinking basic sequence in the span of a tight with constants basis is again tight with constants.

There is a huge difference between the fact that no *subspace* of E is K-crudely finitely representable in all tails of E and then that no *space* is K-crudely finitely representable in all tails of E. For example, we shall see that while the former holds for Tsirelson's space, by Dvoretzky's theorem (see e.g. [16]), ℓ_2 is always finitely representable in any Banach space.

Recall that a basis (e_n) is said to be *strongly asymptotically* ℓ_p, $1\leqslant p\leqslant+\infty$ [9], if there exist a constant C and a function $f:\mathbb{N}\to\mathbb{N}$ such that for any n, any family of n unit vectors which are disjointly supported in $[e_k \mid k\geqslant f(n)]$ is C-equivalent to the canonical basis of ℓ_p^n.

Proposition 4.2. *Let E be a Banach space with a strongly asymptotically ℓ_p basis (e_n), $1\leqslant p<+\infty$, and not containing a copy of ℓ_p. Then (e_n) is tight with constants.*

Proof. Assume that some Banach space Y embeds with constant K in any tail subspace of E. We may assume that Y is generated by a block sequence (y_n) of E and, since any strongly asymptotically ℓ_p basis is unconditional, (y_n) is unconditional. By renorming E we may assume it is 1-unconditional. By a result of W.B. Johnson [23] for any n there is a constant $d(n)$ such that $(y_0,\dots,y_n)$ is $2K$-equivalent to a sequence of vectors in the linear span of $d(n)$ disjointly supported unit vectors in any tail subspace of E, in particular in $[e_k \mid k\geqslant f(d(n))]$. Therefore $[y_0,\dots,y_n]$ $2KC$-embeds into ℓ_p. This means that Y is crudely finitely representable in ℓ_p and therefore embeds into L_p, and since (y_n) is unconditional asymptotically ℓ_p, that Y contains a copy of ℓ_p (details of the last part of this proof may be found in [9]). □

Corollary 4.3. *Tsirelson's space T and its convexifications $T^{(p)}$, $1<p<+\infty$, are tight with constants.*

Observe that on the contrary, the dual T^* of T, which is strongly asymptotically ℓ_∞ and does not contain a copy of c_0, is minimal and therefore does not contain any tight subspace.

Suppose a space X is crudely finitely representable in all of its subspaces. Then there is some constant K and a subspace Y such that X is K-crudely finitely representable in all of the subspaces of Y. For if not, we would be able to construct a sequence of basic sequences (x_n^K) in X such that (x_n^{K+1}) is a block sequence of (x_n^K) and such that X is not K^2-crudely finitely representable in $[x_n^K]$. By Lemma 2.2, we can then find a block sequence (y_n) of (x_n^0) that is $\sqrt{K}$-equivalent with a block sequence of (x_n^K) for any K and hence if X were K-crudely finitely representable in $[y_n]$ for some K, then it would also be $K^{3/2}$-crudely finitely representable in $[x_n^K]$, which is a contradiction.

When a space X is K-crudely finitely representable in any of its subspaces for some K, we say that X is *locally minimal*. For example, by the universality properties of c_0, any space with an asymptotically ℓ_∞ basis is locally minimal.

Theorem 4.4 *(5th dichotomy). Let E be an infinite-dimensional Banach space with basis (e_n). Then there is a block sequence (x_n) satisfying one of the following two properties, which are mutually exclusive and both possible.*

(1) *(x_n) is tight with constants,*
(2) *$[x_n]$ is locally minimal.*

Proof. If E contains c_0, the result is trivial. So suppose not and find by the solution to the distortion problem a block sequence (y_n) and inevitable, positively separated, closed subsets F_0 and F_1 of the unit sphere of $[y_n]$. Define for each integer $K \geqslant 1$ the set

$$\mathbb{A}_K = \big\{(z_n) \leqslant (y_n) \;\big|\; z_{2n} \in F_0 \cup F_1 \text{ and } (z_{2n}) \text{ codes by 0's and 1's a block sequence } (v_n) \text{ of } (z_{2n+1}) \text{ such that for all } N, [v_n] \sqsubseteq_K [z_{2n+1}]_{n\geqslant N} \text{ and moreover } 1/2 < \|v_n\| < 2\big\}.$$

Clearly $\mathbb{A}_K$ is analytic, so we can apply Gowers' Determinacy Theorem to get one of two cases:

(i) either there is a block sequence (x_n) and a K such that player II has a strategy to play inside $(\mathbb{A}_K)_\Delta$ whenever I plays a block sequence of (x_n), where Δ will be determined later,
(ii) or we can choose inductively a sequence of block sequences (x_n^K) such that $(x_n^{K+1}) \leqslant (x_n^K)$ and such that no block sequence of (x_n^K) belongs to $\mathbb{A}_K$.

Consider first case (ii). Set $w_n = x_{2n}^n$ and choose now further block sequences (x_n) and (h_n) of (w_n) such that

$$x_0 < h_0 < h_1 < x_1 < h_2 < h_3 < x_4 < \cdots$$

and $h_{2n} \in F_0$, $h_{2n+1} \in F_1$.

We claim that (x_n) is tight with constants. If not, we can find some block sequence (u_n) of (x_n) and a K such that $[u_n]$ embeds with constant K into any tail subspace of $[x_n]$. By passing to tails of (x_n) and of (u_n), we can suppose that (x_n) is a block sequence of (x_n^K), (u_n) is a block sequence of (x_n) and $[u_n]$ K-embeds into all tails of $[x_n]$. By filling in with appropriate h_i between x_n and x_{n+1}, we can now produce a block sequence (z_n) of (x_n^K) such that (z_{2n}) codes by 0's and 1's the block sequence (u_n) of (z_{2n+1}) with the property that for all N, $[u_n] \sqsubseteq_K [z_{2n+1}]_{n\geqslant N}$. In other words, we have produced a block sequence of (x_n^K) belonging to $\mathbb{A}_K$, which is impossible. Thus, (x_n) is tight with constants.

Consider now case (i) instead and let II play according to her strategy. We suppose that Δ is chosen sufficiently small so that $\delta_i < \operatorname{dist}(F_0, F_1)/3$ and if two block sequences are Δ-close then they are 2-equivalent. Let $(y_n) \in (\mathbb{A}_K)_\Delta$ be the response by II to the sequence (x_n) played by I and let $(z_n) \in \mathbb{A}_K$ be such that $\|z_n - y_n\| < \delta_n$ for all n. Then (z_{2n}) codes by 0's and 1's a block sequence (v_n) of (z_{2n+1}). Let (u_n) be the block sequence of (y_{2n+1}) constructed in the same way as (v_n) is constructed over (z_{2n+1}). We claim that (u_n) is $4K$-crudely finitely representable in any block subspace of $[x_n]$.

For this, let $[u_0, \dots, u_m]$ be given and suppose that (f_n) is any block subspace of (x_n). Find a large k such that $(z_0, z_2, \dots, z_{2k})$ codes the block sequence $(v_0, \dots, v_m)$ of $(z_1, \dots, z_{2k+1})$ and let l be large enough so that when I has played $x_0, \dots, x_l$ then II has played $y_0, \dots, y_{2k+1}$. Consider now the game in which player I plays

$$x_0, x_1, \dots, x_l, f_{l+1}, f_{l+2}, \dots.$$

Then, following the strategy, II will play a block sequence

$$y_0, \dots, y_{2k+1}, g_{2k+2}, g_{2k+3}, \dots \in (\mathbb{A}_K)_\Delta.$$

So let $(h_n) \in \mathbb{A}_K$ be such that $\|h_n - y_n\| < \delta_n$ for all $n \leqslant 2k+1$ and $\|h_n - g_n\| < \delta_n$ for all $n \geqslant 2k+2$. For $n \leqslant k$, we have, as $\|h_{2n} - z_{2n}\| < 2\delta_n < \frac{2}{3}\operatorname{dist}(F_0, F_1)$, that $h_{2n} \in F_i \Leftrightarrow z_{2n} \in F_i$. Also, $(h_{2n+1})_{n=0}^k$ and $(y_{2n+1})_{n=0}^k$ are 2-equivalent and $(h_{2n+1})_{n=k+1}^\infty$ and $(g_{2n+1})_{n=k+1}^\infty$ are 2-equivalent, so (h_{2n}) will code a block sequence (w_n) of (h_{2n+1}) such that $(w_0, \dots, w_m)$ is 2-equivalent to $(u_0, \dots, u_m)$. Moreover, since $(h_n) \in \mathbb{A}_K$, $[w_n]$ will K-embed into every tail subspace of $[h_{2n+1}]$, and hence $2K$-embed into every tail subspace of $[g_{2n+1}]$. Therefore, since (g_{2n+1}) is block sequence of (f_n), $[u_0, \dots, u_m]$ will $4K$-embed into $[f_n]$, which proves the claim. It follows that $[u_n]$ is locally minimal, which proves the theorem. □

Local minimality can be reformulated in a way that makes the relation to local theory clearer. For this, let $\mathbb{F}_n$ be the metric space of all n-dimensional Banach spaces up to isometry equipped with the Banach–Mazur metric

$$d(X, Y) = \inf\bigl(\log\bigl(\|T\| \cdot \|T^{-1}\|\bigr) \mid T : X \to Y \text{ is an isomorphism}\bigr).$$

Then for every Banach space X, the set of n-dimensional Y that are almost isometrically embeddable into X form a closed subset $(X)_n$ of $\mathbb{F}_n$. It is well known that this set $(X)_n$ does not always stabilise, i.e., there is not necessarily a subspace $Y \subseteq X$ such that for all further subspaces $Z \subseteq Y$, $(Z)_n = (Y)_n$. However, if instead X comes equipped with a basis and for all block subspaces Y we let $\{Y\}_n$ be the set of all n-dimensional spaces that are almost isometrically embeddable into all tail subspaces of Y, then one can easily stabilise $\{Y\}_n$ on subspaces. Such considerations are for example the basis for [30].

Theorem 4.4 gives a dichotomy for when one can stabilise the set $(X)_n$ in a certain way, which we could call *crude*. Namely, X is locally minimal if and only if there is some constant K such that for all subspaces Y of X and all n, $d_H((X)_n, (Y)_n) \leqslant K$, where d_H is the Hausdorff distance. So by Theorem 4.4, the local structure stabilises crudely on a subspace if and only if a space is not saturated by basic sequences tight with constants.

Often it is useful to have a bit more than local minimality. So we say that a basis (e_n) is *locally block minimal* if it is K-crudely block finitely representable in all of its block bases for some K. As with crude finite representability we see that there then must be a constant K and a block (y_n) such that (e_n) is K-crudely block finitely representable in all block subspaces of (y_n). We now have the following version of Theorem 4.4 for finite block representability.

Theorem 4.5. *Let (e_n) be a Schauder basis. Then (e_n) has a block basis (x_n) with one of the following two properties, which are mutually exclusive and both possible.*

(1) *For all block bases* (y_n) *of* (x_n) *there are intervals* $I_1 < I_2 < I_3 < \cdots$ *such that* $(y_n)_{n\in I_K}$ *is not* K*-equivalent to a block sequence of* $(x_n)_{n\notin I_K}$,
(2) (x_n) *is locally block minimal.*

Finally we note that there exist tight spaces which do not admit subspaces which are tight with constants:

Example 4.6. There exists a reflexive, tight, locally block minimal Banach space.

Proof. E. Odell and T. Schlumprecht [33] have built a reflexive space OS with a basis such that every monotone basis is block finitely representable in any block subspace of OS. It is in particular locally block minimal and therefore contains no basic sequence which is tight with constants. We do not know whether the space OS is tight. Instead, we notice that since the summing basis of c_0 is block finitely representable in any block subspace of OS, OS cannot contain an unconditional block sequence. By Gowers' 1st dichotomy it follows that some block subspace of OS is HI, and by the 3rd dichotomy (Theorem 3.13) and the fact that HI spaces do not contain minimal subspaces, that some further block subspace is tight, which completes the proof. □

It is unknown whether there is an unconditional example with the above property. There exists an unconditional version of OS [32], but it is unclear whether it has no minimal subspaces. However, the dual of a space constructed by Gowers in [17] can be shown to be both tight and locally minimal.

Example 4.7. (See [14].) There exists a space with an unconditional basis which is tight and locally minimal.

5. Local block minimality, asymptotic structure and a dichotomy for containing c_0 or ℓ_p

Recall that a basis (e_n) is said to be *asymptotically* ℓ_p (in the sense of Tsirelson's space) if there is a constant C such that for all normalised block sequences $n < x_1 < \cdots < x_n$, $(x_i)_{i=1}^n$ is C-equivalent with the standard unit vector basis of ℓ_p^n.

When (x_n) is asymptotically ℓ_p and some block subspace $[y_n]$ of $[x_n]$ is K-crudely block finitely representable in all tail subsequences of (x_n), then it is clear that (y_n) must actually be equivalent with the unit vector basis of ℓ_p, or c_0 for $p = \infty$. So this shows that for asymptotically ℓ_p bases (x_n), either $[x_n]$ contains an isomorphic copy of ℓ_p or c_0 or (x_n) itself satisfies condition (1) of Theorem 4.5. This is the counterpart of Proposition 4.2 for block sequences. As an example, we mention that, since T^* does not contain c_0, but has a strongly asymptotically ℓ_∞ basis, it thus satisfies (1). This small observation indicates that one can characterise when ℓ_p or c_0 embeds into a Banach space by characterising when a space contains an asymptotic ℓ_p space.

We first prove a dichotomy for having an asymptotic ℓ_p subspace, for the proof of which we need the following lemma.

Lemma 5.1. *Suppose* (e_n) *is a basic sequence such that for some* C *and all* n *and normalised block sequences* $n < y_1 < y_2 < \cdots < y_{2n}$, *we have*

$$(y_{2i-1})_{i=1}^n \sim_C (y_{2i})_{i=1}^n.$$

Then (e_n) *has an asymptotic* ℓ_p *subsequence for some* $1 \leqslant p \leqslant \infty$.

Proof. By the theorem of Brunel and Sucheston [5], we can, by passing to a subsequence of (e_n), suppose that (e_n) generates a spreading model, i.e., we can assume that for all integers $n < l_1 < l_2 < \cdots < l_n$ and $n < k_1 < k_2 < \cdots < k_n$ we have

$$(e_{l_1}, \ldots, e_{l_n}) \sim_{1+\frac{1}{n}} (e_{k_1}, \ldots, e_{k_n}).$$

Now suppose that $e_n < y_1 < y_2 < \cdots < y_n$ and $e_n < z_1 < \cdots < z_n$ are normalised block sequences of (e_{2i}). Then there are $n < l_1 < l_2 < \cdots < l_n$ and $n < k_1 < k_2 < \cdots < k_n$ such that

$$e_n < y_1 < e_{l_1} < y_2 < e_{l_2} < \cdots < y_n < e_{l_n}$$

and

$$e_n < z_1 < e_{k_1} < z_2 < e_{k_2} < \cdots < z_n < e_{k_n},$$

so $(y_i) \sim_C (e_{l_i}) \sim_{1+\frac{1}{n}} (e_{k_i}) \sim_C (z_i)$ and $(y_i) \sim_{2C^2} (z_i)$. Thus, asymptotically all finite normalised block sequences are $2C^2$-equivalent.

By Krivine's theorem [25], there is some ℓ_p that is block finitely representable in (e_{2i}) and hence asymptotically all finite normalised block sequences are equivalent to ℓ_p^n of the correct dimension and (e_{2i}) is asymptotic ℓ_p. □

Theorem 5.2. *Suppose X is a Banach space with a basis. Then X has a block subspace W, which is either asymptotic ℓ_p, for some $1 \leqslant p \leqslant +\infty$, or such that*

$$\forall M\ \exists n\ \forall U_1, \ldots, U_{2n} \subseteq W\ \exists u_i \in \mathcal{S}_{U_i}\ \big(u_1 < \cdots < u_{2n}\ \&\ (u_{2i-1})_{i=1}^n \nsim_M (u_{2i})_{i=1}^n\big).$$

Proof. Assume first that for some M and $V \subseteq X$ we have

$$\forall n\ \forall Y \subseteq V\ \exists Z \subseteq Y\ \forall z_1 < \cdots < z_{2n} \in \mathcal{S}_Z\ (z_{2i-1})_{i=1}^n \sim_M (z_{2i})_{i=1}^n.$$

Then we can inductively define $V \supseteq Z_1 \supseteq Z_2 \supseteq Z_3 \supseteq \cdots$ such that for each n,

$$\forall z_1 < \cdots < z_{2n} \in \mathcal{S}_{Z_n}\ (z_{2i-1})_{i=1}^n \sim_M (z_{2i})_{i=1}^n.$$

Diagonalising over this sequence, we can find a block subspace $W = [w_n]$ such that for all $m \geqslant n$, $w_m \in Z_n$. Therefore, if $n < z_1 < \cdots < z_{2n}$ is a sequence of normalised blocks of (w_n), then $(z_{2i-1})_{i=1}^n \sim_M (z_{2i})_{i=1}^n$. By Lemma 5.1, it follows that W has an asymptotic ℓ_p subspace.

So suppose on the contrary that

$$\forall M\ \forall V \subseteq X\ \exists n\ \exists Y \subseteq V\ \forall Z \subseteq Y\ \exists z_1 < \cdots < z_{2n} \in \mathcal{S}_Z\ (z_{2i-1})_{i=1}^n \nsim_M (z_{2i})_{i=1}^n.$$

Now find some small $\Delta > 0$ such that two normalised block sequences of X that are Δ-close are $\sqrt{2}$-equivalent. Applying the determinacy theorem of Gowers to Δ and the sets

$$\mathbb{A}(M, n) = \big\{(z_i) \bigm| (z_{2i-1})_{i=1}^n \nsim_M (z_{2i})_{i=1}^n\big\},$$

we have that

$$\forall M\ \forall V \subseteq X\ \exists n\ \exists Z \subseteq V\ \text{ II has a strategy to play normalised } z_1 < \cdots < z_{2n} \text{ such that } (z_{2i-1})_{i=1}^n \nsim_{\frac{M}{2}} (z_{2i})_{i=1}^n \text{ when I is restricted to playing blocks of } Z.$$

Using this we can inductively define a sequence $X \supseteq W_1 \supseteq W_2 \supseteq W_3 \supseteq \cdots$ such that for all M there is an $n = n(2M)$ such that II has a strategy to play normalised $z_1 < \cdots < z_{2n}$ such that $(z_{2i-1})_{i=1}^n \nsim_M (z_{2i})_{i=1}^n$ whenever I is restricted to playing blocks of W_M. Diagonalising over this sequence, we find some $W \subseteq X$ such that for all M, $W \subseteq^* W_M$. So for all M there is n such that II has a strategy to play normalised $z_1 < \cdots < z_{2n}$ such that $(z_{2i-1})_{i=1}^n \nsim_M (z_{2i})_{i=1}^n$ whenever I is restricted to playing blocks of W.

Letting player I play a segment of the block basis of U_i until II plays a vector, we easily see that whenever $U_1, \ldots, U_{2n} \subseteq W$, there are $z_i \in \mathcal{S}_{U_i}$ such that $z_1 < \cdots < z_{2n}$ and $(z_{2i-1})_{i=1}^n \nsim_M (z_{2i})_{i=1}^n$. This finishes the proof. □

Using this, we can now prove the main result of this section.

Theorem 5.3 *(The c_0 and ℓ_p dichotomy). Suppose X is a Banach space not containing a copy of c_0 nor of ℓ_p, $1 \leqslant p < \infty$. Then X has a subspace Y with a basis satisfying one of the following properties.*

(i) $\forall M\ \exists n\ \forall U_1, \ldots, U_{2n} \subseteq Y\ \exists u_i \in \mathcal{S}_{U_i}\ u_1 < \cdots < u_{2n}\ \&\ (u_{2i-1})_{i=1}^n \nsim_M (u_{2i})_{i=1}^n$.

(ii) *For all block bases (z_n) of $Y = [y_n]$ there are intervals $I_1 < I_2 < I_3 < \cdots$ such that $(z_n)_{n\in I_K}$ is not K-equivalent to a block sequence of $(y_n)_{n\notin I_K}$.*

Notice that both (i) and (ii) trivially imply that Y cannot contain a copy of c_0 or ℓ_p, since (i) implies some lack of homogeneity and (ii) some lack of minimality. However, we do not know if there are any spaces satisfying both (i) and (ii) or if, on the contrary, these two properties are incompatible.

Proof. If X has no block subspace satisfying (i), then it must have an asymptotically ℓ_p block subspace Y for some $1 \leqslant p \leqslant \infty$. Let C be the constant of asymptoticity. Suppose now that $Z = [z_n]$ is a further block subspace. Since Z is not isomorphic to ℓ_p, this means that for all L and K, there is some M such that $(z_n)_{n=L}^M$ is not KC-equivalent to ℓ_p^{M-L-1} and hence not K-equivalent with a normalised block sequence of $(y_n)_{n=M}^\infty$ either. So if $I_1 < I_2 < \cdots < I_{K-1}$ have been defined, to define I_K, we let $N = \max I_{K-1} + 1$ and find M such that $(z_n)_{n=2N-1}^M$ is not K-equivalent with a normalised block sequence of $(y_n)_{n=M}^\infty$. It follows that $(z_n)_{n=N}^M$ is not K-equivalent with a normalised block sequence of $(y_1, y_2, \ldots, y_{N-1}, y_{M+1}, y_{M+2}, \ldots)$. Letting $I_K = [N, M]$ we have the result. □

We should mention that G. Androulakis, N. Kalton and Tcaciuc [1] have extended Tcaciuc's dichotomy from [41] to a dichotomy characterising containment of ℓ_p and c_0. The result above implies theirs and moreover provides additional information.

6. Tightness by range and subsequential minimality

Theorem 1.1 shows that if one allows oneself to pass to a basis for a subspace, one can find a basis in which there is a close connection between subspaces spanned by block bases and subspaces spanned by subsequences. Thus, for example, if the basis is tight there can be no space embedding into all the subspaces spanned by subsequences of the basis. On the other hand, any block basis in Tsirelson's space T is equivalent to a subsequence of the basis, and actually every subspace of a block subspace $[x_n]$ in T contains an isomorphic copy of a subsequence of (x_n). In fact, this phenomenon has a deeper explanation and we shall now proceed to show that the connection between block sequences and subsequences can be made even closer.

Lemma 6.1. *If (e_n) is a basis for a space not containing c_0, then for all finite intervals (I_n) such that $\min I_n \xrightarrow[n\to\infty]{} \infty$ and all subspaces Y, there is a further subspace Z such that*

$$\|P_{I_k}|_Z\| \xrightarrow[k\to\infty]{} 0.$$

Proof. By a standard perturbation argument, we can suppose that Y is generated by a normalised block basis (y_n). Let K be the basis constant of (e_n). As $\min I_n \xrightarrow[n\to\infty]{} \infty$ and each I_n is finite, we can choose a subsequence (v_n) of (y_n) such that for all k the interval I_k intersects the range of at most one vector v_m from (v_n). Now, since c_0 does not embed into $[e_n]$, no tail sequence of (v_n) can satisfy an upper c_0 estimate. This implies that for all N and $\delta > 0$ there is a normalised vector

$$z = \sum_{i=N}^{N'} \eta_i v_i,$$

where $|\eta_i| < \delta$. Using this, we now construct a normalised block sequence (z_n) of (v_n) such that there are $m(0) < m(1) < \cdots$ and α_i with

$$z_n = \sum_{i=m(n)}^{m(n+1)-1} \alpha_i v_i$$

and $|\alpha_i| < \frac{1}{n}$ whenever $m(n) \leqslant i < m(n+1)$.

Now suppose $u = \sum_j \lambda_j z_j$ and k are given. Then there is at most one vector z_n whose range intersect the interval I_k. Also, there is at most one vector v_p from the support of z_n whose range intersect I_k. Therefore,

$$\begin{aligned}\big\|P_{I_k}(u)\big\| &= \big\|P_{I_k}(\lambda_n z_n)\big\| = \big\|P_{I_k}(\lambda_n \alpha_p v_p)\big\| \\ &\leqslant 2K\|\lambda_n \alpha_p v_p\| \leqslant |\lambda_n| \cdot \frac{2K}{n} \leqslant \frac{4K^2}{n}\|u\|.\end{aligned}$$

It follows that $\|P_{I_k}|_{[z_l]}\| \leqslant \frac{4K^2}{n_k}$, where n_k is such that I_k intersects the range of z_{n_k} (or $n_k = k$ if I_k intersects the range of no z_n). Since $\min I_k \xrightarrow[k\to\infty]{} \infty$ and (z_n) is a block basis, $n_k \to \infty$ when $k \to \infty$, and hence $\|P_{I_k}|_{[z_l]}\| \xrightarrow[k\to\infty]{} 0$. □

Our next result should be contrasted with the construction by Pełczyński [36] of a basis (f_i) such that every basis is equivalent with a subsequence of it, and hence such that every space contains an isomorphic copy of a subsequence. We shall see that for certain spaces E such constructions cannot be done relative to the subspaces of E provided that we demand that (f_n) lies in E too. Recall that two Banach spaces are said to be *incomparable* if neither of them embeds into the other.

Proposition 6.2. *Suppose that (e_n) is a basis such that any two block subspaces with disjoint ranges are incomparable. Suppose also that (f_n) is either a block basis or a shrinking basic sequence in $[e_n]$. Then $[e_n]$ is saturated with subspaces Z such that no subsequence of (f_n) embeds into Z.*

Proof. Let Y be an arbitrary subspace of $[e_n]$. Suppose first that (f_n) is a normalised shrinking basic sequence. Then, by taking a perturbation of (f_n), we can suppose that each f_n is a finite block vector of (e_i) and, moreover, that $\min \operatorname{range}(f_n) \to \infty$. Let $I_n = \operatorname{range}(f_n)$.

Fix an infinite set $N \subseteq \mathbb{N}$. Then for all infinite subsets $A \subseteq N$ there is an infinite subset $B \subseteq A$ such that $(f_n)_{n\in B}$ is a block sequence and hence, since block subspaces of (e_n) with disjoint ranges are incomparable, $[f_n]_{n\in B} \not\preceq [e_n]_{n\notin \bigcup_{i\in B} I_i}$, and so also $[f_n]_{n\in N} \not\preceq [e_n]_{n\notin \bigcup_{i\in A} I_i}$. Applying Lemma 3.2 to $X = [f_n]_{n\in N}$ and $(I_n)_{n\in N}$, this implies that for all embeddings $T : [f_n]_{n\in N} \to [e_n]_{n\in\mathbb{N}}$, we have $\liminf_{n\in N} \|P_{I_k} T\| > 0$. So find by Lemma 6.1 a subspace $Z \subseteq Y$ such that $\|P_{I_k}|_Z\| \xrightarrow[k\to\infty]{} 0$. Then no subsequence of $(f_n)_{\in\mathbb{N}}$ embeds into Z.

The argument in the case (f_n) is a block basis is similar. We set $I_n = \operatorname{range} f_n$ and repeat the argument above. □

We notice that in the above proof we actually have a measure for how "flat" a subspace Z of $[e_n]$ needs to be in order that the subsequences of (f_n) cannot embed into Z. Namely, it suffices that $\|P_{I_k}|_Z\| \xrightarrow[k\to\infty]{} 0$.

We should also mention that, by similar but simpler arguments, one can show that if (e_n) is a basis such that any two disjoint subsequences span incomparable spaces, then some subspace of $[e_n]$ fails to contain any isomorphic copy of a subsequence of (e_n).

The assumption in Proposition 6.2 that block subspaces with disjoint ranges are incomparable is easily seen to be equivalent to the following property of a basis (e_n), that we call *tight by range*. If (y_n) is a block sequence of (e_n) and $A \subseteq \mathbb{N}$ is infinite, then

$$[y_n]_{n\in\mathbb{N}} \not\preceq \left[e_n \;\middle|\; n \notin \bigcup_{i\in A} \operatorname{range} y_i \right].$$

Thus, (e_n) is tight by range if it is tight and for all block sequences (y_n) of (e_n) the corresponding sequence of intervals I_i is given by $I_i = \operatorname{range} y_i$. This property is also weaker than disjointly supported subspaces being incomparable, which we shall call *tight by support*. It is trivial to see that a basis, tight by range, is continuously tight.

We say that a basic sequence (e_n) is *subsequentially minimal* if any subspace of $[e_n]$ contains an isomorphic copy of a subsequence of (e_n). It is clearly a weak form of minimality.

In [26] the authors study another notion in the context of certain partly modified mixed Tsirelson spaces that they also call subsequential minimality. According to their definition, a basis (e_n) is subsequentially minimal if any block basis has a further block basis equivalent to a subsequence of (e_n). However, in all their examples the basis (e_n) is weakly null and it is easily

seen that whenever this is the case the two definitions agree. They also define (e_n) to be strongly non-subsequentially minimal if any block basis contains a further block basis that has no further block basis equivalent to a subsequence of (e_n). By Proposition 6.2, this is seen to be weaker than tightness by range.

We shall now proceed to show a dichotomy between tightness by range and subsequential minimality.

Theorem 6.3 *(4th dichotomy). Let E be a Banach space with a basis (e_n). Then there exists a block sequence (x_n) of (e_n) with one of the following properties, which are mutually exclusive and both possible*:

(1) *Any two block subspaces of $[x_n]$ with disjoint ranges are incomparable.*
(2) *The basic sequence (x_n) is subsequentially minimal.*

Arguably Theorem 6.3 is not a dichotomy in Gowers' sense, since property (2) is not hereditary: for example the universal basis of Pełczyński [36] satisfies (2) while admitting subsequences with property (1). However, it follows obviously from Theorem 6.3 that any basis (e_n) either has a block basis such that any two block subspaces with disjoint ranges are incomparable or has a block basis (x_n) that is *hereditarily subsequentially minimal*, i.e., such that any block basis has a further block basis that is subsequentially minimal. Furthermore, by an easy improvement of our proof or directly by Gowers' second dichotomy, if the first case of Theorem 6.3 fails, then one can also suppose that $[x_n]$ is quasi minimal.

We shall call a basis (x_n) *sequentially minimal* if it is both hereditarily subsequentially minimal and quasi minimal. This is equivalent to any block basis of (x_n) having a further block basis (y_n) such that every subspace of $[x_n]$ contains an equivalent copy of a subsequence of (y_n). We may therefore see Theorem 6.3 as providing a dichotomy between tightness by range and sequential minimality.

Before giving the proof of Theorem 6.3, we first need to state an easy consequence of the definition of Gowers' game.

Lemma 6.4. *Let E be a space with a basis and assume II has a winning strategy in Gowers' game in E to play in some set $\mathbb{B}$. Then there is a non-empty tree T of finite block sequences such that $[T] \subseteq \mathbb{B}$ and for all $(y_0, \dots, y_m) \in T$ and all block sequences (z_n) there is a block y_{m+1} of (z_n) such that $(y_0, \dots, y_m, y_{m+1}) \in T$.*

Proof. Suppose σ is the strategy for II. We define a pruned tree T of finite block bases $(y_0, \dots, y_m)$ and a function ψ associating to each $(y_0, \dots, y_m) \in T$ a sequence $(z_0, \dots, z_k)$ such that for some $k_0 < \cdots < k_m = k$,

$$\begin{array}{lccccccccccc} \mathbf{I} & z_0 & \cdots & z_{k_0} & & z_{k_0+1} & \cdots & z_{k_1} & \cdots & z_{k_{m-1}+1} & \cdots & z_{k_m} \\ \mathbf{II} & & & & y_0 & & & & y_1 \ \cdots & & & & y_m \end{array}$$

has been played according to σ.

- The empty sequence $\emptyset$ is in T and $\psi(\emptyset) = \emptyset$.
- If $(y_0, \dots, y_m) \in T$ and

$$\psi(y_0, \dots, y_m) = (z_0, \dots, z_k),$$

then we let $(y_0, \dots, y_m, y_{m+1}) \in T$ if there are some $z_k < z_{k+1} < \cdots < z_l$ and $k_0 < \cdots < k_m = k$ such that

$$\begin{array}{lccccccccc} \mathbf{I} & z_0 & \cdots & z_{k_0} & z_{k_0+1} & \cdots & z_{k_1} & \cdots & z_{k_m+1} & \cdots & z_l \\ \mathbf{II} & & & y_0 & & & y_1 & \cdots & & & y_{m+1} \end{array}$$

has been played according to σ and in this case we let

$$\psi(y_0, \dots, y_m, y_{m+1}) = (z_0, \dots, z_k, z_{k+1}, \dots, z_l)$$

be some such sequence.

Now, if $(y_0, y_1, y_2, \dots)$ is such that $(y_0, \dots, y_m) \in T$ for all m, then $\psi(\emptyset) \subseteq \psi(y_0) \subseteq \psi(y_0, y_1) \subseteq \cdots$ and (y_i) is the play of II according to the strategy σ in response to $(z_i) = \bigcup_n \psi(y_0, \dots, y_n)$ being played by I. So $[T] \subseteq \mathbb{B}$. It also follows by the construction that for each $(y_0, \dots, y_m) \in T$ and block sequence (z_i) there is a block y_{m+1} of (z_i) such that $(y_0, \dots, y_m, y_{m+1}) \in T$. $\square$

We now pass to the proof of Theorem 6.3.

Proof. If E contains c_0 the theorem is trivial. So suppose not. By the solution to the distortion problem and by passing to a subspace, we can suppose there are two positively separated inevitable closed subsets F_0 and F_1 of the unit sphere of E, i.e., such that $\operatorname{dist}(F_0, F_1) > 0$ and every block basis has block vectors belonging to both F_0 and F_1.

Suppose that (e_n) has no block sequence satisfying (1). Then for all block sequences (x_n) there are further block sequences (y_n) and (z_n) with disjoint ranges such that $[y_n] \sqsubseteq [z_n]$. We claim that there is a block sequence (f_n) and a constant K such that for all block sequences (x_n) of (f_n) there are further block sequences (y_n) and (y_n) with disjoint ranges such that $[y_n] \sqsubseteq_K [z_n]$. If not, we can construct a sequence of block sequences (f_n^K) such that (f_n^{K+1}) is a block of (f_n^K) and such that any two block sequences of (f_n^K) with disjoint ranges are K^2-incomparable. By Lemma 2.2, we then find a block sequence (g_n) of (e_n) that is $\sqrt{K}$-equivalent with a block sequence of (f_n^K) for every $K \geqslant 1$. Find now block subspaces (y_n) and (z_n) of (g_n) with disjoint ranges and a K such that $[y_n] \sqsubseteq_{\sqrt{K}} [z_n]$. Then (g_n) is $\sqrt{K}$-equivalent with a block sequence of (f_n^K) and hence we can find $K^{3/2}$-comparable block subspaces of (f_n^K) with disjoint ranges, contradicting our assumption.

So suppose (f_n) and K are chosen as in the claim. Then for all block sequences (x_n) of (f_n) we can find an infinite set $B \subseteq \mathbb{N}$ and a block sequence (y_n) of (x_n) such that $[y_n]_{n \in B}$ K-embeds into $[y_n]_{n \notin B}$.

We claim that any block basis of (f_n) has a further block basis in the following set of normalised block bases of (f_n):

$$\mathbb{A} = \left\{ (y_n) \;\middle|\; \forall n\ y_{2n} \in F_0 \cup F_1 \ \&\ \exists^\infty n\ y_{2n} \in F_0 \ \&\ [y_{2n+1}]_{y_{2n} \in F_0} \sqsubseteq_K [y_{2n+1}]_{y_{2n} \in F_1} \right\}.$$

To see this, suppose that (x_n) is a block sequence of (f_n) and let (z_n) be a block sequence of (x_n) such that $z_{3n} \in F_0$ and $z_{3n+1} \in F_1$. We can now find an infinite set $B \subseteq \mathbb{N}$ and a block sequence (v_n) of (z_{3n+2}) such that $[v_n]_{n \in B} \sqsubseteq_K [v_n]_{n \notin B}$. Let now $y_{2n+1} = v_n$ and notice that we

can choose $y_{2n} = z_i \in F_0$ for $n \in B$ and $y_{2n} = z_i \in F_1$ for $n \notin B$ such that $y_0 < y_1 < y_2 < \cdots$. Then $(y_n) \in \mathbb{A}$.

Choose now a sequence $\Delta = (\delta_n)$ of positive reals, $\delta_n < \operatorname{dist}(F_0, F_1)/3$, such that if (x_n) and (y_n) are block bases of (e_n) with $\|x_n - y_n\| < \delta_n$, then $(x_n) \sim_2 (y_n)$. Since $\mathbb{A}$ is clearly analytic, it follows by Gowers' determinacy theorem that for some block basis (x_n) of (f_n), II has a winning strategy to play in $\mathbb{A}_\Delta$ whenever I plays a block basis of (x_n). We now show that some block basis (v_n) of (x_n) is such that any subspace of $[v_n]$ contains a sequence $2K$-equivalent to a subsequence of (v_n), which will give us case (2).

Pick first by Lemma 6.4 a non-empty tree T of finite block sequences of (x_n) such that $[T] \subseteq \mathbb{A}_\Delta$ and for all $(u_0, \ldots, u_m) \in T$ and all block sequences (z_n) there is a block u_{m+1} of (z_n) such that $(u_0, \ldots, u_m, u_{m+1}) \in T$. Since T is countable, we can construct inductively a block sequence (v_n) of (x_n) such that for all $(u_0, \ldots, u_m) \in T$ there is some v_n with $(u_0, \ldots, u_m, v_n) \in T$.

We claim that (v_n) works. For if (z_n) is any block sequence of (v_n), we construct inductively a sequence $(u_n) \in \mathbb{A}_\Delta$ as follows. Using inductively the extension property of T, we can construct an infinite block sequence (h_n^0) of (z_n) that belongs to $[T]$. Since $[T] \subseteq \mathbb{A}_\Delta$, there is a shortest initial segment $(u_0, \ldots, u_{2k_0}) \in T$ of (h_n^0) such that $d(u_{2k_0}, F_0) < \delta_{2k_0}$. Pick now a term u_{2k_0+1} from (v_n) such that $(u_0, \ldots, u_{2k_0}, u_{2k_0+1}) \in T$.

Again, using the extension property of T, there is an infinite block sequence (h_n^1) of (z_n) such that

$$(u_0, \ldots, u_{2k_0}, u_{2k_0+1})^\frown \big(h_n^1\big)_n \in [T].$$

Also, as $[T] \subseteq \mathbb{A}_\Delta$, there is a shortest initial segment

$$(u_0, \ldots, u_{2k_0}, u_{2k_0+1}, \ldots, u_{2k_1}) \in T$$

of

$$(u_0, \ldots, u_{2k_0}, u_{2k_0+1})^\frown \big(h_n^1\big)_n$$

that properly extends $(u_0, \ldots, u_{2k_0}, u_{2k_0+1})$ and such that $d(u_{2k_1}, F_0) < \delta_{2k_1}$. We then pick a term u_{2k_1+1} of (v_n) such that $(u_0, \ldots, u_{2k_1}, u_{2k_1+1}) \in T$. We continue in the same fashion.

At infinity, we then have a block sequence $(u_n) \in \mathbb{A}_\Delta$ and integers $k_0 < k_1 < \cdots$ such that $d(u_{2n}, F_0) < \delta_{2n}$ if and only if $n = k_i$ for some i and such that for every i, u_{2k_i+1} is a term of (v_n). Let now $(w_n) \in \mathbb{A}$ be such that $\|w_n - u_n\| < \delta_n$. Then, as $\delta_n < \operatorname{dist}(F_0, F_1)/3$, we have that $w_{2n} \in F_0$ if and only if $n = k_i$ for some i and $w_{2n} \in F_1$ otherwise. Moreover, as $(w_n) \in \mathbb{A}$,

$$[w_{2k_i+1}]_{i\in\mathbb{N}} = [w_{2n+1}]_{w_{2n}\in F_0} \sqsubseteq_K [w_{2n+1}]_{w_{2n}\in F_1} = [w_{2n+1}]_{n\neq k_i}.$$

So by the choice of δ_n we have

$$[u_{2k_i+1}]_{i\in\mathbb{N}} \sqsubseteq_2 [w_{2k_i+1}]_{i\in\mathbb{N}} \sqsubseteq_K [w_{2n+1}]_{n\neq k_i} \sqsubseteq_2 [u_{2n+1}]_{n\neq k_i}.$$

Since $[u_{2n+1}]_{n\neq k_i}$ is a subspace of $[z_n]$ and (u_{2k_i+1}) a subsequence of (v_n) this finishes the proof. □

If, for some constant C, all subspaces of $[e_n]$ contain a C-isomorphic copy of a subsequence of (e_n), we say that (e_n) is *subsequentially C-minimal*. Our proof shows that condition (2) in

Theorem 6.3 may be improved to "For some constant C the basic sequence (x_n) is subsequentially C-minimal".

We notice that if (x_n) is hereditarily subsequentially minimal, then there is some C and a block sequence (v_n) of (x_n) such that (v_n) is hereditarily subsequentially C-minimal with the obvious definition. To see this, we first notice that by Proposition 6.2, (x_n) can have no block bases (y_n) such that further block subspaces with disjoint ranges are incomparable. So, by the proof of Theorem 6.3, for any block base (y_n) there is a constant C and a further block basis (z_n) which is subsequentially C-minimal. A simple diagonalisation using Lemma 2.2 now shows that by passing to a block (v_n) the C can be made uniform. Recall that Gowers also proved that a quasi minimal space must contain a further subspace which is C-quasi minimal [20].

We also indicate a variation on Theorem 6.3, relating the Casazza property to a slightly stronger form of sequential minimality. This answers the original problem of Gowers left open in [20], which was mentioned in Section 1. This variation is probably of less interest than Theorem 6.3 because the Casazza property does not seem to imply tightness and also because the stronger form of sequential minimality may look somewhat artificial (although it is satisfied by Tsirelson's space and is reminiscent of Schlumprecht's notion of Class 1 space [40]).

We say that two block sequences (x_n) and (y_n) *alternate* if either $x_0 < y_0 < x_1 < y_1 < \cdots$ or $y_0 < x_0 < y_1 < x_1 < \cdots$.

Proposition 6.5. *Let E be a Banach space with a basis (e_n). Then there exists a block sequence (x_n) with one of the following properties, which are exclusive and both possible*:

(1) *$[x_n]$ has the Casazza property, i.e., no alternating block sequences in $[x_n]$ are equivalent.*
(2) *There exists a family $\mathcal{B}$ of block sequences saturating $[x_n]$ and such that any two block sequences in $\mathcal{B}$ have subsequences which alternate and are equivalent.*

In particular, in case (2)*, E contains a block subspace $U = [u_n]$ such that for every block sequence of U, there is a further block sequence equivalent to, and alternating with, a subsequence of (u_n).*

Proof. If (e_n) does not have a block sequence satisfying (1), then any block sequence of (e_n) has a further block sequence in $\mathbb{A} = \{(y_n) \mid (y_{2n}) \sim (y_{2n+1})\}$. Let Δ be small enough so that $\mathbb{A}_\Delta = \mathbb{A}$. By Gowers' theorem, let (x_n) be some block sequence of (e_n) so that II has a winning strategy to play in $\mathbb{A}_\Delta$ whenever plays a block sequence of (x_n). Let T be the associated tree given by Lemma 6.4. By construction, for any block sequence (z_n) of (x_n), we may find a further block sequence (v_n) such that for any $(y_0, \ldots, y_m) \in T$, there exists some v_n with $(y_0, \ldots, y_m, v_n) \in T$. We set $f((z_n)) = (v_n)$ and $\mathcal{B} = \{f((z_n)) \mid (z_n) \leqslant (x_n)\}$. Given (v_n) and (w_n) in $\mathcal{B}$, it is then clear that we may find subsequences (v'_n) and (w'_n) so that $(v'_0, w'_0, v'_1, w'_1, \ldots) \in T$, and therefore (v'_n) and (w'_n) are equivalent. □

We may also observe that there is no apparent relation between tightness by range and tightness with constants. Indeed Tsirelson's space is tight with constants and sequentially minimal. Similarly, Example 4.7 is tight by support and therefore by range, but is locally minimal. Using the techniques of [3], one can construct a space combining the two forms of tightness.

Example 6.6. (See [14].) There exists a space with a basis which is tight with constants and tight by range.

Finally, if a space X is locally minimal and equipped with a basis which is tight by support and therefore unconditional (such as Example 4.7), then the reader will easily check the following. The canonical basis of $X \oplus X$ is tight (for a block subspace $Y = [y_n]$ of $X \oplus X$ use the sequence of intervals associated to the ranges of y_n with respect to the canonical 2-dimensional decomposition of $X \oplus X$), but neither tight by range nor with constants. However, a more interesting question remains open: does there exist a tight space which does not contain a basic sequence which is tight by range or with constants?

There is a natural strengthening of sequential minimality that has been considered in the literature, namely, the *blocking principle* (also known as the *shift property* in [7]) due to Casazza, Johnson, and Tzafriri [6]. It is known that for a normalized unconditional basis (e_n) the following properties are equivalent (see, e.g., [12]).

(1) Any block sequence (x_n) spans a complemented subspace of $[e_n]$.
(2) For any block sequence (x_n), $(x_{2n}) \sim (x_{2n+1})$.
(3) For any block sequence (x_n) and integers $k_n \in \operatorname{supp} x_n$, $(x_n) \sim (e_{k_n})$.

Moreover, any of the above properties necessarily hold uniformly. We say that (e_n) satisfies the blocking principle if the above properties hold for (e_n). The following proposition can be proved along the lines of the proof of the minimality of T^* in [6] (Theorem 14).

Proposition 6.7. *Let (e_n) be an unconditional basis satisfying the blocking principle and spanning a locally minimal space. Then (e_n) spans a minimal space.*

Thus, by the 5th dichotomy (Theorem 4.4), we have

Corollary 6.8. *Let (e_n) be an unconditional basis satisfying the blocking principle. Then there is a subsequence (f_n) of (e_n) such that either $[f_n]$ is minimal or (f_n) is tight with constants.*

7. Chains and strong antichains

The results in this section are in response to a question of Gowers from his fundamental study [20] and concern what types of quasi orders can be realised as the set of (infinite-dimensional) subspaces of a fixed Banach space under the relation of isomorphic embeddability.

Problem 7.1. (See Problem 7.9. in [20].) Given a Banach space X, let $\mathbb{P}(X)$ be the set of all equivalence classes of subspaces of X, partially ordered by isomorphic embeddability. For which posets P does there exist a Banach space X such that every subspace Y of X contains a further subspace Z with $\mathbb{P}(Z) = P$?

Gowers noticed himself that by a simple diagonalisation argument any such poset $\mathbb{P}(X)$ must either have a minimal element, corresponding to a minimal space, or be uncountable. We shall now use our notion of tightness to show how to attack this problem in a uniform way and improve on several previous results.

Suppose X is a separable Banach space and let $SB(X)$ denote the set of all closed linear subspaces of X. We equip $SB(X)$ with the so-called *Effros–Borel* structure, which is the σ-algebra generated by sets on the form

$$\{Y \in SB(X) \mid Y \cap U \neq \emptyset\},$$

where U is an open subset of X. In this way, $SB(X)$ becomes a standard Borel space, i.e., isomorphic as a measurable space to the real line equipped with its Borel algebra. We refer to the measurable subsets of $SB(X)$ as Borel sets. Let also $SB_\infty(X)$ be the subset of $SB(X)$ consisting of all infinite-dimensional subspaces of X. Then $SB_\infty(X)$ is a Borel subset of $SB(X)$ and hence a standard Borel space in its own right.

Definition 7.2. Suppose that X is a separable Banach space and E is an analytic equivalence relation on a Polish space $\mathcal{Z}$. We say that X has an *E-antichain*, if there is a Borel function $f : \mathcal{Z} \to SB(X)$ such that for $x, y \in \mathcal{Z}$

(1) if xEy, then $f(x)$ and $f(y)$ are biembeddable,
(2) if $x \not E y$, then $f(x)$ and $f(y)$ are incomparable.

We say that X has a *strong E-antichain* if there is a Borel function $f : \mathcal{Z} \to SB(X)$ such that for $x, y \in \mathcal{Z}$

(1) if xEy, then $f(x)$ and $f(y)$ are isomorphic,
(2) if $x \not E y$, then $f(x)$ and $f(y)$ are incomparable.

For example, if $=_{\mathbb{R}}$ is the equivalence relation of identity on $\mathbb{R}$, then $=_{\mathbb{R}}$-antichains and strong $=_{\mathbb{R}}$-antichains simply correspond to a perfect antichain in the usual sense, i.e., an uncountable Borel set of pairwise incomparable subspaces. Also, having a strong E-antichain implies, in particular, that E Borel reduces to the isomorphism relation between the subspaces of X.

The main result of [11] reformulated in this language states that if $E_{\mathbf{\Sigma}^1_1}$ denotes the complete analytic equivalence relation, then $C[0, 1]$ has a strong $E_{\mathbf{\Sigma}^1_1}$-antichain.

We will now prove a result that simultaneously improves on two results due respectively to the first and the second author. In [12], the authors proved that a Banach space not containing a minimal space must contain a perfect set of non-isomorphic subspaces. This result was improved by Rosendal in [37], in which it was shown that if a space does not contain a minimal subspace it must contain a perfect set of pairwise incomparable spaces. And Ferenczi proved in [10] that if X is a separable space without minimal subspaces, then E_0 Borel reduces to the isomorphism relation between the subspaces of X. Recall that E_0 is the equivalence relation defined on $2^{\mathbb{N}}$ by xE_0y if and only if $\exists m\ \forall n \geqslant m\ x_n = y_n$.

Theorem 7.3. *Let X be a separable Banach space. Then X either contains a minimal subspace or has a strong E_0-antichain.*

Proof. Suppose X has no minimal subspace. By Theorem 3.13 and Lemma 3.11, we can find a basic sequence (e_n) in X and a continuous function $f : [\mathbb{N}] \to [\mathbb{N}]$ such that for all $A, B \in [\mathbb{N}]$, if B is disjoint from an infinite number of intervals $[f(A)_{2i}, f(A)_{2i+1}]$, then $[e_n]_{n\in A}$ does not embed into $[e_n]_{n\in B}$. We claim that there is a continuous function $h : 2^{\mathbb{N}} \to [\mathbb{N}]$ such that

(1) if xE_0y, then $|h(x) \setminus h(y)| = |h(y) \setminus h(x)| < \infty$,
(2) if $x \not E_0 y$, then $[e_n]_{n\in h(x)}$ and $[e_n]_{n\in h(y)}$ are incomparable spaces.

This will clearly finish the proof using the fact that subspaces of the same finite codimension in a common superspace are isomorphic.

We will construct a partition of $\mathbb{N}$ into intervals

$$I_0^0 < I_0^1 < I_0^2 < I_1^0 < I_1^1 < I_1^2 < \cdots$$

such that if we set $J_n^0 = I_n^0 \cup I_n^2$ and $J_n^1 = I_n^1$, the following conditions hold:

(1) for all n, $|J_n^0| = |J_n^1|$,
(2) if $s \in 2^n$, $a = J_0^{s_0} \cup J_1^{s_1} \cup \cdots \cup J_{n-1}^{s_{n-1}} \cup I_n^0$, and $A \in [a, \mathbb{N}]$, then for some i,

$$\big[f(A)_{2i}, f(A)_{2i+1}\big] \subseteq I_n^0,$$

(3) if $s \in 2^n$, $a = J_0^{s_0} \cup J_1^{s_1} \cup \cdots \cup J_{n-1}^{s_{n-1}} \cup I_n^1$, and $A \in [a, \mathbb{N}]$, then for some i,

$$\big[f(A)_{2i}, f(A)_{2i+1}\big] \subseteq I_n^1.$$

Assuming this is done, for $x \in 2^{\mathbb{N}}$ we set $h(x) = J_0^{x_0} \cup J_1^{x_1} \cup \cdots$. Then for all n there is an i such that

$$\big[f\big(h(x)\big)_{2i}, f\big(h(x)\big)_{2i+1}\big] \subseteq J_n^{x_n}.$$

Therefore, if $x \not\mathrel{E_0} y$, then $h(y) = J_0^{y_0} \cup J_1^{y_1} \cup \cdots$ is disjoint from an infinite number of $J_n^{x_n}$ and thus also from an infinite number of intervals $[f(h(x))_{2i}, f(h(x))_{2i+1}]$, whence $[e_n]_{n\in h(x)}$ does not embed into $[e_n]_{n\in h(y)}$. Similarly, $[e_n]_{n\in h(y)}$ does not embed into $[e_n]_{n\in h(x)}$.

On the other hand, if $x E_0 y$, then clearly $|h(x) \setminus h(y)| = |h(y) \setminus h(x)| < \infty$.

It therefore only remains to construct the intervals I_n^i. So suppose by induction that $I_0^0 < I_0^1 < I_0^2 < \cdots < I_{n-1}^0 < I_{n-1}^1 < I_{n-1}^2$ have been chosen (the initial step being $n = 0$) such that the conditions are satisfied. Let $m = \max J_{n-1}^0 + 1 = \max I_{n-1}^2 + 1$. For each $s \in 2^n$ and $a = J_0^{s_0} \cup J_1^{s_1} \cup \cdots \cup J_{n-1}^{s_{n-1}}$, there are by continuity of f some $k_s > m$, some interval $m \leqslant M_s \leqslant k_s$ and an integer i_s such that for all $A \in [a \cup [m, k_s], \mathbb{N}]$, we have

$$\big[f(A)_{2i_s}, f(A)_{2i_s+1}\big] = M_s.$$

Let now $k = \max_{s\in 2^n} k_s$ and $I_n^0 = [m, k]$. Then if $s \in 2^n$ and $a = J_0^{s_0} \cup \cdots \cup J_{n-1}^{s_{n-1}}$, we have for all $A \in [a \cup I_n^0, \mathbb{N}]$ some i such that

$$\big[f(A)_{2i}, f(A)_{2i+1}\big] \subseteq I_n^0.$$

Again for each $s \in 2^n$ and $a = J_0^{s_0} \cup J_1^{s_1} \cup \cdots \cup J_{n-1}^{s_{n-1}}$ there are by continuity of f some $l_s > k+1$, some interval $k+1 \leqslant L_s \leqslant l_s$ and an integer j_s such that for all $A \in [a \cup [k+1, l_s], \mathbb{N}]$, we have

$$\big[f(A)_{2j_s}, f(A)_{2j_s+1}\big] = L_s.$$

Let now $l = \max_{s\in 2^n} l_s + k$ and $I_n^1 = [k+1, l]$. Then if $s \in 2^n$ and $a = J_0^{s_0} \cup \cdots \cup J_{n-1}^{s_{n-1}}$, we have for all $A \in [a \cup I_n^1, \mathbb{N}]$ some j such that

$$\big[f(A)_{2j}, f(A)_{2j+1}\big] \subseteq I_n^1.$$

Finally, we simply let $I_n^2 = [l+1, l + |I_n^1| - |I_n^0|]$. This finishes the construction. □

Definition 7.4. We define a quasi order $\subseteq^*$ and a partial order $\subseteq_0$ on the space $[\mathbb{N}]$ of infinite subsets of $\mathbb{N}$ by the following conditions:

$$A \subseteq^* B \quad \Leftrightarrow \quad A \setminus B \text{ is finite}$$

and

$$A \subseteq_0 B \quad \Leftrightarrow \quad \big(A = B \text{ or } \exists n \in B \setminus A\colon\ A \subseteq B \cup [0, n[\big).$$

Also, if (a_n) and (b_n) are infinite sequences of integers, we let

$$(a_n) \leqslant^* (b_n) \quad \Leftrightarrow \quad \forall^\infty n\ a_n \leqslant b_n.$$

We notice that $\subseteq_0$ is a partial order refining the quasi order $\subseteq^*$, namely, whenever $A \subseteq^* B$ we let $A \subseteq_0 B$ if $B \not\subseteq^* A$ or $A = B$ or $A \triangle B$ admits a greatest element which belongs to B.

Proposition 7.5.

(1) *Any closed partial order on a Polish space Borel embeds into* $\subseteq_0$.
(2) *Any partial order on a set of size at most* $\aleph_1$ *embeds into* $\subseteq_0$.
(3) *The quasi order* $\subseteq^*$ *embeds into* $\subseteq_0$*, but does not Borel embed.*
(4) *And finally* $\subseteq_0$ *Borel embeds into* $\subseteq^*$.

Proof. (1) By an unpublished result of A. Louveau [29], any closed partial order on a Polish space Borel embeds into $(\mathcal{P}(\mathbb{N}), \subseteq)$. And if we let (J_n) be a partition of $\mathbb{N}$ into countable many infinite subsets, we see that $(\mathcal{P}(\mathbb{N}), \subseteq)$ Borel embeds into $\subseteq^*$ and $\subseteq_0$ by the mapping $A \mapsto \bigcup_{n\in A} J_n$.

(2) & (3) It is well known that any partial order of size at most $\aleph_1$ embeds into $\subseteq^*$ and if we let $s : [\mathbb{N}] \to [\mathbb{N}]$ be any function such that $|A \triangle B| < \infty \Leftrightarrow s(A) = s(B)$ and $|A \triangle s(A)| < \infty$, i.e., s is a selector for E_0, then s embeds $\subseteq^*$ into $\subseteq_0$. To see that there cannot be a Borel embedding of $\subseteq^*$ into $\subseteq_0$, we notice that if $h : [\mathbb{N}] \to [\mathbb{N}]$ was a Borel function such that $A \subseteq^* B \Leftrightarrow h(A) \subseteq_0 h(B)$, then, in particular, $|A \triangle B|$ is finite $\Leftrightarrow h(A) = h(B)$, contradicting that E_0 is a non-smooth equivalence relation on $[\mathbb{N}]$.

(4) To see that $\subseteq_0$ Borel embeds into $\subseteq^*$, we define for an infinite subset A of $\mathbb{N}$ a sequence of integers $g(A) = (a_n)$ by

$$a_n = \sum_{i\in A\cap[0,n]} 2^i.$$

Suppose now that $g(A) = (a_n)$ and $g(B) = (b_n)$. Then for each n,

$$a_n = b_n \quad \Leftrightarrow \quad A \cap [0, n] = B \cap [0, n]$$

and

$$a_n < b_n \quad \Leftrightarrow \quad \exists m \in B \setminus A,\ m \leqslant n,\ A \cap [0, n] \subseteq B \cup [0, m[.$$

Thus, we have $a_n = b_n$ for infinitely many n if and only if $A = B$, and if $a_n < b_n$ for infinitely many n, then either $B \setminus A$ is infinite or for some $m \in B \setminus A$ we have $A \subseteq B \cup [0, m[$. Moreover, if $B \setminus A$ is infinite, then for infinitely many n, $a_n < b_n$. So

$$B \not\subseteq^* A \quad \Rightarrow \quad (b_n) \not\leqslant^* (a_n) \quad \Rightarrow \quad (B \not\subseteq^* A \text{ or } A \subseteq_0 B),$$

and thus by contraposition

$$(b_n) \leqslant^* (a_n) \quad \Rightarrow \quad B \subseteq^* A.$$

Also, if $(b_n) \not\leqslant^* (a_n)$, then $B \not\subseteq^* A$ or $A \subseteq_0 B$, so if moreover $B \subseteq_0 A$, we would have $A \subseteq_0 B$ and hence $A = B$, contradicting $g(B) = (b_n) \not\leqslant^* (a_n) = g(A)$. Thus,

$$B \subseteq_0 A \quad \Rightarrow \quad (b_n) \leqslant^* (a_n).$$

To see that also

$$(b_n) \leqslant^* (a_n) \quad \Rightarrow \quad B \subseteq_0 A,$$

notice that if $(b_n) \leqslant^* (a_n)$ but $B \not\subseteq_0 A$, then, as $B \subseteq^* A$, we must have $A \subseteq_0 B$ and hence $(a_n) \leqslant^* (b_n)$. But then $a_n = b_n$ for almost all n and thus $A = B$, contradicting $B \not\subseteq_0 A$. Therefore,

$$B \subseteq_0 A \quad \Leftrightarrow \quad (b_n) \leqslant^* (a_n),$$

and we thus have a Borel embedding of $\subseteq_0$ into the quasi order $\leqslant^*$ on the space $\mathbb{N}^{\mathbb{N}}$. It is well known and easy to see that this latter Borel embeds into $\subseteq^*$ and hence so does $\subseteq_0$. □

Proposition 7.6. *Any Banach space without a minimal subspace contains a subspace with an F.D.D.* (F_n) *satisfying one of the two following properties*:

(a) *if* $A, B \subseteq \mathbb{N}$ *are infinite, then*

$$\sum_{n \in A} F_n \sqsubseteq \sum_{n \in B} F_n \quad \Leftrightarrow \quad A \subseteq^* B,$$

(b) *if* $A, B \subseteq \mathbb{N}$ *are infinite, then*

$$\sum_{n \in A} F_n \sqsubseteq \sum_{n \in B} F_n \quad \Leftrightarrow \quad A \subseteq_0 B.$$

Proof. Suppose X is a Banach space without a minimal subspace. Then by Theorem 3.13, we can find a continuously tight basic sequence (e_n) in X. Using the infinite Ramsey theorem for analytic sets, we can also find an infinite set $D \subseteq \mathbb{N}$ such that

(i) either for all infinite $B \subseteq D$, $[e_i]_{i\in B}$ embeds into its hyperplanes,
(ii) or for all $B \subseteq D$, $[e_i]_{i\in B}$ is not isomorphic to a proper subspace.

And, by Lemma 3.11, we can after renumbering the sequence $(e_n)_{n\in D}$ as $(e_n)_{n\in\mathbb{N}}$ suppose that there is a continuous function $f : [\mathbb{N}] \to [\mathbb{N}]$ that for $A, B \in [\mathbb{N}]$, if B is disjoint from an infinite number of intervals $[f(A)_{2i}, f(A)_{2i+1}]$, then $[e_n]_{n\in A}$ does not embed into $[e_n]_{n\in B}$.

We now construct a partition of $\mathbb{N}$ into intervals

$$I_0 < I_1 < I_2 < \cdots$$

such that the following conditions hold:

– for all n, $|I_0 \cup \cdots \cup I_{n-1}| < |I_n|$,
– if $A \in [\mathbb{N}]$ and $I_n \subseteq A$, then for some i,

$$\big[f(A)_{2i}, f(A)_{2i+1}\big] \subseteq I_n.$$

Suppose by induction that $I_0 < I_1 < \cdots < I_{n-1}$ have been chosen such that the conditions are satisfied. Let $m = \max I_{n-1} + 1$. For each $a \subseteq [0, m[$ there are by continuity of f some $l_a > m$, some interval $m \leqslant M_a \leqslant l_a$ and an integer i_a such that for all $A \in [a \cup [m, l_a], \mathbb{N}]$, we have

$$\big[f(A)_{2i_a}, f(A)_{2i_a+1}\big] = M_a.$$

Let now $l > \max_{a\subseteq[0,m[} l_a$ be such that $|I_0 \cup \cdots \cup I_{n-1}| < l - m$, and set $I_n = [m, l[$. Then if $a \subseteq [0, m[$, we have for all $A \in [a \cup I_n, \mathbb{N}]$ some i such that

$$\big[f(A)_{2i}, f(A)_{2i+1}\big] \subseteq I_n,$$

which ends the construction.

Let now $F_n = [e_i]_{i\in I_n}$. Clearly, $\sum_{i=0}^{n-1} \dim F_i < \dim F_n$, and if $A \setminus B$ is infinite and we let $A^* = \bigcup_{n\in A} I_n$ and $B^* = \bigcup_{n\in B} I_n$, then B^* will be disjoint from an infinite number of the intervals defined by $f(A^*)$ and hence $\sum_{n\in A} F_n = [e_n]_{n\in A^*}$ does not embed into $\sum_{n\in B} F_n = [e_n]_{n\in B^*}$.

In case of (i) we have that for all infinite $C \subseteq \mathbb{N}$,

$$(e_n)_{n\in C} \sqsubseteq (e_n)_{n\in C'} \sqsubseteq (e_n)_{n\in C''} \sqsubseteq (e_n)_{n\in C'''} \sqsubseteq \cdots,$$

where D' denotes $D \setminus \min D$. So, in particular, for any infinite $A \subseteq \mathbb{N}$, $\sum_{n\in A} F_n$ embeds into all of its finite codimensional subspaces and thus if $A \setminus B$ is finite, then $\sum_{n\in A} F_n \sqsubseteq \sum_{n\in B} F_n$. This gives us (a).

In case (ii), if $A \subseteq_0 B$ but $B \not\subseteq_0 A$, we have, as $\dim F_n > \sum_{i=0}^{n-1} \dim F_i$, that $\sum_{n\in A} F_n$ embeds as a proper subspace of $\sum_{n\in B} F_n$. Conversely, if $\sum_{n\in A} F_n \sqsubseteq \sum_{n\in B} F_n$, then $A \setminus B$ is finite and so either $A \subseteq_0 B$ or $B \subseteq_0 A$. But if $B \subseteq_0 A$ and $A \not\subseteq_0 B$, then $\sum_{n\in B} F_n$ embeds as a proper subspace into $\sum_{n\in A} F_n$ and thus also into itself, contradicting (ii). Thus, $A \subseteq_0 B$. So assuming (ii) we have the equivalence in (b). □

We may observe that Tsirelson's space satisfies case (a) of Proposition 7.6, while case (b) is verified by Gowers–Maurey's space, or more generally by any space of type (1) to (4).

By Propositions 7.5 and 7.6 we now have the following result.

Theorem 7.7. *Let X be an infinite-dimensional separable Banach space without a minimal subspace and let $SB_\infty(X)$ be the standard Borel space of infinite-dimensional subspaces of X ordered by the relation $\sqsubseteq$ of isomorphic embeddability. Then $\subseteq_0$ Borel embeds into $SB_\infty(X)$ and by consequence*

(a) *any partial order of size at most $\aleph_1$ embeds into $SB_\infty(X)$,*
(b) *any closed partial order on a Polish space Borel embeds into $SB_\infty(X)$.*

We notice that this proves a strong dichotomy for the partial orders of Problem 7.1, namely, either they must be of size 1 or must contain any partial order of size at most $\aleph_1$ and any closed partial order on a Polish space. In particular, in the second case we have well-ordered chains of length ω_1 and also $\mathbb{R}$-chains. This completes the picture of [13].

8. Refining Gowers' dichotomies

We recall the list of inevitable classes of subspaces contained in a Banach space given by Gowers in [20]. Remember that a space is said to be quasi minimal if any two subspaces have a common $\sqsubseteq$-minorant, and strictly quasi minimal if it is quasi minimal but does not contain a minimal subspace. Also two spaces are incomparable in case neither of them embeds into the other, and totally incomparable if no space embeds into both of them.

Theorem 8.1. *(See Gowers [20].) Let X be an infinite-dimensional Banach space. Then X contains a subspace Y with one of the following properties, which are all possible and mutually exclusive.*

(i) *Y is hereditarily indecomposable,*
(ii) *Y has an unconditional basis such that any two disjointly supported block subspaces are incomparable,*
(iii) *Y has an unconditional basis and is strictly quasi minimal,*
(iv) *Y has an unconditional basis and is minimal.*

Here the condition of (ii) that any two disjointly supported block subspaces are incomparable, i.e., tightness by support, is equivalent to the condition that any two such subspaces are totally incomparable or just non-isomorphic.

Theorem 1.1 improves the list of Gowers in case (iii). Indeed, any strictly quasi minimal space contains a tight subspace, but the space $S(T^{(p)})$, $1 < p < +\infty$ is strictly quasi minimal and not tight: it is saturated with subspaces of $T^{(p)}$, which is strictly quasi minimal, and, as was already observed, it is not tight because its canonical basis is symmetric.

Concerning case (i), properties of HI spaces imply that any such space contains a tight subspace, but it remains open whether every HI space with a basis is tight.

Question 8.2. Is every HI space with a basis tight?

Using Theorems 1.1 and 1.4, we refine the list of inevitable spaces of Gowers to 6 main classes as follows.

Theorem 8.3. *Let X be an infinite-dimensional Banach space. Then X contains a subspace Y with one of the following properties, which are all mutually exclusive.*

(1) *Y is hereditarily indecomposable and has a basis such that any two block subspaces with disjoint ranges are incomparable,*
(2) *Y is hereditarily indecomposable and has a basis which is tight and sequentially minimal,*
(3) *Y has an unconditional basis such that any two disjointly supported block subspaces are incomparable,*
(4) *Y has an unconditional basis such that any two block subspaces with disjoint ranges are incomparable, and is quasi minimal,*
(5) *Y has an unconditional basis which is tight and sequentially minimal,*
(6) *Y has an unconditional basis and is minimal.*

We conjecture that the space of Gowers and Maurey is of type (1), although we have no proof of this fact. Instead, in [14] we prove that an asymptotically unconditional HI space constructed by Gowers [18] is of type (1).

We do not know whether type (2) spaces exist. If they do, they may be thought of as HI versions of type (5) spaces, i.e., of Tsirelson like spaces, so one might look for an example in the family initiated by the HI asymptotically ℓ_1 space of Argyros and Deliyanni, whose "ground" space is a mixed Tsirelson's space based on the sequence of Schreier families [2].

The first example of type (3) was built by Gowers [17] and further analysed in [22]. Other examples are constructed in [14].

Type (4) means that for any two block subspaces Y and Z with disjoint ranges, Y does not embed into Z, but some further block subspace Y' of Y does (Y' therefore has disjoint support but not disjoint range from Z). It is unknown whether there exist spaces of type (4). Gowers sketched the proof of a weaker result, namely the existence of a strictly quasi minimal space with an unconditional basis and with the Casazza property, i.e., such that for no block sequence the sequence of odd vectors is equivalent to the sequence of even vectors, but his example was never actually checked. Alternatively, results of [26, Section 4] suggest that a mixed Tsirelson space example might be looked for.

The main example of a space of type (5) is Tsirelson's space. Actually since spaces of type (1) to (4) are either HI or satisfy the Casazza property, they are never isomorphic to a proper subspace. Therefore, for example, spaces with a basis saturated with block subspaces isomorphic to their hyperplanes must contain a subspace of type (5) or (6). So our results may reinforce the idea that Tsirelson's space is the canonical example of classical space without a minimal subspace.

It is worth noting that as a consequence of the theorem of James, spaces of type (3), (4) and (5) are always reflexive.

Using some of the additional dichotomies, one can of course refine this picture even further. We shall briefly consider how this can be done using the 5th dichotomy plus a stabilisation theorem of A. Tcaciuc [41] generalising a result of [15].

We state a slightly stronger version of the theorem of Tcaciuc than what is proved in his paper and also point out that there is an unjustified use of a recent result of Junge, Kutzarova and Odell in his paper; their result only holds for $1 \leqslant p < \infty$. Tcaciuc's theorem states that any Banach

space contains either a strongly asymptotically ℓ_p subspace, $1 \leqslant p \leqslant +\infty$, or a subspace Y such that

$$\forall M\ \exists n\ \forall U_1, \dots, U_{2n} \subseteq Y\ \exists x_i \in \mathcal{S}_{U_i}\ (x_{2i-1})_{i=1}^n \nsim_M (x_{2i})_{i=1}^n,$$

where the U_i range over infinite-dimensional subspaces of Y. The second property in this dichotomy will be called *uniform inhomogeneity*. As strongly asymptotically ℓ_p bases are unconditional, while the HI property is equivalent to uniform inhomogeneity with $n = 2$ for all M, Tcaciuc's dichotomy is only relevant for spaces with an unconditional basis.

When combining Theorem 8.3, Tcaciuc's result, Proposition 4.2 (see also [9]), the 5th dichotomy, the fact that asymptotically ℓ_∞ spaces are locally minimal, and the classical theorem of James, we obtain 19 inevitable classes of spaces and examples for 8 of them. The class (2) is divided into two subclasses and the class (4) into four subclasses, which are not made explicit here for lack of an example of space of type (2) or (4) to begin with. Recall that the spaces contained in any of the 12 subclasses of type (1)–(4) are never isomorphic to their proper subspaces, and in this sense these subclasses may be labeled "exotic". On the contrary "classical", "pure" spaces must belong to one of the 7 subclasses of type (5)–(6).

Theorem 8.4. *Any infinite-dimensional Banach space contains a subspace with a basis of one of the following types*:

Type	*Properties*	*Examples*
(1a)	*HI, tight by range and with constants*	*?*
(1b)	*HI, tight by range, locally minimal*	G^*
(2)	*HI, tight, sequentially minimal*	*?*
(3a)	*tight by support and with constants, uniformly inhomogeneous*	*?*
(3b)	*tight by support, locally minimal, uniformly inhomogeneous*	G_u^*
(3c)	*tight by support, strongly asymptotically* ℓ_p, $1 \leqslant p < \infty$	X_u
(3d)	*tight by support, strongly asymptotically* ℓ_∞	X_u^*
(4)	*unconditional basis, quasi minimal, tight by range*	*?*
(5a)	*unconditional basis, tight with constants, sequentially minimal, uniformly inhomogeneous*	*?*
(5b)	*unconditional basis, tight, sequentially and locally minimal, uniformly inhomogeneous*	*?*
(5c)	*tight with constants, sequentially minimal, strongly asymptotically* ℓ_p, $1 \leqslant p < \infty$	T, $T^{(p)}$
(5d)	*tight, sequentially minimal, strongly asymptotically* ℓ_∞	*?*
(6a)	*unconditional basis, minimal, uniformly inhomogeneous*	S
(6b)	*minimal, reflexive, strongly asymptotically* ℓ_∞	T^*
(6c)	*isomorphic to* c_0 *or* l_p, $1 \leqslant p < \infty$	c_0, ℓ_p

We know of no space close to being of type (5a) or (5b). A candidate for (5d) could be the dual of some partly modified mixed Tsirelson's space not satisfying the blocking principle

(see [26]). Schlumprecht's space S [39] does not contain an asymptotically ℓ_p subspace, therefore it contains a uniformly inhomogeneous subspace, which implies by minimality that S itself is of type (6a). The definition and analysis of the spaces G^*, G_u^*, X_u and X_u^* can be found in [14].

For completeness we should mention that R. Wagner has also proved a dichotomy between asymptotic unconditionality and a strong form of the HI property [43]. His result could be used to further refine the cases of type (1) and (2).

9. Open problems

Problem 9.1.

(1) Does there exist a tight Banach space admitting a basis which is not tight?
(2) Does there exist a tight, locally block minimal and unconditional basis?
(3) Find a locally minimal and tight Banach space with finite cotype.
(4) Does there exist a tight Banach space which does not contain a basic sequence that is either tight by range or tight with constants? In other words, does there exist a locally and sequentially minimal space without a minimal subspace?
(5) Suppose $[e_n]$ is sequentially minimal. Does there exist a block basis all of whose subsequences are subsequentially minimal?
(6) Is every HI space with a basis tight?
(7) Is every tight basis continuously tight?
(8) Do there exist spaces of type (2), (4), (5a), (5b), (5d)?
(9) Suppose (e_n) is tight with constants. Does (e_n) have a block sequence that is (strongly) asymptotically ℓ_p for some $1 \leqslant p < \infty$?
(10) Does there exist a separable HI space X such that $\subseteq^*$ Borel embeds into $SB_\infty(X)$?
(11) If X is a separable Banach space without a minimal subspace, does $\subseteq^*$ Borel embed into $SB_\infty(X)$? What about more complicated quasi orders, in particular, the complete analytic quasi order $\leqslant_{\mathbf{\Sigma}^1_1}$?

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