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Ergodic Banach spaces

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Abstract

We show that any Banach space contains a continuum of non-isomorphic subspaces or a minimal subspace. We define an ergodic Banach space X as a space such that E_0 Borel reduces to isomorphism on the set of subspaces of X, and show that every Banach space is either ergodic or contains a subspace with an unconditional basis which is complementably universal for the family of its block-subspaces. We also use our methods to get uniformity results. We show that an unconditional basis of a Banach space, of which every block-subspace is complemented, must be asymptotically c_0 or ℓ_p, and we deduce some new characterisations of the classical spaces c_0 and ℓ_p.

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1. Introduction

The following question was asked the authors by G. Godefroy: how many non-isomorphic subspaces must a given Banach space contain? By the results of Gowers [9,10] and Komorowski and Tomczak-Jaegermann [18] solving the homogeneous space problem, if X is not isomorphic to ℓ_2 then it must contain at least two non-isomorphic subspaces. Except ℓ_2, no examples of spaces with only finitely, or even countably

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many, isomorphism classes of subspaces are known, so we may ask what the possible number of non-isomorphic subspaces of a given Banach space is, supposing it being non-isomorphic to ℓ_2. This question may also be asked in the setting of the classification of analytic equivalence relations up to Borel reducibility. If X is not isomorphic to ℓ_2, when can we classify the relation of isomorphism on subspaces of X?

Stated as above not much is known about our problem. Certainly, there is a number of particular results scattered throughout the literature implying that particular spaces have a great number of subspaces. For example, the spaces c_0 and ℓ_p, $p \neq 2$ have $\aleph_1$ non-isomorphic subspaces [19]. But there seems to have been no results on the problem in this generality. However, from Gowers's dichotomy theorem [10], one easily sees that a space without a minimal subspace must at least have uncountably many non-isomorphic subspaces. Moreover, assuming the consistency of large cardinals, Bagaria and Lopez-Abad [2] showed it to be consistent that any space without a minimal subspace must contain $2^{\aleph_0}$ many non-isomorphic subspaces. But firstly, this should be a fact of ZFC, and secondly, one would like to have a more constructive result saying that there is an uncountable Borel set of non-isomorphic subspaces.

A topological space X is said to be Polish if it is separable and its topology can be generated by a complete metric. Its Borel subsets are those belonging to the smallest σ-algebra containing the open sets. A subset is analytic if it is the continuous direct image of a Polish space or equivalently of a Borel set in a Polish space. All uncountable Polish spaces turn out to be Borel isomorphic, i.e., isomorphic by a function that is Borel bimeasurable.

A C-measurable set is one belonging to the smallest σ-algebra containing the open sets and closed under the Souslin operation, in particular all analytic sets are C-measurable. All C-measurable sets are universally measurable, i.e., measurable with respect to any σ-finite Borel measure on the space. Furthermore, they have the Baire property, i.e., can be written on the form $A = U \triangle M$, where U is open and M is meagre and are completely Ramsey. In fact they satisfy almost any regularity property satisfied by Borel sets (see [17, 29.D] for more on C-measurable sets) Moreover, as C-measurable functions are closed under composition, these form a useful extension of the class of Borel functions.

Most results contained in this article are centered around the notion of Borel reducibility. This notion turns out to be extremely useful as a mean of measuring complexity in analysis. It also gives another refined view of cardinality, in that it provides us with a notion of the number of classes of an equivalence relation before everything gets muddled up by the well-orderings provided by the axiom of choice.

Definition 1. Suppose that E and F are analytic equivalence relations on Polish spaces X and Y, respectively. Then we write $E \leqslant_B F$ iff there is a Borel function $f : X \longrightarrow Y$, such that $xEy \longleftrightarrow f(x)Ff(y)$. Moreover, we denote by $E \sim_B F$ the fact that the relations are Borel bireducible, i.e., $E \leqslant_B F$ and $F \leqslant_B E$.

Then $E \leqslant_B F$ means that there is an injection from X/E into Y/F admitting a Borel lifting. Intuitively, this says that the objects in X are simpler to classify with respect to E than the objects in Y with respect to F. Or again that Y objects modulo F provide complete invariants for X objects with respect to E-equivalence, and furthermore, these invariants can be calculated in a Borel manner from the initial objects.

We call an equivalence relation E on a Polish space X *smooth* if it Borel reduces to the identity relation on $\mathbb{R}$, or in fact to the identity relation on any uncountable Polish space. This is easily seen to be equivalent to admitting a countable separating family (A_n) of Borel sets, i.e., such that for any $x, y \in X$ we have $xEy \Longleftrightarrow \forall n\ (x \in A_n \leftarrow \rightarrow y \in A_n)$.

A Borel probability measure μ on X is called *E-ergodic* if for any μ-measurable $A \subset X$ that is E-invariant, i.e., $x \in A \ \wedge\ xEy \longrightarrow y \in A$, either $\mu(A) = 0$ or $\mu(A) = 1$. We call μ *E-non-atomic* if every equivalence class has measure 0.

Suppose μ was E-ergodic and (A_n) a separating family for E. Then by ergodicity and the fact that the A_n are invariant either A_n or A_n^c has measure 1, so $\bigcap \{A_n \,|\mu(A_n) = 1\} \cap \bigcap \{A_n^c \,|\mu(A_n) = 0\}$ is an E class of full measure and μ is atomic. So a smooth equivalence relation cannot carry an ergodic, non-atomic probability measure.

The minimal non-smooth Borel equivalence relation is the relation of eventual agreement of infinite binary sequences, E_0. This is defined on $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ by

$$xE_0y \longleftrightarrow \exists n \quad \forall m \geqslant n\ x_m = y_m$$

To see that E_0 is non-smooth just notice that the usual coin-flipping measure on $2^{\mathbb{N}}$ is E_0 non-atomic and ergodic by the zero-one law. Furthermore, any perfect set of almost disjoint infinite subsets of $\mathbb{N}$ shows that E_0 has a perfect set of classes.

If E is an equivalence relation on a set X and $A \subset X$, then we call A a *transversal for E on X* if it intersects every E-equivalence class in exactly one point. We notice that if E is an equivalence relation and A a transversal for E, both of them analytic, then E is smooth. An analytic equivalence relation is said to have a *perfect* set of classes if there is an uncountable Borel set consisting of pairwise inequivalent elements. This is a very rigid notion that does not depend on the cardinality of the continuum and is stronger that just demanding that it should have uncountable many classes. In fact there are analytic equivalence relations that have an uncountable set of classes, but in models violating the continuum hypothesis do not have $2^{\aleph_0}$ many classes.

Our general reference for descriptive set theory and Ramsey theory is [17] of which we adopt the notation wholesale. A friendly introduction to modern combinatorial set theory can be found in [14].

It is natural to try to distinguish some class of Banach spaces by a condition on the number of non-isomorphic subspaces. A step up from homogeneity would be when the subspaces would at least admit some classification in terms of real numbers, i.e., something resembling type or entropy. This would say that in some sense the space could not be too wild and one would expect such a space to have more regularity properties than those of a more generic space, in particular than those of a hereditarily indecomposable space.

A number of results in the 1970s and 1990s showed that there was essentially no hope for a general isomorphic classification of Banach spaces, nor even for finding nice subspaces of a certain type. The first of these result were Tsirelson's construction of a Banach space not containing any copies of c_0 or ℓ_p (see [4]) and the proof by Enflo that not every separable Banach space has a basis (see [19]). The second amount of evidence came with the construction of a space without any unconditional basic sequence by Gowers and Maurey [11]. There were, however two more encouraging results, namely the solution to the homogeneous space problem and Gowers's dichotomy [10] saying that either a Banach space contains a hereditarily indecomposable subspace or a subspace with an unconditional basis, that is, either a very rigid space (with few isomorphisms and projections) or a somewhat nice space (with many isomorphisms and projections).

We isolate another class of separable Banach spaces, namely those on which the isomorphism relation between subspaces does not reduce E_0, the non-ergodic ones, in particular this class includes those admitting classification by real numbers, and show that if a space belongs to this class, then it must satisfy some useful regularity properties.

Let $\mathcal{B}_X$ be the space of closed linear subspaces of a Banach space X, equipped with its Effros–Borel structure (see [17] or [6]). We note that isomorphism is analytic on $\mathcal{B}_X^2$. Let us define a Banach space X to be *ergodic* if the relation E_0 Borel reduces to isomorphism on subspaces of X. In [6,24], the authors studied spaces generated by subsequences of a space X with a basis: for X a Banach space with an unconditional basis, either X is ergodic or X is isomorphic to its hyperplanes, to its square, and more generally to any direct sum $X \oplus Y$ where Y is generated by a subsequence of the basis, and satisfies other regularity properties.

Note that it is easily checked that Gowers's construction of a space with a basis, such that no disjointly supported subspaces are isomorphic ([8,12]), provides an example of a space for which the complexity of isomorphism on subspaces generated by subsequences is exactly E_0.

In the main part of this article, we shall consider a Banach space with a basis, and restrict our attention to subspaces generated by block-bases. As long as we consider only block-subspaces, there are more examples of spaces with low complexity, for example ℓ_p, $1 \leqslant p < +\infty$ or c_0 has only one class of isomorphism for block-subspaces. After noting a few facts about the number of non-isomorphic subspaces of a Banach space, that come as consequences of Gowers's dichotomy theorem (Lemma 2 to Theorem 4), we prove that block-subspaces in a non-ergodic Banach space satisfy regularity properties (Theorems 9, 12, Corollary 10). We then show how our methods yield uniformity results (Propositions 16, 17). We find new characterisations of the classical spaces c_0 and ℓ_p (Corollary 20, Proposition 21). Finally, we show how to generalise our results to subspaces with a finite-dimensional decomposition on the basis (Theorem 22, Proposition 23).

2. A dichotomy for minimality of Banach spaces

Let us recall a definition of H. Rosenthal: we say that a space X is *minimal* if X embeds in any of its subspaces. Minimality is hereditary. In the context of block-subspaces, there are two natural definitions: we define a space X with a basis to be *block-minimal* if every block-subspace of X has a further block-subspace isomorphic to X; it is *equivalence block-minimal* if every block-subspace of X has a further block-subspace equivalent to X. The second property is hereditary, but the first one is not, so we also define a *hereditarily block-minimal* space as a space X with a basis such that any of its block-subspaces is block-minimal.

Let X be a Banach space with a basis $\{e_i\}$. If $y=(y_n)_{n\in\mathbb{N}}$ is a block-sequence of X, we denote by $Y=[y_n]_{n\in\mathbb{N}}=[y]$ the closed linear span of y. For two finite or infinite block-bases z and y of $\{e_i\}$, write $z\leqslant y$ if z is a blocking of y (and write $Z\leqslant Y$ for the corresponding subspaces). If $y=(y_i)_{i\in\mathbb{N}}$, $z=(z_i)_{i\in\mathbb{N}}$ and $N\in\mathbb{N}$, write $z\leqslant^* y$ iff there is an N such that $(z_i)_{i\geqslant N}\leqslant y$ (and write $Z\leqslant^* Y$ for the corresponding subspaces). If $s=(s_1,\dots,s_n)$ and $t=(t_1,\dots,t_k)$ are two finite block-bases, i.e., $supp(s_i)<supp(s_{i+1})$ and $supp(t_i)<supp(t_{i+1})$, then we write $s\preccurlyeq t$ iff s is an initial segment of t, i.e., $n\leqslant k$ and $s_i=t_i$ for $i\leqslant n$. In that case we write $t\backslash s$ for $(t_{n+1},\dots,t_k)$. If s is a finite block-basis and y is a finite or infinite block-basis supported after s, denote by $s^\frown y$ the concatenation of s and y.

We denote by $bb(X)$ the set of normalised block-bases on X. This set can be equipped with the product topology of the norm topology on X, in which case it becomes a Polish space that we denote by $bb_N(X)$.

Sometimes we want to work with blocks with rational coordinates, though we no longer can demand these to be normalised (by rational, we shall always mean an element of $\mathbb{Q}+i\mathbb{Q}$ in the case of a complex Banach space). We identify the set of such blocks with the set $\mathbb{Q}_*^{<\mathbb{N}}$ of finite, not identically zero, sequences of rational numbers. We shall denote by $(\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$ the set of (not necessarily successive) infinite sequences of rational blocks. Again when needed we will give $\mathbb{Q}_*^{<\mathbb{N}}$ the discrete topology and $(\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$ the product topology. The set of rational block-bases may be seen as a subset of $(\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$ and is denoted by $bb_{\mathbb{Q}}$. The set of finite rational block-bases is then denoted by $fbb_{\mathbb{Q}}$.

Finally for the topology that interests us the most: let $\mathbf{Q}$ be the set of normalised blocks of the basis that are a multiple of some block with rational coordinates; we denote by $bb_d(X)$ the set of block-bases of vectors in $\mathbf{Q}$, equipped with the product topology of the discrete topology on $\mathbf{Q}$. As $\mathbf{Q}$ is countable, this topology is Polish and epsilon matters may be forgotten until the applications; when we deal with isomorphism classes, they are not relevant since a small enough perturbation preserves the class. Note also that the canonical embedding of $bb_d(X)$ into $\mathcal{B}_X$ is Borel, and this allows us to forget about the Effros–Borel structure when checking ergodicity. Unless specified otherwise, from now on we work with this topology.

We first prove a Lemma about uniformity for these properties. For $C\geqslant 1$, we say that a space X with a basis is *C block-minimal* (*resp.*, *C equivalence block-minimal*) if any block-subspace of X has a further block-subspace which is C-isomorphic (resp., C-equivalent) to X.

Lemma 2. (i) *Let X be a Banach space with a basis and assume X is* (*equivalence*) *block-minimal. Then there exists $C\geqslant 1$ such that X is C* (*equivalence*) *block-minimal.*

(ii) *Suppose* $\{e_n\}$ *is a basis in a Banach space, such that any subsequence of* $\{e_n\}$ *has a block-sequence equivalent to* $\{e_n\}$. *Then there is a subsequence* $\{f_n\}$ *of* $\{e_n\}$ *and a constant* $C \geqslant 1$, *such that any subsequence of* $\{f_n\}$ *has a block-sequence C-equivalent to* $\{e_n\}$.

Proof. We will only prove (i) as the proof of (ii) is similar. Let for $n \in \mathbb{N}$ $c(n)$ denote a constant such that any n-codimensional subspaces of any Banach space are $c(n)$-isomorphic [6, Lemma 3]. Let X be block-minimal. We want to construct by induction a decreasing sequence of block-subspaces $X_n, n \geqslant 1$ and successive block-vectors x_n such that the first vectors of X_n are $x_1, \ldots, x_{n-1}$ and such that no block-subspace of X_n is n isomorphic to X. Assume we may carry out the induction: then for all $n \in \mathbb{N}$, no block-subspace of $[x_n]_{n \in \mathbb{N}}$ is n-isomorphic to X, and this contradicts the block-minimality of X. So the induction must stop at some n, meaning that every block-subspace of X_n whose first vectors are $x_1, \ldots, x_n$ has a further block-subspace n isomorphic to X. Then by definition of $c(n)$, every block-subspace of X_n has a further block-subspace $nc(n)$ isomorphic to X. By block-minimality we may assume that X_n is K-isomorphic to X for some K. Take now any block-subspace Y of X, it is K-isomorphic to a subspace of X_n; by standard perturbation arguments, we may find a block-subspace of Y which is $2K$-equivalent to a block-subspace of X_n, and by the above, an even further block-subspace $2K$ equivalent to a $nc(n)$-isomorphic copy of X; so finally Y has a block-subspace $2Knc(n)$ isomorphic to X and so, X is $2Knc(n)$ block-minimal.

We may use the same proof for equivalence block-minimality, using instead of $c(n)$ a constant $d(n) = (1 + (n + 1)c)^2$, such that any two normalised block-sequences differing by only the n first vectors are $d(n)$-equivalent (c stands for the constant of the basis). □

Let us recall a version of the Gowers's game $G_{A,Y}$ shown to be equivalent to Gowers's original game by Bagaria and Lopez-Abad [2]: Player I plays in the kth move a normalised block-vector y_k of Y such that $y_{k-1} < y_k$ and Player II responds by either doing nothing or playing a normalised block-vector $x \in [y_{i+1}, \ldots, y_k]$ if i was the last move where she played a vector. Player II wins the game if in the end she has produced an infinite sequence $(x_k)_{k \in \mathbb{N}}$ which is a block-sequence in A. If Player II has a winning strategy for $G_{A,Y}$ we say that she has a winning strategy for Gowers's game in Y for producing block-sequences in A. Gowers proved that if A is analytic in $bb_N(X)$, such that any normalised block-sequence contains a further normalised block-sequence in A, then II has a winning strategy in some Y to produce a block-sequence arbitrarily close to a block-sequence in A.

As an application of Gowers's theorem one can mention that if X is (equivalence) block-minimal, then there is a constant C, such that for every block-subspace $Y \leqslant X$, Player II has a winning strategy for Gowers's game in some $Z \leqslant Y$ for producing block-sequences spanning a space C-isomorphic to X (respectively, C-equivalent to the basis of X).

We recall that a space with a basis is said to be *quasi-minimal* if any two block-subspaces have further isomorphic block-subspaces. On the contrary, two spaces are said to be *totally incomparable* if no subspace of the first one is isomorphic to a subspace of the second. Using his dichotomy theorem, Gowers [10] proved the following result about Banach spaces.

Theorem 3 (*Gowers's "trichotomy"*). *Let X be a Banach space. Then X either contains*

- *a hereditarily indecomposable subspace,*
- *a subspace with an unconditional basis such that no disjointly supported block-subspaces are isomorphic,*
- *a subspace with an unconditional basis which is quasi-minimal.*

Using his game we prove:

Theorem 4. *Let X be a separable Banach space. Then*

(i) *X is ergodic or contains a quasi-minimal subspace with an unconditional basis.*
(ii) *X contains a perfect set of mutually totally incomparable subspaces or a quasi-minimal subspace.*
(iii) *X contains a perfect set of non-isomorphic subspaces or a block-minimal subspace with an unconditional basis.*

Proof. First notice that because of the hereditary nature of the properties, each of the subspaces above may be chosen to be spanned by block-bases of a given basis. Rosendal proved that any hereditarily indecomposable Banach space X is ergodic, and this can be proved using subspaces generated by subsequences of a basic sequence in X [24]. Following Bossard (who studied the particular case of a space defined by Gowers [1]), we may prove that a space X such that no disjointly supported block-subspaces are isomorphic is ergodic (map $\alpha \in 2^{\mathbb{N}}$ to $[e_{2n+\alpha(n)}]_{n\in\mathbb{N}}$, where (e_n) is the unconditional basis of X). This takes care of (i).

A space such that no disjointly supported block-subspaces are isomorphic contains $2^{\mathbb{N}}$ totally incomparable block-subspaces (take subspaces generated by subsequences of the basis corresponding to a perfect set of almost disjoint infinite subsets of $\mathbb{N}$). Also any hereditarily indecomposable space is quasi-minimal, so (ii) follows.

Finally, for the proof of (iii) we will first show that the statement we want to prove is $\boldsymbol{\Sigma}^1_2$. This will be done by showing that given a block-minimal space X, there is a further block-subspace Y such that for $Z \leqslant Y$ we can find continuously in Z an $X' \leqslant Z$ and an isomorphism of X' with X. The proof uses ideas of coding with asymptotic sets which are at the basis of many recent constructions such as the space of Gowers and Maurey [11], and more specifically some ideas of Lopez-Abad [20].

By Shoenfield's absoluteness theorem (see [13, Theorem 25.20] or [16, Theorem 13.15]) it will then be sufficient to show the statement under Martin's axiom and the negation of the continuum hypothesis. This was almost done by Bagaria and Lopez-Abad who showed it to be consistent relative to the existence of a weakly compact cardinal, see [2], but we will see that it can be done in a simple manner directly from $MA + \neg CH$.

Note first that having a perfect set of non-isomorphic subspaces or containing a copy of c_0 are both $\mathbf{\Sigma}^1_2$, and that if X contains a copy of c_0 then it has a block-sequence equivalent to the unit vector basis of c_0, in which case the theorem holds. If on the contrary it does not contain c_0 then by passing to a subspace, by the solution to the distortion problem by Odell and Schlumprecht, we may assume that it contains two closed, positively separated, asymptotic subsets of the unit sphere A_0 and A_1 [22]. Suppose that $Y = [y] \leqslant X$ is block-minimal. Fix a bijection π between $\mathbb{N}$ and $\mathbb{Q}_*^{<\mathbb{N}}$, the set of finite sequences of rational numbers not identically zero. Then for any $\alpha = 0^{n_0}10^{n_1}10^{n_2}1\dots$ in $2^{\mathbb{N}}$ there is associated a unique sequence $(\pi(n_0), \pi(n_1), \pi(n_2), \dots)$ in $(\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$. Furthermore, any element of $(\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$ gives a unique sequence of block-vectors of Y simply by taking the corresponding finite linear combinations.

Let $D := \{(z_n) \in bb_N(X) \,|\, (z_n) \leqslant y \;\wedge\; \forall n\; z_n \in A_0 \cup A_1 \;\wedge\; \exists^\infty n\; z_n \in A_1\}$ which is Borel in $bb_N(X)$. Then if $(z_n) \in D$ it codes a unique infinite sequence of block-vectors (not necessarily consecutive) of Y, by first letting $(z_n) \mapsto \alpha \in 2^{\mathbb{N}}$ where $\alpha(n) = 1 \longleftrightarrow z_n \in A_1$ and then composing with the other coding. Notice that this coding is continuous from D to $(\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$, when $\mathbb{Q}_*^{<\mathbb{N}}$ is taken discrete.

Let E be the set of $(z_n) \leqslant y$ such that $(z_{2n+1}) \in D$ and the function sending (z_{2n}) to the sequence of block-vectors of Y coded by (z_{2n+1}) is an isomorphism of $[z_{2n}]_{n\in\mathbb{N}}$ with Y.

E is clearly Borel in $bb_N(X)$ and we claim that any block-sequence contains a further block-sequence in E. For suppose that $z \leqslant y$ is given. Then we first construct a further block-sequence (z_n) such that $z_{3n+1} \in A_0$ and $z_{3n+2} \in A_1$. By block-minimality of Y there are a block-sequence (x_n) of (z_{3n}) isomorphic to Y and a sequence $\alpha \in 2^{\mathbb{N}}$ coding a sequence of block-vectors (y_n) of Y such that $x_n \mapsto y_n$ is an isomorphism of $[x_n]$ with Y (a standard perturbation argument shows that we can always take our y_n to be a finite rational combination on Y).

Now in between x_n and x_{n+1} there are z_{3m+1} and z_{3m+2}, so we can code α by a corresponding subsequence (z'_n) of these such that $x_n < z'_n < x_{n+1}$. The combined sequence is then in E. So by Gowers's theorem there is for any $\Delta > 0$ a winning strategy τ for II for producing blocks in E_Δ in some $Y' \leqslant Y$. By choosing Δ small enough and modifying τ a bit we can suppose that the vectors of odd index played by II are in $A_0 \cup A_1$. So if Δ is chosen small enough, a perturbation argument shows that τ is in fact a strategy for playing blocks in E. By changing the strategy again we can suppose that II responds to block-bases in $bb_d(Y')$ by block-bases in $bb_d(Y')$. So finally we see that X has a block-minimal subspace iff there are $Y' = [y']$ and $Y = [y]$ with $Y' \leqslant Y \leqslant X$ and a continuous function $(f_1, f_2) = f : bb_d(Y') \longrightarrow bb_d(Y') \times (\mathbb{Q}_*^{<\mathbb{N}})^{\mathbb{N}}$ such that for all $z \leqslant y'$, $f_2(z)$ codes a sequence (w_n) of blocks of Y such that $[w_n]_{n\in\mathbb{N}} = Y$, and $f_1(z) = (v_n) \leqslant z$ with $v_n \mapsto w_n$ being an isomorphism between $[v_n]_{n\in\mathbb{N}}$ and Y.

The statement is therefore $\mathbf{\Sigma}^1_2$, and to finish the proof we now need the following lemma:

Lemma 5 ($MA_{\sigma-centered}$). *Let $A \subset bb_{\mathbb{Q}}$ be linearly ordered under $\leqslant^*$ of cardinality strictly less than the continuum. Then there is an $y_\infty \in bb_{\mathbb{Q}}$ such that $x_0 \leqslant^* y$ for all $y \in A$.*

Proof. For $s \in fbb_{\mathbb{Q}}$ and $y \in bb_{\mathbb{Q}}$, denote by (s, y) the set of block-bases in $bb_{\mathbb{Q}}$ of the form $s^\frown z$ for $z \leqslant y$. Let $\mathcal{P} = \left\{(s, y)\,\middle|\, s \in fbb_{\mathbb{Q}} \wedge y \in A\right\}$, ordered by the inclusion. As a preliminary remark, note that if $(t, z) \subset (s, y)$, then $s \preccurlyeq t$, $t \setminus s \leqslant y$, and $z \leqslant^* y$. Conversely, if $s \in fbb_{\mathbb{Q}}$ and $z \leqslant^* y$, then extensions t of s, with $t \setminus s \leqslant z$ far enough, are such that $(t, z) \subset (s, y)$.

Put $D_n = \{(s, y) \in \mathcal{P} \,|\, |s| \geqslant n\}$ and $D_y = \{(t, z) \in \mathcal{P} \,|\, z \leqslant^* y\}$. Then D_n and D_y, for $y \in A$, are dense in $\mathcal{P}$, i.e., any element in $\mathcal{P}$ has a minorant in D_n (resp., D_y). To see that D_n is dense, just take for any given $(s, y) \in \mathcal{P}$ some extension s' of s such that $s' \setminus s \leqslant y$ and $|s'| \geqslant n$, then $(s', y) \in D_n$ and $(s', y) \subset (s, y)$. On the other hand, to see that D_y is dense for $y \in A$, suppose $(s, z) \in \mathcal{P}$ is given. Then as A is linearly ordered by $\leqslant^*$, let w be the minimum of z and y. By the preliminary remark, $(s', w) \subset (s, z)$ for a long enough extension s' of s such that $s' \setminus s \leqslant w$, and as $w \leqslant^* y$, (s', w) is in D_y.

Let $\mathcal{P}_s = \{(s, y) \,|\, y \in A\}$, which is centered in $\mathcal{P}$, i.e., every finite subset of $\mathcal{P}_s$ has a common minorant in $\mathcal{P}$. This follows from the same argument as above, using the preliminary remark. So since s is supposed to be rational, we see that $\mathcal{P}$ is σ-centered, i.e., a countable union of centered subsets. Notice that as $|A| < 2^{\aleph_0}$, there are less than continuum many dense sets D_n and D_y. So by $MA_{\sigma-centered}$ there is a filter G on $\mathcal{P}$ intersecting each of these sets.

Suppose that (s, y) and $(t, z) \in G$ then as G is a filter, they have a common minorant $(v, w) \in G$, but then $s \preccurlyeq v$ and $t \preccurlyeq v$, so either $s \preccurlyeq t$ or $t \preccurlyeq s$. Therefore $y_\infty := \bigcup \left\{s \in fbb_{\mathbb{Q}} \,|\, \exists y\ (s, y) \in G\right\}$ is a block-basis. Furthermore as G intersects all of D_n for $n \in \mathbb{N}$ we see that y_∞ is an infinite block-basis.

We now prove that $y_\infty \leqslant^* y$ for all $y \in A$. Since G intersects D_y, without loss of generality we may assume that $(s, y) \in G$ for some s. Then $y_\infty \setminus s \leqslant y$. For if $t \preccurlyeq y_\infty$ and $(t, z) \in G$, take $(v, w) \in G$ such that $(v, w) \subset (t, z)$ and $(v, z) \subset (s, y)$, then $s, t \preccurlyeq v \preccurlyeq y_\infty$ and $t \setminus s \preccurlyeq v \setminus s \leqslant y$, and therefore as t was arbitrary $y_\infty \setminus s \leqslant y$.

Suppose now that X does not have a perfect set of non-isomorphic subspaces. Then by Burgess's theorem (see [17, (35.21)]), it has at most $\aleph_1$ many isomorphism classes of subspaces, and in particular as we are supposing the continuum hypothesis not to hold, less than continuum many. Let $(X_\xi)_{\xi<\omega_1}$ be an enumeration of an element from each class. Then if none of these are minimal, we can construct inductively a $\leqslant^*$ decreasing sequence $(Y_\xi)_{\xi<\omega_1}$ of rational block-subspace such that X_ξ does not embed into Y_ξ and using the above Lemma find some Y_{ω_1} diagonalising the whole sequence. By taking, e.g., the subsequence consisting of every second term of the basis of Y_{ω_1} one can suppose that Y_{ω_1} embeds into every term of the sequence $(Y_\xi)_{\xi<\omega_1}$ and that therefore in particular Y_{ω_1} is isomorphic to no X_ξ, $\xi < \omega_1$, which is impossible. This finishes the proof of the theorem. $\square$

We remark that if X does not contain a minimal subspace, there is in fact a perfect set of subspaces such each two of them do not both embed into each other. This is slightly stronger than saying that they are non-isomorphic.

3. Residual isomorphism classes of block-subspaces

We recall our result from [6,24] in a slightly modified form.

Theorem 6. *Let X be a Banach space with a basis $\{e_i\}$. Then E_0 Borel reduces to isomorphism on subspaces spanned by subsequences of the basis, or there exists a sequence $(F_n)_{n\geqslant 1}$ of successive finite subsets of $\mathbb{N}$ such that for any infinite subset N of $\mathbb{N}$, if $N \cap [\min(F_n), \max(F_n)] = F_n$ for infinitely many n's, then the space $[e_i]_{i\in N}$ is isomorphic to X. It follows that if X is non-ergodic with an unconditional basis, then it is isomorphic to its hyperplanes, to its square, and more generally to $X \oplus Y$ for any subspace Y spanned by a subsequence of the basis.*

Indeed, by [6], improved in [24], either E_0 Borel reduces to isomorphism on subspaces spanned by subsequences of the basis, or the set of infinite subsets of $\mathbb{N}$ spanning a space isomorphic to X is residual in $2^{\mathbb{N}}$; the characterisation in terms of finite subsets of $\mathbb{N}$ is then a classical characterisation of residual subsets of $2^{\mathbb{N}}$ (see [17], or the remark at the end of Lemma 7 in [6]). Both proofs are similar to (and simpler than) the following proofs of Proposition 7 and Proposition 8 for block-subspaces. The last part of the theorem is specific to the case of subspaces spanned by subsequences and is also proved in [6].

We now wish to extend this result to the set of block-bases, for which it is useful to use the Polish space $bb_d(X)$. Unless stated otherwise this is the topology referred to.

As before, the notation $x = (x_n)_{n\in\mathbb{N}}$ will be used to denote an infinite block-sequence; $\tilde{x}$ will denote a finite block-sequence, and $|\tilde{x}|$ its length as a sequence, $supp(\tilde{x})$ the union of the supports of the terms of $\tilde{x}$. For two finite block-sequences $\tilde{x}$ and $\tilde{y}$, write $\tilde{x} < \tilde{y}$ to mean that they are successive. For a sequence of successive finite block-sequences $(\tilde{x}_i)_{i\in I}$, we denote the concatenation of the block-sequences by $\tilde{x}_1^\frown \dots ^\frown \tilde{x}_n$ if the sequence is finite or $\tilde{x}_1^\frown \tilde{x}_2^\frown \dots$ if it is infinite, and we denote by $supp(\tilde{x}_i, i \in I)$ the support of the concatenation, by $[\tilde{x}_i]_{i\in I}$ the closed linear span of the concatenation. For a finite block-sequence $\tilde{x} = (x_1, \dots, x_n)$, we denote by $N(\tilde{x})$ the set of elements of $bb_d(X)$ whose first n vectors are $(x_1, \dots, x_n)$.

Proposition 7. *Let X be a Banach space with a Schauder basis. Then either X is ergodic, or there exists $K \geqslant 1$ such that a residual set of block-sequences in $bb_d(X)$ span spaces mutually K-isomorphic.*

Proof. The relation of isomorphism is either meagre or non-meagre in $bb_d(X)^2$. First assume that it is meagre. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing sequence of dense open subsets of $bb_d(X)^2$ so that $\cap_{n\in\mathbb{N}} U_n$ does not intersect $\simeq$. We build by induction successive finite blocks $\{\tilde{a}_n^0, n \in \mathbb{N}\}$ and $\{\tilde{a}_n^1, n \in \mathbb{N}\}$ such that for all n, $|\tilde{a}_n^0| = |\tilde{a}_n^1|$, and $supp(\tilde{a}_n^i) < supp(\tilde{a}_{n+1}^j)$ for all $(i, j) \in 2^2$. For $\alpha \in 2^{\mathbb{N}}$, we let $x(\alpha)$ be the concatenated infinite block-sequence $\tilde{a}_0^{\alpha(0)} {}^\frown \tilde{a}_1^{\alpha(1)} {}^\frown \dots$. And for $n \in \mathbb{N}$ and $\beta \in 2^n$, we let $\tilde{x}(\beta)$ be the concatenated finite block-sequence $\tilde{a}_0^{\beta(0)} {}^\frown \dots {}^\frown \tilde{a}_{n-1}^{\beta(n-1)}$. We require furthermore of the sequences $\{\tilde{a}_n^0\}$ and $\{\tilde{a}_n^1\}$ that for each $n \in \mathbb{N}$, each β and β' in 2^n,

$$N(\tilde{x}(\beta^\frown 0)) \times N(\tilde{x}(\beta'^\frown 1)) \subset U_n.$$

Before explaining the construction, let us check that with these conditions, the map $\alpha \mapsto x(\alpha)$ realises a Borel reduction of E_0 to $(bb_d(X), \simeq)$. Indeed, when $\alpha E_0 \alpha'$, the corresponding sequences differ by at most finitely many vectors, and since we took care that $|\tilde{a}_n^0| = |\tilde{a}_n^1|$ for all n, $x(\alpha)$ and $x(\alpha')$ span isomorphic subspaces. On the other hand, when α and α' are not E_0-related, without loss of generality there is an infinite set I such that for all $i \in I$, $\alpha(i) = 0$ and $\alpha'(i) = 1$; it follows that for all $i \in I$, $(x(\alpha), x(\alpha'))$ belongs to U_i, and so by choice of the $U_n's$, $(x(\alpha), x(\alpha'))$ does not belong to $\simeq$.

Now let us see at step n how to construct the sequences: given a pair β_0, β_0' in $(2^n)^2$, using the fact that U_n is dense and open, the pair $\tilde{x}(\beta_0), \tilde{x}(\beta_0')$ may be extended to a pair of finite successive block-sequences which are of the form $(\tilde{x}(\beta_0)^\frown \tilde{z}_0, \tilde{x}(\beta_0')^\frown \tilde{z}_0')$ with $N(\tilde{x}(\beta_0)^\frown \tilde{z}_0) \times N(\tilde{x}(\beta_0')^\frown \tilde{z}_0')) \subset U_n$, and we may require that $supp(\tilde{x}(\beta_0)) \cup supp(\tilde{x}(\beta_0')) < supp(\tilde{z}_0) \cup supp(\tilde{z}_0')$. Given an other pair β_1, β_1' in $(2^n)^2$, the pair $(\tilde{x}(\beta_1)^\frown \tilde{z}_0, \tilde{x}(\beta_1')^\frown \tilde{z}_0')$ may be extended to a pair of finite successive block-sequences $(\tilde{x}(\beta_1)^\frown \tilde{z}_1, \tilde{x}(\beta_1')^\frown \tilde{z}_1')$ such that $N(\tilde{x}(\beta_1)^\frown \tilde{z}_1) \times N(\tilde{x}(\beta_1')^\frown \tilde{z}_1') \subset U_n$. Here with our notation $\tilde{z}_1$ extends $\tilde{z}_0$ and $\tilde{z}_1'$ extends $\tilde{z}_0'$. Repeat this $(2^n)^2$ times to get $\tilde{z}_{4^n-1}, \tilde{z}_{4^n-1}'$ such that for all β and β' in 2^n,

$$N(\tilde{x}(\beta)^\frown \tilde{z}_{4^n-1}) \times N(\tilde{x}(\beta')^\frown \tilde{z}_{4^n-1}') \subset U_n.$$

Finally extend $(\tilde{z}_{4^n-1}, \tilde{z}_{4^n-1}')$ to $(\tilde{a}_n^0, \tilde{a}_n^1)$ such that $|\tilde{a}_n^0| = |\tilde{a}_n^1|$; we still have that, for all β and β' in 2^n, $N(\tilde{x}(\beta)^\frown \tilde{a}_n^0) \times N(\tilde{x}(\beta')^\frown \tilde{a}_n^1) \subset U_n$, i.e. with our notation,

$$N(\tilde{x}(\beta^\frown 0)) \times N(\tilde{x}(\beta'^\frown 1)) \subset U_n.$$

Now assume the relation of isomorphism is non-meagre in $bb_d(X)^2$. As the relation is analytic it has the Baire property and $bb_d(X)$ is Polish, so by Kuratowski–Ulam [17, Theorem 8.41], there must be some non-meagre section, that is, some isomorphism class $\mathcal{A}$ is non-meagre. Fix a block-sequence x in this class, then clearly, for some constant C, the set $\mathcal{A}_C$ of blocks-sequences spanning a space C-isomorphic to $[x]$ is non-meagre. Now being analytic, this set has the Baire property, so is residual in some basic open set U, of the form $N(\tilde{x})$, for some finite block-sequence $\tilde{x}$.

We now prove that $\mathcal{A}_k$ is residual in $bb_d(X)$ for $k = Cc(2\max(supp(\tilde{x})))$; The conclusion of the proposition then holds for $K = k^2$. Recall that for $n \in \mathbb{N}$, $c(n)$ denotes a constant such for any Banach space X, any n-codimensional subspaces of X are $c(n)$-isomorphic [6, Lemma 3]. So let us assume $V = N(\tilde{y})$ is some basic open set in $bb_d(X)$ such that $\mathcal{A}_k$ is meagre in V. We may assume that $|\tilde{y}| > |\tilde{x}|$ and write $\tilde{y} = \tilde{x}'^\frown \tilde{z}$ with $\tilde{x} < \tilde{z}$ and $|\tilde{x}'| \leqslant \max(supp(\tilde{x}))$. Choose $\tilde{u}$ and $\tilde{v}$ to be finite sequences of blocks such that $\tilde{u}, \tilde{v} > \tilde{z}$, $|\tilde{u}| = |\tilde{x}'|$ and $|\tilde{v}| = |\tilde{x}|$, and such that

$\max(supp(\tilde{u})) = max(supp(\tilde{v}))$. Let U' be the basic open set $N(\tilde{x}^\frown\tilde{z}^\frown\tilde{u})$ and let V' be the basic open set $N(\tilde{x}'^\frown\tilde{z}^\frown\tilde{v})$. Again $\mathcal{A}_C$ is residual in U' while $\mathcal{A}_k$ is meagre in V'.

Now let T be the canonical map from U' to V'. For all u in U', $T(u)$ differs from at most $|\tilde{x}| + \max(supp(\tilde{x})) \leqslant 2\max(supp(\tilde{x}))$ vectors from u, so $[T(u)]$ is $c(2\max(supp(\tilde{x})))$ isomorphic to $[u]$. Since $k = Cc(2\max(supp(\tilde{x})))$ it follows that $\mathcal{A}_k$ is residual in $V' \subset V$. The contradiction follows by choice of V. $\square$

By analogy with the definition of atomic measures, we may see Proposition 7 as stating that a non-ergodic Banach spaces with a basis must be "atomic" for its block-subspaces.

We now want to give a characterisation of residual subsets of $bb_d(X)$. If $\mathcal{A}$ is a subset of $bb_d(X)$ and $\Delta = (\delta_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers, we denote by $\mathcal{A}_\Delta$ the usual Δ-expansion of $\mathcal{A}$ in $bb_d(X)$, that is $x = (x_n) \in \mathcal{A}_\Delta$ iff there exists $y = (y_n) \in \mathcal{A}$ such that $\|y_n - x_n\| \leqslant \delta_n, \forall n \in \mathbb{N}$. Given a finite block-sequence $\tilde{x} = (x_1, \ldots, x_n)$, we say that a (finite or infinite) block-sequence (y_i) *passes through* $\tilde{x}$ if there exists some integer m such that $\forall 1 \leqslant i \leqslant n,\ y_{m+i} = x_i$.

Proposition 8. *Let $\mathcal{A}$ be residual in $bb_d(X)$. Then for all $\Delta > 0$, there exist successive finite block-sequences $(\tilde{x}_n), n \in \mathbb{N}$ such that any element of $bb_d(X)$ passing trough infinitely many of the $\tilde{x}_n$'s is in $\mathcal{A}_\Delta$.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of dense open sets, which we may assume to be decreasing, such that $\cap_{n\in\mathbb{N}} U_n \subset \mathcal{A}$. Without loss of generality we may also assume Δ to be decreasing. In the following, block-vectors are always taken in **Q**, in the intention of building elements of $bb_d(X)$.

First, U_0 is open so there exists $\tilde{x}_0$ a finite block-sequence such that $N(\tilde{x}_0) \subset U_0$. Now let us choose some $N_1 > \max(supp(\tilde{x}_0))$ and let us take an arbitrary block-vector z_1 such that $N_1 = \min(supp(z_1))$. Let $F_{<1}$ be a finite set of finite block-sequences forming an δ_{N_1}-net for all finite block-sequences supported before N_1 and let F_{01} be a finite set of finite block-sequences forming an δ_{N_1}-net for all finite block-sequences supported after $\tilde{x}_0$ and before N_1. Let $G_1 = \{\tilde{x}_0^\frown\tilde{y}, \tilde{y} \in F_{01}\}$ and let $F_1 = F_{<1} \cup G_1$. Using the fact that U_1 is dense open, we may construct successively a finite block-sequence $\tilde{x}_1$ which extends z_1, so that $\min(supp(\tilde{x}_1)) = N_1 > \max(supp(\tilde{x}_0))$, and such that for any $\tilde{f}_1 \in F_1$, $N(\tilde{f}_1^\frown\tilde{x}_1)$ is a subset of U_1.

Let us now write what happens at the kth step. We choose some $N_k > \max(supp(\tilde{x}_{k-1}))$ and an arbitrary block z_k whose support starts at N_k. We let $F_{<k}$ be a finite set of finite block-sequences forming an δ_{N_k}-net for all finite block-sequences supported before N_k and for all $i < k$, we let F_{ik} be a finite set of finite block-sequences forming an δ_{N_k}-net for all finite block-sequences supported after $\tilde{x}_i$ and before N_k. For any $I = \{i_1 < i_2 < \cdots < i_m = k\}$, we let G_I be the set of finite block-sequences $\tilde{z}$ passing through every i in I, such that the finite sequence of blocks of $\tilde{z}$ supported before $\tilde{x}_{i_0}$ is in $F_{<i_0}$ and such that for all $j < m$, the finite sequence of blocks of $\tilde{z}$ supported between $\tilde{x}_{i_j}$ and $\tilde{x}_{i_{j+1}}$ is in $F_{i_j i_{j+1}}$. And we let F_k be the union of all G_I over all possible subsets of $\{1, 2, \ldots, k\}$ containing k. Using the fact that U_k is dense open, we may construct successively a finite block-sequence $\tilde{x}_k$ which extends z_k, so that $\min(supp(\tilde{x}_k)) = N_k > \max(supp(\tilde{x}_{k-1}))$, and such that for any $\tilde{f}_k \in F_k$, $N(\tilde{f}_k{}^\frown\tilde{x}_k)$ is a subset of U_k.

Repeat this construction by induction, and now let z be a block-sequence passing through $\tilde{x}_n$ for n in an infinite set $\{n_k, k \in \mathbb{N}\}$. We may write $z = \tilde{y}_0{}^\frown\tilde{x}_{n_0}{}^\frown\tilde{y}_1{}^\frown\tilde{x}_{n_1}{}^\frown\ldots$, where $\tilde{y}_0$ is supported before $\tilde{x}_{n_0}$ (we may assume that $n_0 > 0$) and for $k > 0$, $\tilde{y}_k$ is supported between $\tilde{x}_{n_{k-1}}$ and $\tilde{x}_{n_k}$.

Let $\tilde{f}_0 \in F_{<n_0}$ be $\delta_{N_{n_0}}$ distant from $\tilde{y}_0$, and for any $k > 0$, let $\tilde{f}_k \in F_{n_{k-1}n_k}$ be $\delta_{N_{n_k}}$ distant from $\tilde{y}_k$. Then it is clear that z is Δ distant from $f = \tilde{f}_0{}^\frown\tilde{x}_{n_0}{}^\frown\tilde{f}_1{}^\frown\tilde{x}_{n_1}{}^\frown\ldots$. Indeed, consider a term z_n of the block-sequence z: If it appears as a term of some finite sequence $\tilde{x}_{n_k}$ then its distance to the corresponding block f_n of f is 0. If it appears as a term of some $\tilde{y}_k$ then it is less than δ_{N_k}-distant from the block f_n, and $N_k > \max(supp(z_n)) \geqslant n$, so it is less than δ_n-distant from f_n.

It remains to check that f is in $\mathcal{A}$. But for all K, the finite sequence $\tilde{g}_K = \tilde{f}_0{}^\frown\tilde{x}_{n_0}{}^\frown\ldots{}^\frown\tilde{f}_k{}^\frown\tilde{x}_{n_k}$ is an element of $G_{\{n_1,\ldots,n_K\}}$ so is an element of F_K; it follows that $N(\tilde{g}_K)$ is a subset of U_{n_K} and so that f is in U_{n_K}. Finally, f is in $\cap_{k\in\mathbb{N}}U_{n_k}$ so is in $\mathcal{A}$. $\square$

Conversely, given successive blocks $\tilde{x}_n$, the set of block-sequences passing through infinitely many of the $\tilde{x}_n$'s is residual: for a given $\tilde{x}_n$, "$(y_k)_{k\in\mathbb{N}}$ passes through $\tilde{x}_n$" is open and "$(y_k)_{k\in\mathbb{N}}$ passes through infinitely many of the $\tilde{x}_n$'s" is equivalent to "$\forall m \in \mathbb{N}, \exists n > m \in \mathbb{N}: \ (y_k)$ passes through $\tilde{x}_n$", so is G_δ, and clearly dense. If the set $\mathcal{A}$ considered is an isomorphism class, then it is invariant under small enough Δ-perturbations, and so we get an equivalence: $\mathcal{A}$ is residual iff there exist successive finite block-sequences $(\tilde{x}_n), n \in \mathbb{N}$ such that any element of $bb_d(X)$ passing through infinitely many of the $\tilde{x}_n$'s is in $\mathcal{A}$.

Finally, as any element of $bb(X)$ is arbitrarily close to an element of $bb_d(X)$, the following theorem holds:

Theorem 9. *Let X be a Banach space with a basis. Then either X is ergodic or there exists $K \geqslant 1$, and a sequence of successive finite block-sequences $\{\tilde{x}_n\}$ such that all block-sequences passing through infinitely many of the $\{\tilde{x}_n\}$'s span mutually K-isomorphic subspaces.*

If in addition the basis is unconditional, then we may use the projections to get further properties of the residual class.

Corollary 10. *Let X be a non-ergodic Banach space with an unconditional basis. Denote by A an element of the residual class of isomorphism in $bb_d(X)$. Then for any block-subspace Y of X, $A \simeq A \oplus Y$. If X is hereditarily block-minimal, then all residual classes in $bb_d(Y)$, for block-subspaces Y of X, are isomorphic.*

Proof. Let $\{e_i\}$ be the unconditional basis of X and let $\{\tilde{x}_n\}$ be given by Theorem 9. Consider an arbitrary block-subspace Y of X. Its natural basis is unconditional and $Y = [y_i]_{i\in\mathbb{N}}$ is not ergodic as well. Let, by Theorem 6, $(F_n)_{n\geqslant 1}$ be successive finite subsets of $\mathbb{N}$ such that for any infinite subset N of $\mathbb{N}$, if $N \cap [\min(F_n), \max(F_n)] = F_n$ for infinitely many n's, then the space $[y_i]_{i\in N}$ is isomorphic to Y. Passing to subsequences we may assume that for all n in $\mathbb{N}$, $\tilde{x}_n < \cup_{i\in F_n} supp(y_i) < \tilde{x}_{n+1}$. Then

$$A \simeq [\tilde{x}_n]_{n\in\mathbb{N}} \oplus [y_i]_{i\in\cup_{n\in\mathbb{N}} F_n} \simeq A \oplus Y.$$

If now X is hereditarily block-minimal, and B belongs to the residual class in $bb_d(Y)$, for $Y \leqslant X$, then by the above $A \simeq A \oplus B$; but also A is block-minimal so some copy of A embeds as a block-subspace of Y, so $B \simeq B \oplus A$. $\square$

A Banach space is said to be *countably homogeneous* if it has at most countably many non-isomorphic subspaces. By Theorem 4, a countably homogeneous space has a block-minimal subspace with an unconditional basis, and one easily diagonalises to get a hereditarily block-minimal subspace.

Proposition 11. *Let X be a countably homogeneous, hereditarily block-minimal Banach space with an unconditional basis. Then elements in the residual class of isomorphism for $bb_d(X)$ are isomorphic to a (possibly infinite) direct sum of an element of each class.*

Proof. We write the proof in the denumerable case. We partition X in a direct sum of subspaces $X_n, n \in \mathbb{N}$ by partitioning the basis. So each X_n embeds into X. For each n, choose a representative E_n of the nth isomorphism class which is a block of X_n (it is possible because X_n is block-minimal as well). By applications of Gowers's theorem in each X_n, we may pick each vector forming the basis of each E_n far enough, to ensure that $E = \sum_{n\in\mathbb{N}} \oplus E_n$ is a block-subspace of X. We show that E is in the residual class $\mathcal{A}$. Indeed, if m is such that $E_m \in \mathcal{A}$, then $E \simeq E_m \oplus \sum_{n\neq m} \oplus E_n \simeq E_m$ by Corollary 10. $\square$

It follows from the proof above that for any two block-subspaces A and B of X, $A \oplus B$ may be embedded as a block-subspace of X; i.e., under the assumptions of Proposition 11, isomorphism classes of block-subspaces of X form a countable (commutative) semi-group.

Consider the property that every block-subspace Y satisfies $A \simeq A \oplus Y$. We may think of this property as an algebraic property characterising large subspaces in the sense that a large subspace should intuitively "contain" other subspaces, and more importantly, a space should have at most one large subspace (here if A and A' satisfy the property, $A \simeq A \oplus A' \simeq A'$!). Notice that as X is not ergodic, all block-subspaces are isomorphic to their squares by Ferenczi and Rosendal [6] and so the property above is equivalent to saying that every block of X embeds complementably in A (i.e. A is *complementably universal for $bb(X)$*). Generally, a space A is said to be complementably universal for a class $\mathcal{C}$ of Banach spaces if every element of $\mathcal{C}$ is isomorphic to a complemented subspace of X. It is known that no separable Banach space is complementably universal for the class of all separable Banach spaces ([19, Theorem 2.d.9]), but there exists a Banach space X_U with an unconditional basis which is complementably universal for the class of all Banach spaces with an unconditional basis [23], and so for the class of its block-subspaces in particular.

Combining Theorem 4 and Corollary 10, we get

Theorem 12. *Any Banach space is ergodic or contains a subspace with an unconditional basis which is complementably universal for the family of its block-subspaces.*

We now study this property in more detail. We also see how Theorem 9 may be used to obtain uniformity results.

Definition 13. Let X and Y be Banach spaces such that X has a Schauder basis. Y is said to be complementably universal for $bb(X)$ if every block-subspace of X embeds complementably in Y.

Lemma 14. *Let X be a Banach space. Any Banach space complementably universal for $bb(X)$ is decomposable.*

Proof. Let A be complementably universal for $bb(X)$ and indecomposable. First note that X embeds complementably in A, so must be isomorphic to a finite-codimensional subspace of A. As well, any block-subspace of X is isomorphic to a finite-codimensional subspace of A and so none of them is decomposable either. It follows easily that no subspace of X is decomposable. In other words, X is hereditarily indecomposable. It follows also that X is isomorphic to a proper (infinite-dimensional) subspace, and this is a contradiction with properties of hereditarily indecomposable spaces. □

To quantify the property of complementable universality, let us define $dec_X(Y) = \inf KK'$, where the infimum runs over all couples (K, K') such that Y is K-isomorphic to a K'-complemented subspace of X. Of course, $dec_X(Y) = +\infty$ iff Y does not embed complementably in X. We shall say that a space A is *C-complementably universal* for $bb(X)$ if every block-subspace of X is K-isomorphic to some K'-complemented subspace of A, for some K and K' such that $KK' \leqslant C$, that is, if $\sup_{Y \leqslant X} dec_A(Y) \leqslant C$.

Lemma 15. *Assume A, B, C are Banach spaces with bases. Then*

$$dec_A(C) \leqslant dec_A(B)^2 dec_B(C).$$

Proof. Let ε be positive. Let P_B be a projection defined on B and α_{BC} be an isomorphism from $P_B(B)$ onto C such that $\|P_B\|.\|\alpha_{BC}\|.\|\alpha_{BC}^{-1}\| \leqslant dec_B(C) + \varepsilon$. Let P_A be a projection defined on A and α_{AB} be an isomorphism from $P_A(A)$ onto B such that $\|P_A\|.\|\alpha_{AC}\|.\|\alpha_{AB}^{-1}\| \leqslant dec_A(B) + \varepsilon$.

We let $P = \alpha_{AB}^{-1} P_B \alpha_{AB} P_A$, defined on A; it is easily checked that P is a projection. We let $\alpha = \alpha_{BC}\alpha_{AB}$: it is an isomorphism from $P(A)$ onto C. Then

$$dec_A(C) \leqslant \|P\|.\|\alpha\|.\|\alpha^{-1}\| \leqslant (dec_A(B) + \varepsilon)^2 (dec_B(C) + \varepsilon). \qquad \square$$

Proposition 16. *Let X be a Banach space with a Schauder basis and let A be complementably universal for $bb(X)$. Then there exists $C \geqslant 1$ such that every finite-dimensional block-subspace of X C-embeds complementably in A.*

Proof. First it is clear that it is enough to restrict ourselves to elements of $bb_d(X)$ with the previously defined topology. We let for $k \in \mathbb{N}$, $\mathcal{A}_k$ denote the set of block-subspaces of X which are k-isomorphic to some k-complemented subspace of A. Now it is clear that one of the $\mathcal{A}_k$ must be non-meagre. This set is analytic, so has the Baire property, so is residual in some basic open set U, of the form $N(\tilde{u})$. We now show that $\mathcal{A}_K$ is residual for $K = kc(2\max(supp(\tilde{u}))$. Otherwise, as in Proposition 7, we may assume $\mathcal{A}_K$ is meagre in $V = N(\tilde{y})$, and $\mathcal{A}_k$ is residual in $U' = N(\tilde{x})$ where $\tilde{x}$ extends $\tilde{u}$, $|\tilde{x}| \leqslant 2\max(supp(\tilde{u}))$ and $\max(supp(\tilde{x})) = \max(supp(\tilde{y}))$.

Now let T be the canonical map from U' to V. For all u in U', $T(u)$ differs from at most $q \leqslant 2\max(supp(\tilde{u}))$ vectors from u, so the space $[T(u)]$ is $c(2\max(supp(\tilde{u}))$ isomorphic to $[u]$. So $T(u)$ is in $\mathcal{A}_K$ whenever u is in $\mathcal{A}_k$. It follows that $\mathcal{A}_K$ is residual in V, a contradiction.

Now consider any finite-dimensional space F generated by a finite block-sequence of A, $F = [x_1, \ldots, x_p]$: it may be extended to a block-sequence $x = (x_i)_{i \in \mathbb{N}}$ in $\mathcal{A}_K$, that is $dec_A([x]) \leqslant K^2$. But also $dec_{[x]}(F) \leqslant c$, where c is the constant of the basis, so by Lemma 15, $dec_A(F) \leqslant cK^4$. $\square$

Proposition 17. *Let X be a space with an unconditional basis. If A is complementably universal for $bb(X)$ and isomorphic to its square, then A is C-complementably universal for $bb(X)$ for some $C \geqslant 1$.*

Proof. The first part of the proof is as above to get $K \in \mathbb{N}$ such that $\mathcal{A}_K$, the set of block-subspaces of X which are K-isomorphic to some K-complemented subspace of A, is residual. So by Proposition 8, there exists a sequence $\tilde{x}_n$ of successive finite blocks such that any block passing through infinitely many of the $\tilde{x}_n's$ is in $\mathcal{A}_{2K}$.

Let now $Y = [y_n]_{n \in \mathbb{N}} = [y]$ be an arbitrary block-subspace of X. We may define a sequence $(\tilde{y}_i)$ of finite block-sequences with $\tilde{y}_1 ^\frown \tilde{y}_2 ^\frown \ldots = y$ and a subsequence of $\{\tilde{x}_n\}$, denoted $\{\tilde{x}_i\}$, such that for all i,

$$supp(\tilde{y}_{i-1}) < supp(\tilde{x}_i) < supp(\tilde{y}_{i+1}).$$

We let $w = \tilde{y}_1 ^\frown \tilde{x}_2 ^\frown \tilde{y}_3 ^\frown \ldots$ and $w' = \tilde{x}_1 ^\frown \tilde{y}_2 ^\frown \tilde{x}_3 ^\frown \ldots$ It is clear that $Y_1 = [\tilde{y}_{2i-1}]_{i \in \mathbb{N}}$ is c-complemented in $[w]$, where c is the constant of unconditionality of the basis; so $dec_{[w]}(Y_1) \leqslant c$. But we know that $dec_A([w]) \leqslant 4K^2$, so by Lemma 15, $dec_A(Y_1) \leqslant 16K^4c$. Likewise if we denote $[\tilde{y}_{2i}]_{i \in \mathbb{N}}$ by Y_2 and use $[w']$, we prove that $dec_A(Y_2) \leqslant 16K^4c$. It follows that $Y_1 \oplus Y_2$ satisfies $dec_{A \oplus_1 A}(Y_1 \oplus Y_2) \leqslant (16K^4c)^2$ (here $\oplus_1$ denotes the ℓ_1-sum) , and, if D is such that A is D isomorphic to its square $A \oplus_1 A$, that $dec_A(Y) \leqslant 2^9 D^2 K^8 c^3$. $\square$

In view of the fact that by Theorem 6, any unconditional basic sequence in a non-ergodic Banach space spans a space isomorphic to its square, the previous proposition may be applied to Theorem 12: a non-ergodic Banach space contains a subspace X with an unconditional basis which is *uniformly* complementably universal for $bb(X)$.

4. Asymptotically ℓ_p spaces

Consider now spaces with a basis with the stronger property that every block-subspace is complemented. It is well-known that every block-subspace of ℓ_p or c_0 is complemented, and the same is true for spaces $(\sum_{n=1}^{+\infty} \oplus \ell_s^n)_p$, the relevant case being $s \neq p$ (see [19]), or for Tsirelson's spaces $T_{(p)}$ (see [4]). All these examples are asymptotically ℓ_p or c_0, and we shall now see that this is not by chance.

We recall the definition of an asymptotically ℓ_p space with a basis. Consider the so-called *asymptotic game* in X, where Player I plays integers (n_k) and Player II plays successive unit vectors (x_k) in X such that $supp(x_k) > n_k$ for all k. Then X is asymptotically ℓ_p if there exists a constant C such that for any $n \in \mathbb{N}$, Player I has a winning strategy in the asymptotic game of length n for forcing II to play a sequence C-equivalent to the unit basis of ℓ_p^n. The similar definition holds for c_0.

Our reference for asymptotic structure in Banach spaces will be the paper of Maurey et al. [21]. Note that there are two natural notions of asymptotic structure for Banach spaces: the first is associated to the set of finite-codimensional subspaces of X, and the second to tail subspaces of X taken with a given basis. Our definition obviously corresponds to the second notion. Note also that, if formally slightly different from the definition in [21] (Definition 1.7), our definition is easily seen to be equivalent to it (use [21, Definition 1.3.3 and Proposition 1.5]).

We start by a uniformity result similar to Proposition 17, for Banach spaces with a basis for which every block-subspace is complemented.

Proposition 18. *Let X be a space with an unconditional basis $\{e_i\}$, and assume that every block-subspace of X is complemented. Then there exists $C \geqslant 1$ such that every block-subspace of X is C-complemented.*

Proof. Without loss of generality, assume $\{e_i\}$ is 1-unconditional. Given any finite or infinite block-sequence $\{x_n\}$ of X, we define for any $n \in \mathbb{N}$, $E_n = [e_i, \min(supp(x_n)) \leqslant i \leqslant \max(supp(x_n))]$. We note that by [19], 1.c.8, Remark 1, the projection onto $[x_n]$ may be chosen to be block-diagonal with respect to E_n (just replace P by $\sum_k E_k P E_k$, and the norm of the projection is preserved).

We shall prove that if for some C and every $n \in \mathbb{N}$, $[x_1, \dots, x_n]$ is C-complemented by a block-diagonal projection with respect to E_n, then $[x_n]_{n \in \mathbb{N}}$ is C-complemented (by a block-diagonal projection with respect to E_n). The proposition follows by an easy induction.

So let for each $n \in \mathbb{N}$, P_n be a projection on $[x_1, \dots, x_n]$, of norm less than C, which is block-diagonal with respect to the E_k's. Passing to a diagonal subsequence we may assume that for each k, $E_k(P_n)_{|E_k}$ converges to some projection Q_k defined from E_k onto the 1-dimensional space generated by x_k. Define Q on X by $Qx = \sum_{k\in\mathbb{N}} Q_k E_k x$. It is easy to check that Q is a projection onto $[x_n]_{n\in\mathbb{N}}$ of norm less than C. $\square$

A few comments before the next proposition. Our original proof of Proposition 18 was similar to the one of Proposition 17. We are thankful to the referee for indicating to us that a much more direct proof existed. The property of complementation for block-subspaces is too regular to require the strength of the theorem of Baire.

We then used Theorem 5.3 in [21] to conclude that X must be asymptotically c_0 or ℓ_p. However, the referee showed us that if X does not contain c_0, there is a chain of classical properties equivalent to our property, which imply that X is asymptotically c_0 or ℓ_p by a more direct and more informative proof. We write this chain of equivalences in the next proposition.

Let X be a Banach space with a basis $\{e_i\}$. Given a block-subspace $[x_n]$ of X, where $\{x_n\}$ is supposed normalised, we let as before, for $n \in \mathbb{N}$, $E_n = [e_i, \min(supp(x_n)) \leqslant i \leqslant \max(supp(x_n))]$. We shall call *canonical projection onto* $[x_n]_{n\in\mathbb{N}}$ a projection P defined on X by $Px = \sum_{n\in\mathbb{N}} x_n^*(E_n x)x_n$, where for all $n \in \mathbb{N}$, $x_n^* \in E_n^*$ is a norm 1 functional such that $x_n^*(x_n) = 1$.

Finally, following [3], we say that a finite-dimensional decomposition $X = \sum_{n\in\mathbb{N}} E_n$ of a Banach space is *absolute* if there exists a constant C such that for every $x_n, y_n \in E_n$ such that for all $n \in \mathbb{N}$, $\|y_n\| \leqslant \|x_n\|$, it follows that $\|\sum_{n\in\mathbb{N}} y_n\| \leqslant C\|\sum_{n\in\mathbb{N}} x_n\|$.

Proposition 19. *Let X be a Banach space with an unconditional basis $\{e_i\}$ which does not contain a copy of c_0. The following are equivalent*:

(i) *every block-subspace of X is complemented*,
(ii) *every block-subspace $[x_n]_{n\in\mathbb{N}}$ of X is complemented by any canonical projection onto $[x_n]_{n\in\mathbb{N}}$*,
(iii) *$\{e_i\}$ has the shift property, i.e. for any normalised block-sequence $\{x_n\}$ of $\{e_i\}$, $\{x_n\}_{n\in\mathbb{N}}$ is equivalent to $\{x_{n+1}\}_{n\in\mathbb{N}}$*,
(iv) *every blocking $E_n = [e_i, r_n \leqslant i < r_{n+1}]$ of $\{e_i\}$ (where (r_n) is an increasing sequence of integers) is absolute*.

Proof. (iii) $\Rightarrow$ (iv) is immediate and was already observed by Casazza and Kalton in [3]. From (iv) we get that any choices of functionals x_n^* in the definition of a canonical projection will give a bounded projection, so (ii) follows, and (ii) $\Rightarrow$ (i) is trivial.

Assume (i). Let $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ be normalised block-sequences such that for all $n \in \mathbb{N}$, $x_n < y_n < x_{n+1}$. We prove that $\{x_n\}$ and $\{y_n\}$ are equivalent and (iii) follows.

Following the method of Lindenstrauss and Tzafriri, [19] Theorem 2.a.10, we apply their Lemma 2.a.11: if $\sum_{n\in\mathbb{N}} \mu_n x_n$ converges, then for any sequence λ_n converging to 0, $\sum_{n\in\mathbb{N}} \lambda_n \mu_n y_n$ converges. If $\sum_{n\in\mathbb{N}} \mu_n y_n$ does not converge, we easily construct a block-sequence of $\{y_n\}$ which is equivalent to the canonical basis of c_0 (see the proof of Theorem 2.a.10), so c_0 embeds in X, a contradiction. By the same proof, if $\sum_{n\in\mathbb{N}} \mu_n y_n$ converges, then so does $\sum_{n\in\mathbb{N}} \mu_n x_n$. $\square$

Corollary 20. *Let X be a Banach space with an unconditional basis $\{e_i\}$, such that every block-subspace of X is complemented. Then X is asymptotically c_0 or $\ell_p, 1 \leqslant p < +\infty$.*

Proof. First assume c_0 does not embed in X. We apply Proposition 19, and we note that uniformity of the constants of equivalence is easily obtained in (iii). Krivine's theorem implies that ℓ_p (or c_0) is asymptotic in X ([21, Remark 1.6.3]): that is, for all $\varepsilon > 0$ and $n \in \mathbb{N}$, Player II has a winning strategy in the asymptotic game of length n for producing a sequence $1+\varepsilon$ equivalent to the basis of l_p^n (or l_∞^n). This, in combination with (iii), implies very directly that X is asymptotically c_0 or ℓ_p (this was already essentially observed by Kalton in [15]).

If c_0 embeds in X, let $\{u_n\}$ be a normalised block-basic sequence in X which is equivalent to the canonical basis of c_0. We again apply Lemma 2.a.11 from [19]: we deduce easily that if $\{u_{n_i}\}$ is any subsequence of $\{u_n\}$, and $\{v_i\}$ a normalised block-basic sequence in X, such that for all $i \in \mathbb{N}$, $u_{n_i} < v_i < u_{n_{i+1}}$, then $\{v_i\}$ is equivalent to the unit vector basis of c_0. Furthermore, by an easy induction, there is uniformity in the equivalence of these sequences $\{v_i\}$ with the basis of c_0. From this, it is straightforward to see that X is asymptotically c_0. $\square$

We now pass to a proposition of independent interest concerning asymptotically c_0 or ℓ_p spaces.

Proposition 21. *Let $\{e_i\}$ be a basic sequence asymptotically c_0 or ℓ_p such that every subsequence of $\{e_i\}$ has a block-sequence equivalent to $\{e_i\}$* (*in particular, if $\{e_i\}$ is subsymmetric or equivalence-block-minimal*). *Then $\{e_i\}$ is equivalent to the unit basis of c_0 or ℓ_p.*

Proof. Let p be such that $\{e_i\}_{i\in\mathbb{N}}$ is asymptotically ℓ_p (the case of c_0 is similar). Assume every subsequence of $\{e_i\}$ has a block-sequence equivalent to $\{e_i\}$. Then as shown in Lemma 2 we may (by passing to a subsequence) assume that for some $C \geqslant 1$, every subsequence of $\{e_i\}$ has a block-sequence C-equivalent to $\{e_i\}$.

We fix $n \in \mathbb{N}$ and build a winning strategy for Player II in the asymptotic game of infinite length for producing a block-sequence $2C$-equivalent to (e_i). This strategy may then be opposed to a winning strategy for Player I for producing length n block-sequences C'-equivalent to the unit basis of ℓ_p^n. We get that $e_1, \ldots, e_n$ is $2CC'$-equivalent to ℓ_p^n for all n which will conclude the proof.

Let $\mathcal{A} = \left\{(n_k) \in [\mathbb{N}]^{\mathbb{N}} : \exists (x_k) \in bb_N(X)\ \forall k\ n_{2k} < x_k < n_{2k+1} \ \wedge\ (x_k) \sim^C (e_k)\right\}$. We claim that any sequence $(m_k) \in [\mathbb{N}]^{\mathbb{N}}$ contains a further subsequence in $\mathcal{A}$, for we can suppose that $I_k =]m_{2k}, m_{2k+1}[\neq \emptyset$ for all k and therefore take $(y_i) \sim^C (e_i)$ with $supp(y_i) \subset \bigcup I_n$. Then in between y_i and y_{i+1} there are $n_{2i+1} := m_{2k+1}$ and $n_{2i+2} := m_{2k+2}$, whereby $(n_k) \in \mathcal{A}$. So by the infinite Ramsey theorem there is some infinite $A \subset \mathbb{N}$ such that $[A]^{\mathbb{N}} \subset \mathcal{A}$ and there is a C-measurable $f : [A]^{\mathbb{N}} \longrightarrow bb_N(X)$ choosing witnesses (x_k) for being in $\mathcal{A}$.

Let $\Delta = (\delta_n)$, $\delta_n > 0$ be such that if two normalised block-sequences are less than Δ apart, then they are 2-equivalent. We choose inductively $n_i < m_i < n_{i+1}$ and sets $B_i \subset \mathbb{N}$ such that $n_i, m_i \in B_i \subset B_{i-1}/m_{i-1}$ and such that for all $C, D \in \left[\{n_{j_1}, m_{j_1}, \ldots, n_{j_k}, m_{j_k}\}, B_{j_k+1}\right]$ $(j_1 < \ldots < j_k)$ we have $\|f(C)(k) - f(D)(k)\| < \delta_k$. This can be done as the unit sphere of $[e_{n_{j_k}+1}, \ldots, e_{m_{j_k}-1}]$ is compact.

Now in the asymptotic game of infinite length, we can demand that I plays numbers from the sequence (n_i) and then II replies to $n_{j_1}, \ldots, n_{j_k}$ played by I with some $n_{j_k} < x < m_{j_k}$ such that for all $C \in \left[\{n_{j_1}, m_{j_1}, \ldots, n_{j_k}, m_{j_k}\}, B_{j_k+1}\right]$ we have $\|f(C)(k) - x\| < \delta_k$.

Then in the end of the infinite game, supposing that I has played (n_{j_k}) and II has followed the above strategy responding by (x_k), we have $n_{j_k} < x_k < m_{j_k}$. Let $(y_k) := f(\{n_{j_k}, m_{j_k}\}_{\mathbb{N}})$, then $\|x_k - y_k\| < \delta_k$ and $(x_k) \sim^2 (y_k) \sim^C (e_i)$, so II wins. □

B. Sari drew our attention to the fact that the dual T^* of Tsirelson's space is a good illustration of Proposition 21: it is asymptotically c_0, and is minimal [4], but by the Proposition cannot be equivalence block-minimal.

Lindenstrauss and Tzafriri have proved that if a Banach space has a symmetric basis and all its block-subspaces are complemented, then this basis must be equivalent to the canonical basis of c_0 or ℓ_p ([19, Theorem 2.a.10]). An immediate combination of Corollary 20 and Proposition 21 improves their result: if a Banach space has an unconditional basis $\{e_i\}$ such that all block-subspaces are complemented, and such that every subsequence has a block-sequence equivalent to $\{e_i\}$, then $\{e_i\}$ is equivalent to the canonical basis of c_0 or ℓ_p.

5. Subspaces with a finite-dimensional decomposition

We now want to generalise the previous results, considering more general types of subspaces. So we fix X to be a Banach space with a basis $\{e_n\}$ and we denote by c the constant of the basis.

We first notice that we could consider subspaces generated by disjointly supported (but not necessarily successive) vectors, and get similar results: we find a residual class characterised by a "passing through" property. It follows that in a non-ergodic Banach space with an unconditional basis, subspaces generated by disjointly supported vectors embed complementably in a given element of the residual class. We recall that if a Banach space has an unconditional basis such that any subspace generated by disjointly supported vectors is complemented, then the space is c_0 or ℓ_p ([19, Theorem 2.a.10]); however there does exist a space X_U cited above [23], not isomorphic to c_0 or ℓ_p, and such that every subspace generated by disjointly supported vectors embeds complementably in X_U.

Then we want to represent any subspace of X, possibly up to small perturbations, on the basis $\{e_n\}$, and get similar results as for the case of block-subspaces. We shall call *triangular sequences of blocks* the normalised sequences of (possibly infinitely supported) vectors in the product $X^{\mathbb{N}}$'s, satisfying for all k, $\min(supp(x_k)) <$

$\min(supp(x_{k+1}))$ equipped with the product of the norm topology on X. The set of triangular sequences of blocks will be denoted by tt. By Gaussian elimination method, it is clear that any subspace of X may be seen as the closed linear space generated by some sequence of tt. Once again it is possible to discretise the problem by considering the set tt_d of sequences of vectors in $\mathbf{Q}$ such that for all k, $\min(supp(x_k)) < \min(supp(x_{k+1}))$, and by showing that for any $x \in tt$ and any $\varepsilon > 0$, there exists x_d in tt_d such that $[x_d]$ is $1+\varepsilon$-isomorphic to $[x]$.

Our usual method does generalise to this setting. However, the characterisation of a residual set turns out to be only expressed in terms of particular subspaces of tt, namely those with a finite-dimensional decomposition on the basis (or FDD). So it gives more information, and it is actually easier, to work directly with spaces with a finite-dimensional decomposition on the basis.

Note that a space with a FDD on the basis must have the bounded approximation property. So our methods only allow us to study subspaces with that property. In fact, it is easy to check that the set of sequences in tt_d spanning a space with an FDD on the basis is residual in tt_d, and so, the set of spaces without the (bounded) approximation property is meagre in our topology, which explains why with our methods, we do not seem to be able to "see" spaces without the approximation property.

We say that two finite-dimensional spaces F and G are *successive*, and write $F < G$, if for any $x \in F$, $y \in G$, x and y are successive. A space with a *finite-dimensional decomposition on the basis* is a space of the form $\oplus_{k\in\mathbb{N}} E_k$, with successive, finite-dimensional spaces E_k; such a space *passes through* E if $E_k = E$ for some k. We let *fdd* be the set of infinite sequences of successive finite-dimensional subspaces, and fdd_d be the set of infinite sequences of successive finite-dimensional subspaces which are spanned by a collection of vectors with rational coordinates—equipped with the product of the discrete topology on the set of finite collections of rational vectors.

Theorem 22. *Let X be a Banach space with a basis. Then either X is ergodic, or there exists $K \geqslant 1$, and a sequence of successive finite-dimensional spaces F_n such all spaces with finite-dimensional decomposition on the basis passing through infinitely many F_n's are mutually K-isomorphic.*

Proof. Most of the previous proof may be taken word by word; instead of working with block-subspaces, i.e. subspaces with 1-dimensional decomposition on the basis, we work with subspaces with arbitrary FDD on the basis; we just have to define the Δ-expansion of a subspace with a FDD on the basis using an appropriate distance between finite-dimensional subspaces. We then end up with characterisation in terms of "passing through" some finite sequences of finite-dimensional spaces $\{\tilde{E}_i, i \in \mathbb{N}\}$, and this may be simplified to get Theorem 22, noting that we may choose these sequences to be of length 1 (replace each $\tilde{E}_i = (E_i^1, \ldots, E_i^{n_i})$ by $E_i^1 \oplus \ldots \oplus E_i^{n_i}$). □

If the basis is unconditional then we can use the many projections to get additional results concerning the residual class.

Proposition 23. *Let X have an unconditional basis and be non-ergodic. Let A belong to the residual class in fdd_d. Then every subspace with a finite-dimensional decomposition on the basis embeds complementably in A. Either A fails to have l.u.s.t. or X is C-ℓ_2 saturated.*

Proof. Let (F_n) be given by Theorem 22 and let $Y = \sum \oplus B_n$ be an arbitrary space with a finite-dimensional decomposition on the basis. Passing to a subsequence of (F_n), we may assume that there is a partition of $\mathbb{N}$ in successive intervals $J_i, i \in \mathbb{N}$, such that for all i,

$$F_{i-1} < B'_i = \oplus_{n \in J_i} B_n < F_{i+1}.$$

We have that

$$A \simeq \left(\sum \oplus F_i\right)_{i \in \mathbb{N}} \simeq \left(\sum \oplus F_{2k-1}\right)_{k \in \mathbb{N}} \oplus \left(\sum \oplus F_{2k}\right)_{k \in \mathbb{N}},$$

and so it follows that

$$A \oplus Y \simeq \left(\sum \oplus F_{2k-1}\right)_{k \in \mathbb{N}} \oplus \left(\sum \oplus B'_{2k}\right)_{k \in \mathbb{N}} \oplus \left(\sum \oplus F_{2k}\right)_{k \in \mathbb{N}} \oplus \left(\sum \oplus B'_{2k-1}\right)_{k \in \mathbb{N}}$$
$$\simeq A \oplus A \simeq A.$$

For the last part of the Proposition, if X is C-ℓ_2 saturated for no C, then it follows from the Theorem of Komorowski and Tomczak-Jaegermann ([18,25] for a survey and an improved result) that X has a finitely supported subspace L_n without n-l.u.s.t. for each n: indeed Komorowski–Tomczak-Jaegermann's result takes care of the finite cotype case, and if X has not finite cotype then it contains ℓ_∞^k's uniformly, and so a finitely supported if you wish, finite-dimensional space without n-l.u.s.t. (see the remark after Theorem 2.3. in [25], and e.g. [7]). We may by Theorem 22 extend L_n to a space with a finite-dimensional decomposition on the basis which is K isomorphic to A. Then if c is the unconditional constant of the basis, A must fail n/cK-l.u.s.t. and as n was arbitrary, A fails l.u.s.t. □

Note that it is a consequence of the solution of Gowers and Komorowski-Tomczak to the Homogeneous Banach Space Problem, that the following strengthening holds: a Banach space isomorphic to all its subspaces with a FDD must be isomorphic to ℓ_2. This can also be seen as a consequence of the previous Proposition (combined with Gowers's dichotomy theorem). Note also that by results of Kadec and Pelczynski [19, Theorem 2.d.8], there exists a Banach space which is complementably universal for the set of Banach spaces with the Bounded Approximation Property; but as mentioned before, there is no complementably universal space for the class of separable Banach spaces. The Bounded Approximation Property seems to draw the fine line for positive results with our methods.

We conjecture that ℓ_2 is the only non-ergodic Banach space. However, we are not even able to prove that c_0 and $\ell_p, p \neq 2$ are ergodic—although it is known that those spaces have at least $\aleph_1$ non-isomorphic subspaces [19]. To answer this question, evidently one would have to consider other types of subspaces than those generated by successive blocks, or disjointly supported blocks; one could consider spaces of the form $(\sum_{n=1}^{+\infty} \oplus B_n)_p$, with carefully chosen finite-dimensional B_n so that this direct sum is isomorphic to a subspace of ℓ_p, and play with the possible choices for (B_n) (see [19, Proposition 2.d.7]).

On the other hand, it is more relevant to restrict the question of ergodicity to block-subspaces, if one is looking for a significant dichotomy between "regular" and "wild" spaces with a basis: in this setting, c_0 and ℓ_p are, as they should be, on the regular (i.e. non-ergodic) side of the dichotomy.

Remark. After this article was submitted, the first author and E.M. Galego [5] proved that the spaces c_0 and $\ell_p, 1 \leqslant p < 2$, are ergodic, reinforcing the conjecture that ℓ_2 is the only non-ergodic Banach space. The case of ℓ_p, $p > 2$ is still open.

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