GEOMETRIES OF TOPOLOGICAL GROUPS

CHRISTIAN ROSENDAL

Contents

The aim of the present paper is to organise and put into a coherent form a number of old and new results, ideas and research programmes regarding topological groups and their linear counterparts, namely Banach spaces. As the title indicates, our focus will be on geometries by which we understand the various types of geometric structures that a topological group or Banach spaces may be equipped with, e.g., Lipschitz structure or the quasimetric structure underlying geometric group theory. We shall attempt to provide a common framework and language for several different currently very active disciplines, including geometric nonlinear functional analysis and geometric group theory, and varied objects, e.g., Banach spaces, finitely generated, Lie, totally disconnected locally compact and Polish groups. For this reason, it will be useful not to restrict our objects initially.

1. Banach spaces as geometric objects

1.1. **Categories of geometric structures.** Our model example of topological groups, namely, the additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$ is perhaps somewhat unconventional. Certainly, the Banach space $(X, \|\cdot\|)$ is far more structured that $(X, +)$ and thus one misses much important information by leaving out the normed linear structure. Moreover, algebraically $(X, +)$ is just too simple to be of much interest. However, Banach spaces are good examples since they are objects that have classically been studied under a variety of different perspectives, e.g., as topological vector spaces, as metric or uniform spaces. So, apart from their intrinsic interest, Banach spaces will illustrate some of the appropriate categories in which to study topological groups and also will provide a valuable lesson in how rigidity allows us to reconstruct forgotten structure.

The language of category theory will be convenient to formulate the various geometric structures we shall be studying. So recall that to define a category we need to specify the objects and the morphisms between them. In that way, we derive the concept of isomorphism, namely, an *isomorphism* between objects X and Y is a morphism $X \xrightarrow{\phi} Y$ so that, for some morphism $Y \xrightarrow{\psi} X$, both $\psi\phi$ and $\phi\psi$ equal the unique identities on X and Y respectively.

On the other hand, *embedding*, i.e., isomorphism with a substructure, is not readily a categorical notion as it relies on the model theoretical concept of substructure. However, in all our examples, what constitutes a substructure is evident, e.g., a substructure of a topological vector space is a linear subspace with the induced topology, while a substructure of a metric space is just a subset with the restricted metric. So, for example, an embedding of topological vector spaces is continuous linear map $X \xrightarrow{T} Y$, which is a linear homeomorphism onto its image $V \subseteq Y$.

1.2. **Metric spaces.** Recall that a *Banach space* is a complete normed vector space $(X, \|\cdot\|)$. Thus, the norm is part of the given data. For simplicity, **all Banach spaces are assumed to be real**, i.e., over the field $\mathbb{R}$. In the strictest sense, an isomorphism should be a surjective linear isometry between Banach spaces and the proper notion of morphism is thus *linear isometry*, i.e., a linear operator $X \xrightarrow{T} Y$ so that $\|Tx\| = \|x\|$.

However, instead of *normed vector spaces*, quite often, Banach spaces are considered in the weaker category of *topological vector spaces* with morphisms simply being continuous linear operators. The procedure of dropping the norm from a normed linear space while retaining the topology thus amounts to a forgetful functor

$$\mathsf{NVS} \xrightarrow{\mathbf{F}} \mathsf{TVS}$$

from the category of normed vector spaces to the category of topological vectors spaces. Similarly, rather than entirely eliminating the norm, we may instead erase the linear structure while recording the induced norm metric and thus obtain a forgetful functor

$$\mathsf{NVS} \xrightarrow{\mathbf{G}} \mathsf{MetricSpaces}$$

to the category of metric spaces whose morphisms are (not necessarily surjective) isometries. Observe also that these functors preserve embeddings.

This latter erasure however points to our first rigidity phenomenon, namely, the Mazur–Ulam theorem. Indeed, S. Mazur and S. Ulam [29] showed that if $X \xrightarrow{\phi} Y$ is a surjective isometry between Banach spaces, then ϕ is necessarily affine, i.e., the map $Tx = \phi(x) - \phi(0)$ is a surjective linear isometry between X and Y. In particular, any two isometric Banach spaces are automatically linearly isometric.

In a more recent breakthrough, G. Godefroy and N. J. Kalton established a similar rigidity result for *separable* Banach spaces.

Theorem 1.1. [17] *If $X \xrightarrow{\phi} Y$ is an isometric embedding from a separable Banach space X into a Banach space Y, then there is an isometric linear embedding of X into Y.*

Observe that the conclusion here is somewhat weaker than in the Mazur–Ulam theorem, since ϕ itself may not be affine. This is for good reasons, as for example the map $\phi(x) = (x, \sin x)$ is an isometric, but clearly non-affine embedding of $\mathbb{R}$ into $\ell^\infty(2) = (\mathbb{R}^2, \|\cdot\|_\infty)$. Also, the assumption that X is separable is known to be necessary as there are counter-examples in the non-separable setting.

Though these two rigidity results do not provide us with any functor from the category of metric space reducts of separable Banach spaces to the category of normed vector spaces, they do show that an isomorphism or embedding in the weaker category of metric spaces implies the existence of an isomorphism, respectively, embedding in the category of normed vector spaces.

1.3. **Lipschitz structures.** To venture beyond these simple examples, we consider some common types of maps between metric spaces.

Definition 1.2. *A map $X \xrightarrow{\phi} M$ between metric spaces (X, d) and (M, ∂) is*

- Lipschitz *if there is a constant K so that, for all $x, y \in X$,*
$$\partial(\phi x, \phi y) \leqslant K \cdot d(x, y),$$
- Lipschitz for large distances *if there is a constant K so that, for all $x, y \in X$,*
$$\partial(\phi x, \phi y) \leqslant K \cdot d(x, y) + K,$$
- Lipschitz for short distances *if there are constants $K, \delta > 0$ so that,*
$$\partial(\phi x, \phi y) \leqslant K \cdot d(x, y)$$
whenever $x, y \in X$ satisfy $d(x, y) \leqslant \delta$.

A fact that will becomes important later on is that our definitions above provide a splitting of being Lipschitz as the conjunction of two weaker conditions. Namely, we have the following simple fact:

$$\phi \text{ is Lipschitz} \quad \Leftrightarrow \quad \phi \text{ is Lipschitz for both large and short distances.}$$

As the composition of two Lipschitz maps is again Lipschitz, the class of metric spaces also form a category where the morphisms are now Lipschitz maps. Similarly with Lipschitz for both large and short distances. However, for later purposes where there are no canonical metrics, it is better not to treat spaces with specific choices of metrics, but rather equivalence classes of these.

We therefore define three equivalence relations, namely, *bi-Lipschitz*, *quasi-isometric* and *locally bi-Lipschitz equivalence* on the set of metrics on any set X by letting

$$\begin{aligned}
d \sim_{\mathsf{Lip}} \partial \;&\Leftrightarrow\; \big(X, d\big) \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightleftarrows}} \big(X, \partial\big) \text{ are both Lipschitz}\\
&\Leftrightarrow\; \exists K \;\; \frac{1}{K} d \leqslant \partial \leqslant K \cdot d,\\
d \sim_{\mathsf{QI}} \partial \;&\Leftrightarrow\; \big(X, d\big) \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightleftarrows}} \big(X, \partial\big) \text{ are Lipschitz for large distances}\\
&\Leftrightarrow\; \exists K \;\; \frac{1}{K} d - K \leqslant \partial \leqslant K \cdot d + K\\
d \sim_{\mathsf{locLip}} \partial \;&\Leftrightarrow\; \big(X, d\big) \underset{\mathsf{id}}{\overset{\mathsf{id}}{\rightleftarrows}} \big(X, \partial\big) \text{ are Lipschitz for short distances.}
\end{aligned}$$

Example 1.3. The standard euclidean metric $d_1(x,y) = |x-y|$ on $\mathbb{R}$ is locally Lipschitz equivalent with the truncated metric $d_2(x,y) = \min\{1, |x-y|\}$. On the other hand, since the map $x \mapsto \sqrt{x}$ is not Lipschitz for short distances, these are not locally Lipschitz equivalent with the metric

$$d_3(x,y) = \sqrt{|x-y|}.$$

Eventually, when we turn to topological groups, we may occasionally pick out equivalence classes of metrics without being able to choose any particular metric. These thus become objects of the following types.

Definition 1.4. *A* Lipschitz, quasimetric*, respectively* locally Lipschitz space *is a set X equipped with a Lipschitz, quasi-isometric, respectively, locally Lipschitz equivalence class $\mathcal{D}$ of metrics on X.*

In neither of these three cases do we have an easy grasp of what the space actually is. By definition, it is *that which is invariant* under a certain class of transformations. On the other hand, morphisms are simpler. Indeed, a *morphism*

$$\big(X, \mathcal{D}_X\big) \xrightarrow{\phi} \big(M, \mathcal{D}_M\big)$$

between two Lipschitz or locally Lipschitz spaces is a map $X \xrightarrow{\phi} M$ that is Lipschitz, respectively, Lipschitz for short distances, with respect to some or equivalently any choice of metrics from the respective equivalence classes $\mathcal{D}_X$ and $\mathcal{D}_M$. In this way, Lipschitz and locally Lipschitz spaces form categories in which the isomorphisms are bijective functions that are Lipschitz (for short distances) with an inverse that is also Lipschitz (for short distances).

Just as maps that are Lipschitz for large distances need not be continuous and hence fail to capture topological notions, isomorphisms between quasimetric spaces should neither preserve topology nor record spaces' cardinality either. In analogy with homotopy equivalence of topological spaces, we therefore adjust the notion of morphism.

Definition 1.5. *Two maps $X \xrightarrow{\phi,\psi} M$ from a set X to a metric space (M,d) are* close *if*

$$\sup_{x\in X} d\big(\phi x, \psi x\big) < \infty.$$

Observe that, whether ϕ and ψ are close depends only on the quasi-isometry class of the metric d on M. We may therefore define morphisms in the category of quasimetric spaces to be closeness classes of Lipschitz for large distances maps between these spaces and where composition is computed by composing representatives of these classes.

As a consequence, a Lipschitz for large distances map $X \xrightarrow{\phi} M$ between two quasimetric spaces is a closeness representative of an *isomorphism* between X and M exactly when there is $M \xrightarrow{\psi} X$, Lipschitz for large distances, so that both $\psi\phi$ and $\phi\psi$ are close to the identities on X and M respectively, i.e., so that

$$\sup_{x\in X} d\big(\psi\phi(x), x\big) < \infty \quad \text{and} \quad \sup_{z\in M} \partial\big(\phi\psi(z), z\big)$$

for some/any choice of compatible metrics d, ∂ on X and M.

While motivating the discussion of isomorphisms here, in practice we shall often avoid equivalence classes of metrics and maps and simply work with representatives from these classes. In this way, a map between metric spaces is called a *quasi-isometry* if it is a representative for an isomorphism between the associated quasimetric spaces.

Example 1.6. The map $\mathbb{R}^n \xrightarrow{\phi} \mathbb{Z}^n$ given by $\phi(x_1, \dots, x_n) = \big(\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor\big)$ is a quasi-isometry whose inverse is the inclusion map $\mathbb{Z}^n \to \mathbb{R}^n$ when both are given the euclidean metric.

It is obvious that every metric d on a set X induces not only a metric space (X, d) but also a Lipschitz, locally Lipschitz and quasimetric space, by taking the respective equivalence classes of the metric. Moreover, since the morphisms in the category of a metric space are (not necessarily surjective) *isometries*, these are also automatically morphisms in the other categories.

On the other hand, while not every topological vector space X has a Lipschitz structure compatible with its topology, if X happens to be the reduct of a normed vector space, then all norms compatible with the topology on X are bi-Lipschitz equivalent and thus X is naturally equipped with the Lipschitz structure induced by these norms.

Though there are counter-examples in the non-separable case (see Example 7.12 [4]), the outstanding problem regarding Lipschitz structure on Banach spaces is whether this completely determines the linear structure.

Problem 1.7. Suppose X and Y are bi-Lipschitz equivalent separable Banach spaces. Must X and Y also be linearly isomorphic?

1.4. **Banach spaces as uniform spaces.** Of course every map between metric spaces that is Lipschitz for short distances is automatically uniformly continuous. In particular, this means that the uniform structures $\mathcal{U}_d$ and $\mathcal{U}_\partial$ given by two locally Lipschitz equivalent metrics d and ∂ must coincide, i.e., $\mathcal{U}_d = \mathcal{U}_\partial$. However, to give a proper presentation of this and also to motivate the category of coarse spaces, recall the definition of uniform structures.

Definition 1.8 (A. Weil [45])**.** *A* uniform space *is a set X equipped with a filter $\mathcal{U}$ of subsets $E \subseteq X \times X$, called* entourages*, satisfying*

1. $\Delta \subseteq E$ *for all* $E \in \mathcal{U}$*,*
2. *if* $E \in \mathcal{U}$*, then* $E^{-1} = \{(y,x) \mid (x,y) \in E\} \in \mathcal{U}$*,*
3. *if* $E \in \mathcal{U}$*, then* $F \circ F = \{(x,z) \mid \exists y\, (x,y),(y,z) \in F\} \subseteq E$ *for some* $F \in \mathcal{U}$*.*

Here $\Delta = \{(x,x) \mid x \in X\}$ denotes the diagonal in $X \times X$. Recall that if d is an écart (aka. pseudo-, pre- or semimetric) on a set X, i.e., d is a metric except that possibly $d(x,y) = 0$ for distinct $x, y \in X$, then the induced uniform structure $\mathcal{U}_d$ is the filter generated by the family of entourages

$$E_\alpha = \{(x,y) \mid d(x,y) < \alpha\}$$

for $\alpha > 0$.

Also, a morphism between two uniform spaces $(X, \mathcal{U})$ and $(M, \mathcal{V})$ is simply a uniformly continuous map $X \xrightarrow{\phi} M$, that is, satisfying

$$\forall F \in \mathcal{V}\ \exists E \in \mathcal{U} \colon (x,y) \in E \Rightarrow (\phi x, \phi y) \in F.$$

Again, as the notion of substructure is apparent, we obtain a notion of uniform embeddings.

Important early work on the uniform classification of Banach spaces was done by P. Enflo, J. Lindenstrauss and M. Ribe, who established a number of rigidity results for these. For example, the combined results of Lindenstrass [26] and Enflo [9] establish that if $1 \leqslant p < q < \infty$, then the spaces $L^p([0,1])$ and $L^q([0,1])$ are not uniformly homeomorphic. However, while this distinguishes between the L^p spaces, it does not tell an L^p space apart from an arbitrary space. Regarding this, W. B. Johnson, J. Lindenstrauss and G. Schechtman [20] show that if a Banach space X is uniformly homeomorphic to ℓ^p for some $1 < p < \infty$, then X is actually linearly isomorphic to ℓ^p.

For the record, let us mention that, as opposed to the Lipschitz category, it is known that the uniform structure does not determine the linear structure. Namely, by work of Ribe [35], there are examples of separable uniformly homeomorphic Banach spaces that are not linearly isomorphic. Similarly, quasimetric structure does not determine uniform structure. Indeed by a result due to Kalton [24] there are separable quasi-isometric Banach spaces that are not uniformly homeomorphic.

1.5. **Banach spaces as coarse spaces.** While we have not discussed Banach spaces viewed as quasimetric spaces, we shall now consider a weaker category that abstracts large scale content from metric spaces in a manner similar to how uniform spaces abstracts small scale content. In fact, the following definition is an almost perfect large scale counterpart to that of uniform spaces.

Definition 1.9 (J. Roe [36])**.** *A* coarse space *is a set X equipped with an ideal $\mathcal{E}$ of entourages $E \subseteq X \times X$ satisfying*

(1) $\Delta \in \mathcal{E}$,
(2) *if $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$,*
(3) *if $E \in \mathcal{E}$, then $E \circ E \in \mathcal{E}$.*

Again, if (X,d) is a pseudometric space, the associated coarse structure $\mathcal{E}_d$ is then the ideal, generated by the entourages $E_\alpha = \{(x,y) \in X \times X \mid d(x,y) < \alpha\}$, where now we require $\alpha < \infty$ rather than $\alpha > 0$.

In particular, this means that we can define two maps $Y \xrightarrow{\phi,\psi} X$ from a set Y into a coarse space $(X,\mathcal{E})$ to be *close* if there is an entourage $E \in \mathcal{E}$ so that $(\phi y, \psi y) \in E$ for all $y \in Y$. This conservatively extends the definition from the case of metric spaces.

Definition 1.10. *A map $X \xrightarrow{\phi} M$ between two coarse spaces $(X,\mathcal{E})$ and $(M,\mathcal{F})$ is* bornologous *if*

$$\forall E \in \mathcal{E}\ \exists F \in \mathcal{F}\colon (x,y) \in E \Rightarrow (\phi x, \phi y) \in F.$$

It follows that a map $(X,d) \xrightarrow{\phi} (M,\partial)$ between pseudometric spaces is bornologous if and only if there is a monotone increasing function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ so that

$$\partial(\phi x, \phi y) \leqslant \omega\big(d(x,y)\big)$$

for all $x, y \in X$.

Analogously to the category of quasimetric spaces, morphisms between coarse spaces are closeness classes of bornologous maps and so two coarse spaces $(X,\mathcal{E})$ and $(Y,\mathcal{F})$ are *coarsely equivalent* (that is, isomorphic as coarse spaces) if there are bornologous maps $X \underset{\phi}{\overset{\psi}{\rightleftarrows}} Y$ so that $\psi\phi$ and $\phi\psi$ are close to the identities on X and Y respectively.

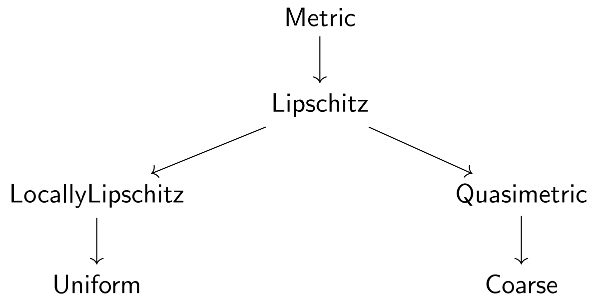

FIGURE 1. Forgetful functors between geometric categories

More concretely, note that a map $X \xrightarrow{\phi} M$ of a metric space (X, d) into a metric space (M, ∂) is a uniform embedding if

$$d(x_n, y_n) \to 0 \;\Leftrightarrow\; \partial(\phi x_n, \phi y_n) \to 0$$

for all sequences $x_n, y_n \in X$. In the same manner, $X \xrightarrow{\phi} M$ is a *coarse embedding* if, for all x_n, y_n,

$$d(x_n, y_n) \to \infty \;\Leftrightarrow\; \partial(\phi x_n, \phi y_n) \to \infty.$$

A coarse embedding is then a coarse equivalence[1] if furthermore $\phi[X]$ is *cobounded* in M, i.e.,

$$\sup_{a \in M} \mathsf{dist}(a, \phi[X]) < \infty.$$

As Lipschitz for short distances entails uniformly continuous and Lipschitz for large distances entails bornologous, we obtain a diagram of forgetful functors between the categories of metric, Lipschitz, locally Lipschitz, uniform, quasimetric and coarse spaces as in Figure 1.

Example 1.11 (Near isometries)**.** Consider the category of metric spaces in which morphisms are closeness classes of *near isometries* i.e., of maps $(X, d) \xrightarrow{\phi} (Y, \partial)$ so that

$$\kappa_\phi = \sup_{x,z \in X} |d(x, z) - \partial(\phi x, \phi z)| < \infty.$$

Then two spaces are isomorphic provided there are near isometries $X \underset{\phi}{\overset{\psi}{\rightleftarrows}} Y$ so that $\psi\phi$ and $\phi\psi$ are close to the identities on X and Y respectively. Observe that it is easy to produce isomorphic spaces that are not isometric and also automorphisms that are not close to any autoisometries.

We remark that if X and Y are Banach spaces that are isomorphic in this category, then there is a surjective near isometry $X \xrightarrow{\phi} Y$ so that furthermore $\phi(0) = 0$. Furthermore, by a result due to J. Gevirtz [15] and P. M. Gruber [19], for any such ϕ, there is a linear isometry $X \xrightarrow{T} Y$ with

$$\sup_x \|Tx - \phi x\| \leqslant 4\kappa_\phi.$$

[1]Strictly speaking, ϕ is a closeness representative of a coarse embedding.

In particular, this shows that any isomorphism is close to a surjective linear isometry and hence that the new notion of isomorphism coincides with linear isometry of spaces.

1.6. **Rigidity of morphisms and embeddability.** So far we have encountered rigidity results for isomorphisms and individual objects in the various categories. The following simple fact, on the other hand, will establish rigidity of morphisms.

Lemma 1.12 (General Corson–Klee lemma)**.** *Suppose $X \xrightarrow{\phi} E$ is a map between normed vector spaces so that, for some $\delta, \Delta > 0$ and all $x, y \in X$,*

$$\|x-y\| < \delta \;\Rightarrow\; \|\phi x - \phi y\| < \Delta.$$

Then ϕ is Lipschitz for large distances.

Proof. Given $x, y \in X$, let n be minimal so that $\|x-y\| < n \cdot \delta$. Then there are $v_0 = x, v_1, \ldots, v_n = y$ so that $\|v_i - v_{i+1}\| < \delta$ for all i. It thus follows that

$$\|\phi x - \phi y\| \leqslant \sum_{i=0}^{n-1} \|\phi v_i - \phi v_{i+1}\| < n \cdot \Delta.$$

Therefore $\|\phi x - \phi y\| < \frac{\Delta}{\delta} \cdot \|x-y\| + \Delta$. □

In particular, both a uniformly continuous and a bornologous map between two Banach spaces is automatically Lipschitz for large distances. Similarly, a uniform homeomorphism or a coarse equivalence between Banach spaces is also a quasi-isometry. On the other hand, since a uniform or coarse subspace of a Banach space need not be the reduct of linear subspace itself, a uniform or coarse embedding between Banach spaces is not in general a quasi-isometric embedding.

Remark 1.13 (Reconstruction functors)**.** The above comments show that, when we restrict the attention to reducts of Banach or just normed vector spaces, there are reconstruction functors going from the categories of uniform, respectively coarse spaces to quasimetric spaces. Namely, suppose $\mathcal{U}$ is the uniform structure induced from some normed vector space structure on the set X. Then we let $\mathbf{F}(X, \mathcal{U}) = (X, \mathcal{D})$ be the quasimetric space induced by some or, equivalently, any normed vector space structure on the set X that is compatible with the uniformity $\mathcal{U}$. Indeed, if $(X, +, \|\cdot\|)$ and $(X, \oplus, \|\!|\cdot|\!\|)$ are two such normed vector space structures, then

$$(X, \|\cdot\|) \xrightarrow{\mathsf{id}} (X, \|\!|\cdot|\!\|)$$

is a uniform homeomorphism and thus a quasi-isometric equivalence. It thus follows that the quasi-isometric equivalence classes of the norm metrics actually coincide.

Similarly, every map between Banach spaces that is Lipschitz for short distances is automatically Lipschitz for large distances and hence actually Lipschitz (for all distances). So this provides a functor from the category of Banach spaces viewed as locally Lipschitz spaces to the category of Banach spaces viewed as Lipschitz spaces.

At this point, we can refer to Figure 2 for a diagram of categories and the functors relating them. All categories refer exclusively to reducts of separable real Banach spaces and the black arrows to functors. Also, blue arrows refer to a rigidity result for isomorphism. For example, an isomorphism in the category of

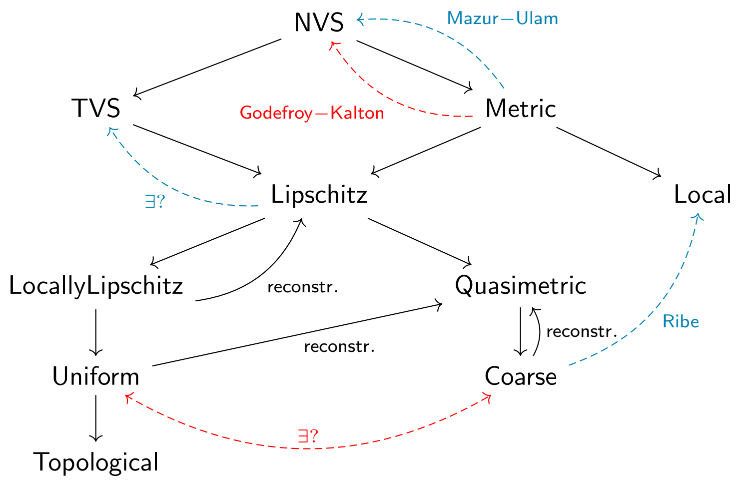

FIGURE 2. Commutative diagram of functors between diverse categories of reducts of separable real Banach spaces. Dashed blue and red arrows refer to rigidity results for isomorphisms, respectively, embeddings.

metric spaces induces another isomorphism in the category of normed vector spaces by the Mazur–Ulam theorem.

Again, while a functor maps isomorphisms to isomorphisms, it need not preserve embeddings, since the latter notion is not intrinsic to the category. Thus, while a uniform embedding between Banach spaces is bornologous, it need not be a coarse embedding. Nevertheless, we do have rigidity results for embeddings not stemming from functors. Indeed, for separable Banach spaces, by the Godefroy–Kalton theorem, isometric embeddings give rise to other linear isometric embeddings. This rigidity is indicated by a red arrow in Figure 2.

Now, though by [24] there are separable quasi-isometric Banach spaces that are not uniformly homeomorphic, it is an open problem due to Kalton whether the notions of uniform and coarse embeddability between Banach spaces coincide.

Problem 1.14 (Kalton)**.** *Are the following two conditions equivalent for all (separable) Banach spaces X and E?*

(1) *X uniformly embeds into E,*
(2) *X coarsely embeds into E.*

Observe that this is far from being trivial, since it is easy to produce uniform embeddings that are not coarse embeddings and vice versa. Also, one cannot hope to replace coarse embeddings by quasi-isometric embeddings, since ℓ^1 embeds into ℓ^2 uniformly, but not quasi-isometrically.

Theorem 1.15. *Assume X and E are Banach spaces and that $E \oplus E$ embeds as a topological vector space into E. Suppose also $X \xrightarrow{\phi} E$ is uniformly continuous and that, for some $\delta, \Delta > 0$,*

$$\|x-y\| > \Delta \;\Rightarrow\; \|\phi x - \phi y\| > \delta.$$

Then there is a simultaneously uniform and coarse embedding $X \xrightarrow{\psi} E$.

Proof. As $E \oplus E$ embeds into E, we may inductively construct three sequences E_n, Z_n, V_n of closed linear subspaces of E so that $E_n \cong Z_n \cong E$ (i.e., isomorphic as topological vector spaces),

$$E_{n+1} \oplus Z_{n+1} \subseteq Z_n$$

and

$$V_n = E_1 \oplus E_2 \oplus \ldots \oplus E_n \oplus Z_n.$$

Indeed, we simply begin with an isomorphic copy V_1 of $E \oplus E$ inside of E and let E_1 and Z_1 be respectively the first and second summand. Again, pick a copy of $E \oplus E$ inside of Z_1 with first and second summand denoted respectively E_2 and Z_2 and let $V_2 = E_1 \oplus E_2 \oplus Z_2 \subseteq V_1$, etc.

Let also P_n denote the projection of V_n onto the summand E_n along the decomposition above. While each P_n is bounded, there need not be any uniform bound on their norms. Note now that $V_1 \supseteq V_2 \supseteq \ldots$, so we can let $V = \bigcap_{n=1}^{\infty} V_n$, which is a closed linear subspace of E containing all of the E_n. Moreover, the P_n all restrict to bounded projections $P_n \colon V \to E_n$ so that $E_m \subseteq \ker P_n$ whenever $n \neq m$.

Composing ϕ with linear isomorphisms between E and E_n, we get a sequence of uniformly continuous maps $X \xrightarrow{\phi_n} E_n$ satisfying $\|x - y\| > \Delta_n \;\Rightarrow\; \|\phi x - \phi y\| > \delta_n$ for some $\Delta_n, \delta_n > 0$ and bounded projections $P_n \colon V \to E_n$ so that $E_m \subseteq \ker P_n$ for $n \neq m$. By Lemma 1 [37], this implies that X admits a simultaneously coarse and uniform embedding into V and thus into E. □

Observe that, if $X \xrightarrow{\phi} E$ is either a uniform or coarse embedding between Banach spaces, then there are $\Delta, \delta > 0$ as in Theorem 1.15. Therefore, apart from the mild assumption that $E \oplus E$ embeds as a topological vector space into E, we have the implication (1)⇒(2) in Problem 1.14.

Corollary 1.16. *Suppose X and E are Banach spaces so that $E \oplus E$ embeds as a topological vector space into E, Then, if X uniformly embeds into E, X also coarsely embeds into E.*

On the other hand, if a coarse embedding could always be strengthened to be uniformly continuous, then we would essentially have proved the converse direction (2)⇒(1). However, one must contend with the following serious obstruction.

Theorem 1.17 (Naor [32])**.** *There is a bornologous map $X \xrightarrow{\phi} E$ between separable Banach spaces that is not close to any uniformly continuous map.*

The above results indicate that the uniform structure of a Banach space is more rigid than the coarse structure. However, once we pass to the underlying topology, almost no information is left. Indeed, it is a result of M. I. Kadets [21] and H. Toruńczyk [43] that any two infinite-dimensional Banach spaces of the same density character are homeomorphic. For example, the separable infinite-dimensional Banach spaces are all homeomorphic to a countable product of lines, $\mathbb{R}^{\mathbb{N}}$.

Remark 1.18 (Universal spaces)**.** In the various categories above, it is interesting to search for *universal spaces*, that is, separable spaces into which every other separable spaces embeds. For example, a classical result states that, for K an uncountable compact metric space, $C(K)$ is universal in the category NVS; every separable Banach space admits an isometric linear embedding into $C(K)$. Similarly, by a result of I. Aharoni [1], c_0 is universal in the category *Lipschitz*.

In contradistinction to this, F. Baudier, G. Lancien and T. Schlumprecht [3] recently showed that the is no infinite-dimensional space that coarsely embeds into all infinite-dimensional spaces. And when combined with a result of Y. Raynaud [33], one sees that the same holds for uniform embeddings.

1.7. **Banach spaces as local objects.** The results of Enflo, Johnson, Lindenstrauss and Schechtman [9, 26, 20] mentioned earlier show rigidity for the uniform structure of the individual spaces $L^p([0,1])$ and ℓ^p. However, there is also a beautiful rigidity result due to Ribe encompassing all Banach spaces. To explain this, we need a technical concept.

Definition 1.19. *A Banach space X is said to be* crudely finitely representable *in a Banach space Y if there is a constant K so that, for every finite-dimensional subspace $E \subseteq X$, there is a finite-dimensional subspace $F \subseteq Y$ and a linear isomorphism $E \xrightarrow{T} F$ with $\|T\| \cdot \|T^{-1}\| \leqslant K$.*

We then say that X and Y are *locally isomorphic* in case they are crudely finitely representable in each other. In [34], Ribe then establishes the surprising fact that any two uniformly homeomorphic spaces must be locally isomorphic. In particular, this implies that all local properties of Banach spaces, i.e., that only depend on the finite-dimensional subspaces (up to some uniform constant of isomorphism) is in principle expressible in terms of the uniform structure of the entire space. This, in turn, has motivated to so called *Ribe programme* (see, e.g., Naor [31]) of identifying exclusively metric expressions for these various local invariants of Banach spaces such as convexity, smoothness, type and cotype, which furthermore then become applicable not only in the linear setting but to metric spaces in general.

Subsequent proofs of Ribe's theorem go by showing that if X and Y are quasi-isometric separable spaces, then X and Y have bi-Lipschitz equivalent ultrapowers $X^{\mathcal{U}}$ and $Y^{\mathcal{U}}$. Moreover, if $V \xrightarrow{\phi} W$ is a bi-Lipschitz embedding of a separable Banach space V into a Banach space W, then using differentiation techniques, V embeds as a topological vector space into W^{**}. In particular, the diagonal copy of X in $X^{\mathcal{U}}$ embeds as a topological vector space into $(Y^{\mathcal{U}})^{**}$. Now, by the principle of local reflexivity, $(Y^{\mathcal{U}})^{**}$ is crudely finitely representable in $Y^{\mathcal{U}}$ and, by the nature of ultrapowers, $Y^{\mathcal{U}}$ is crudely finitely representable in Y. Combined, this shows that if X and Y are quasi-isometric separable spaces, then X is crudely finitely representable in Y and vice versa, i.e., X and Y are locally isomorphic. As uniformly homeomorphic or coarsely equivalent spaces are also quasi-isometric, Ribe's theorem follows.

One may think of Banach spaces as objects in the category Local of *local spaces* in the following sense. The objects of the category are simply Banach spaces and we put an arrow $X \to Y$ from X to Y in case X is crudely finitely representable in Y. Observe that, in this way, an arrow $X \to Y$ does not necessarily correspond to the existence of a special type of function from X to Y. However, if X isometrically embeds into Y, then X also linearly isometrically embeds and thus is crudely finitely representable in Y. This means that we obtain a last functor from the category of metric reducts of separable Banach spaces to Local.

In Figure 2, Ribe's Theorem is indicated as an arrow from the category Coarse to Local. His original rigidity theorem, that is the arrow from Uniform to Local, is then obtained by composition with the functors from Uniform to Quasimetric and further onto Coarse.

When we restrict the category Local to infinite-dimensional spaces, we have initial and terminal objects X and Y, that is, so that for every Z there are (trivially unique) arrows

$$X \to Z \to Y.$$

Indeed, by a result of A. Dvoretzky, Hilbert space ℓ^2 is crudely finitely representable in every infinite-dimensional Banach space (see [13]), while, by a result of S. Kwapień [25], any space crudely finitely representable in ℓ^2 has type and cotype 2 and must be isomorphic to ℓ^2 as a topological vector space. Thus, up to isomorphism, ℓ^2 is the unique initial object.

On the other hand, Y is a terminal object exactly when ℓ^∞ is crudely finitely representable in Y, which by a result of B. Maurey and G. Pisier [28] is equivalent to Y only having trivial cotype. This shows that, for example, c_0 and the reflexive space

$$(\ell^\infty(2) \oplus \ell^\infty(3) \oplus \ldots)_{\ell^2}$$

are terminal.

There is also a concept of locally minimal spaces for which one may established a dichotomy [12]. Specifically, every infinite-dimensional Banach space contains an infinite-dimensional closed linear subspace X satisfying one of

(1) X is crudely finitely representable in all its infinite-dimensional subspaces,
(2) X has a Schauder basis $(x_n)_{n=1}^\infty$ so that no infinite-dimensional subspace $Y \subseteq X$ is crudely finitely representable in all tail subspaces $X_m = [x_n]_{n=M}^\infty$ with a uniform constant.

2. Geometric structures on topological groups

2.1. **Uniform and locally Lipschitz structure.** In the preceding section, we have introduced various geometric structures through the instructive example of Banach spaces. In this case, once the categories are understood, there is no discussion of what the appropriate structure of a Banach space is, since it is just obtained by stripping away information. Also, we saw how one may reconstruct, e.g., affine structure from the metric structure and quasimetric structure from the uniform structure. However, for topological groups, the problem is rather how and when one may endow the group with a canonical structure of a given type.

Recall that a topological group is simply a group G equipped with a topology in which the group operations are continuous. Even a Lie group may be (depending on who you ask) a locally compact locally euclidean group and thus simply a special type of topological group without any further structure. For simplicity, **all groups will be assumed to be Hausdorff**.

Now, apart from being a topological space, a topological group G also has a couple of canonical uniform structures associated with it. The most interesting in this context in the *left-uniform structure* $\mathcal{U}_L$ which is the filter on $G \times G$ generated by entourages

$$E_V = \{(x, y) \in G \times G \mid x^{-1}y \in V\},$$

where V ranges over identity neighbourhoods in G. Observe that if $(X, +)$ is the additive topological group of a Banach space, this is simply the uniform structure of the norm.

As always, with uniform spaces it is often useful to work with écarts generating the uniformity and, in the case of groups, one can even require these to be compatible with the algebraic structure. Indeed, an écart d is said to be *left-invariant* if

$$d(xy, xz) = d(y, z)$$

for all $x, y, z \in G$.

Theorem 2.1 (A. Weil [45]). *The left-uniform structure $\mathcal{U}_L$ on a topological group G is given by*

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d,$$

where the union is over all continuous left-invariant écarts d on G.

In fact, prior to this, independently G. Birkhoff [5] and S. Kakutani [22] showed that if a topological group G satisfies a weak consequence of metrisability, namely, if only it is first countable, then G in fact admits a *compatible* left-invariant metric d, i.e., inducing the topology of G. Moreover, in this case, by left-invariance, this metric will also be compatible with the left-uniform structure, that is, $\mathcal{U}_L = \mathcal{U}_d$. In short, the following properties are equivalent for an arbitrary topological group G.

(1) $\mathcal{U}_L$ is metrisable,
(2) G admits a compatible left-invariant metric,
(3) G is first countable.

Apart from exceptional circumstances, one should not expect that a canonical metric, even up to rescaling, should exist on a metrisable topological group. Nevertheless, it is instructive to look at what such a metric should do for us. Since the general case is not much different, we shall not assume metrisability of G at the outset and hence deal with écarts rather than metrics.

First of all, an écart d should be continuous. This ensures that the induced topology τ_d is coarser than that τ_G of G itself. Secondly, to enforce compatibility with the algebraic structure, we should also require the écart to be left-invariant, which then guarantees that the uniform structure $\mathcal{U}_d$ is coarser than the left-uniform structure $\mathcal{U}_L$. Finally, in case G is metrisable, d can be assumed to be compatible with the topology, whereby actually $\mathcal{U}_L = \mathcal{U}_d$. By the results of Birkhoff, Kakutani and Weil cited above, these requirements can always be fulfilled.

These were the general requirements. Now, how could we identify a canonical locally Lipschitz structure on G? Since a locally Lipschitz structure automatically gives us a metrisable uniform structure, we focus exclusively on metrics.

Definition 2.2. [38] *A compatible left-invariant metric d on a topological group G is said to be* minimal *if, for every other compatible left-invariant metric ∂ on G,*

$$(G, \partial) \xrightarrow{\mathsf{id}} (G, d)$$

is Lipschitz for short distances.

In fact, this definition relies on a quasiordering of metrics by setting $d \ll_L \partial$ if $(G, \partial) \xrightarrow{\mathsf{id}} (G, d)$ is Lipschitz for short distances. Then a minimal metric is just a minimum element in this ordering.

Clearly any two minimal metrics are locally Lipschitz equivalent and thus define a locally Lipschitz structure on G, which furthermore is compatible with the left-uniform structure on the group. On the other hand, the definition of minimal metrics is highly impredicative as it involves a quantification over objects of the same type, namely the class of left-invariant metrics. So is there a characterisation quantifying only over G?

Theorem 2.3. [38] *The following conditions are equivalent for a compatible left-invariant metric d on a topological group G.*

(1) *d is minimal,*
(2) *there are an identity neighbourhood U and a constant K so that, for all $n \geqslant 1$ and $x \in G$,*
$$x, x^2, x^3, \ldots, x^n \in U \Rightarrow n \cdot d(x, 1) \leqslant K \cdot d(x^n, 1),$$
(3) *there are constants K and $\epsilon > 0$ so that, for all $n \geqslant 1$ and $x \in G$,*
$$d(x, 1) \leqslant \frac{\epsilon}{n} \Rightarrow n \cdot d(x, 1) \leqslant K \cdot d(x^n, 1).$$

Observe that by left-invariance we automatically have $d(x^n, 1) \leqslant n \cdot d(x, 1)$. So condition (2) is a linear growth condition on the associated length function $\ell(x) = d(x, 1)$ in an identity neighbourhood of the group. Clearly condition (2) is much simpler that the initial definition of minimality and also has the non-trivial consequence that the restriction of a minimal metric d on G to a subgroup $H \leqslant G$ with the induced topology is also minimal on H.

Nevertheless, while we have a simple characterisation of minimal metrics, we do not have any informative reformulation of which groups admit a minimal metric and hence, equivalently, a locally Lipschitz structure. The language of descriptive set theory allows us to make this question precise at least for the well-behaved class of *Polish* groups, that is, completely metrisable separable topological groups. Concretely, the class of Polish groups can be parametrised by a standard Borel space Gp, e.g., by letting Gp be the Effros–Borel space of closed subgroups of some injectively universal Polish group such as the homeomorphism group $\mathsf{Homeo}([0, 1]^{\mathbb{N}})$ of the Hilbert cube [44].

Problem 2.4. *The class of Polish groups admitting a minimal metric is it Borel in the standard Borel space Gp of Polish groups?*

A positive answer would show that one can characterise these groups without simply asking for an object of the same complicated type as a minimal metric itself. For locally compact second countable groups, the answer to Problem 2.4 is already known. Indeed, condition (3) of Theorem 2.3 appears under the name *weak Gleason metric* in T. Tao's book [42] and from Tao's exposition of A. Gleason, D. Montgomery, H. Yamabe and L. Zippin's solution to Hilbert's fifth problem the following equivalent conditions for a locally compact second countable group emerge.

(1) G is locally Euclidean,
(2) G has *no small subgroups*, that is, there is an identity neighbourhood in G not containing any non-trivial subgroup,
(3) G has a weak Gleason metric,
(4) G is a Lie group.

Thus, by Theorem 2.3, a locally compact second countable group has a canonical locally Lipschitz structure if and only if it is a Lie group.

Beyond locally compact Lie groups, examples of minimal metrics are the operator norm metrics on the unitary groups of complex unital Banach algebras. That is, if

$\mathfrak{A}$ is a complex unital Banach algebra, then the operator norm metric is minimal on

$$\mathcal{U}(\mathfrak{A}) = \{u \in \mathfrak{A} \mid \|u\| = 1 \ \&\ \exists v \in \mathfrak{A}\ \|v\| = 1 \ \&\ uv = vu = \mathbf{1}\}.$$

This is an unpublished result of C. Badea relying essentially on a classical result of I. M. Gelfand [14] characterising the unity $\mathbf{1}$ as the unique doubly power-bounded element $a \in \mathfrak{A}$ with spectrum $\sigma(a) = \{1\}$. For example, $\mathcal{U}(\mathfrak{A})$ could be the group of linear isometries of a complex Banach space with the operator norm metric.

Given that the locally compact second countable groups with minimal metrics are exactly the Lie groups, it is not too surprising that some amount of Lie group structure should follow from having a minimal metric even in the general case. The following result (slightly simplifying a result of [38]) has its origins in work of C. Chevalley [6], Enflo [10] and Gleason [16].

Theorem 2.5. *Let G be a completely metrisable group with a minimal metric and suppose that, for every identity neighbourhood W, the set of squares*

$$\{g^2 \mid g \in W\}.$$

is dense in a neighbourhood of 1. *Then there is an identity neighbourhood V so that, for each $f \in V$, there is a unique* 1*-parameter subgroup $(h^\alpha)_{\alpha\in\mathbb{R}}$ with $f = h^1$ and $h^\alpha \in V$ for all $\alpha \in [-1, 1]$.*

Proof. By Theorem 25 [38], there are open identity neighbourhoods $U \supseteq O$ so that, for every $h \in O$, there is a unique continuous 1-parameter subgroup (h^α) with $h^1 = f$ and $h^\alpha \in U$ for all $\alpha \in [-1, 1]$. Moreover, by the last paragraph of the proof, U can be supposed to be arbitrarily small.

In particular, if d is the minimal metric, we can suppose there is a constant K so that

$$g, g^2, g^3, \ldots, g^n \in U \Rightarrow n \cdot d(g, 1) \leqslant K \cdot d(g^n, 1).$$

Assume that (h^α) is this unique 1-parameter subgroup associated to some element $h = h^1 \in O$. Then

$$\sup_{\alpha\in[-1,1]} d(h^\alpha, 1) \leqslant K \cdot d(h^1, 1).$$

Indeed, if $d(h^\alpha, 1) > \epsilon$ for some $\alpha \in [-1, 1]$ and $\epsilon > 0$, then there is a rational number $\frac{k}{n} \in [-1, 1]$ so that also $d(h^{\frac{k}{n}}, 1) > \epsilon$, whence $d(h^{\frac{1}{n}}, 1) > \frac{\epsilon}{k}$. By symmetry, assume that $\frac{k}{n} > 0$. Then

$$h^{\frac{1}{n}}, h^{\frac{2}{n}}, \ldots, h^{\frac{n}{n}} \in U$$

whereby $n \cdot d(h^{\frac{1}{n}}, 1) \leqslant K \cdot d(h^1, 1)$ and thus $\epsilon \leqslant \frac{n\epsilon}{k} < n \cdot d(h^{\frac{1}{n}}, 1) \leqslant K \cdot d(h^1, 1)$.

It follows that the set

$$V = \{h \in O \mid h^\alpha \in O \text{ for all } \alpha \in [-1, 1]\}$$

is an identity neighbourhood. Indeed, fix $\epsilon > 0$ small enough so that $B_d(K\epsilon) \subseteq O$. Then, if $d(h, 1) < \epsilon$, we have $d(h^\alpha, 1) \leqslant Kd(h^1, 1) < K\epsilon$ and so $h^\alpha \in O$ for all $\alpha \in [-1, 1]$. Thus $B_d(\epsilon) \subseteq V$.

Note finally that, if $h \in V$, then $h^\alpha \in V$ for all $\alpha \in [-1, 1]$, since $g^\beta := h^{\beta\alpha}$ is the unique 1-parameter subgroup associated with $g^1 = h^\alpha$ and $g^\beta = h^{\beta\alpha} \in O$ for all $\beta \in [-1, 1]$, i.e., $h^\alpha = g^1 \in V$. □

2.2. **Coarse and quasimetric structure.** Of course having a minimal metric is already a restrictive condition among locally compact groups and one should not expect it to be ubiquitous in other settings either. So let us instead turn our attention to quasimetric and coarse geometry.

Example 2.6 (Finitely generated groups)**.** The standard and indeed motivating example of a quasimetric geometry is that induced by the word metric

$$\rho_S(x,y) = \min(k \mid \exists s_1,\ldots,s_k \in S^{\pm} \colon x = ys_1\cdots s_k)$$

on a group Γ generated by a finite subset $S \subseteq \Gamma$. The fundamental observation of geometric group theory is that this geometry is independent of the specific finite generating set S. Indeed, if T is another finite generating set, then there is a k so that each element of T can be written as a word of length at most k in S and so one sees that $\rho_T \leqslant k\cdot\rho_S$. By symmetry, it thus follows that the two metrics are bi-Lipschitz equivalent and hence define the same quasimetric and even Lipschitz structure.

Example 2.7 (Compactly generated groups)**.** A similar argument applies to compactly generated locally compact groups. Namely, if M and L are two symmetric compact generating sets containing 1 for a locally compact group G, then $M \subseteq M^2 \subseteq M^3 \subseteq \ldots$ is an exhaustive sequence of compact subsets and thus, by the Baire category theorem, some M^l has non-empty interior and therefore covers L by finitely many left-translates. It thus follows that $L \subseteq M^k$ for some $k \geqslant 1$ and therefore as in Example 2.6 the two word metrics ρ_M and ρ_L are Lipschitz equivalent. However, though left-invariant, the word metrics are no longer compatible with the topology on G unless G itself is discrete. But a simple argument using the construction of Birkhoff and Kakutani allows us to find a continuous left-invariant écart d representing the same quasi-isometry class as ρ_M and ρ_L.

The recent book by Y. de Cornulier and P. de la Harpe [7] provides a complete introduction to the geometric group theory of locally compact groups.

Example 2.8 (Fragmentation metrics on homeomorphism groups)**.** Fix a closed manifold M and let $\mathsf{Homeo}_0(M)$ be the identity component of the homeomorphism group equipped with the compact-open topology. We note that $\mathsf{Homeo}_0(M)$ consists of the isotopically trivial homeomorphisms and hence is connected. Fix also a covering $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ of M by embedded open balls and let $U_i \subseteq \mathsf{Homeo}_0(M)$ be the set of homeomorphisms g with $\mathsf{supp}(g) \subseteq B_i$. By results of R. D. Edwards and R. C. Kirby [8], every $g \in \mathsf{Homeo}_0(M)$ sufficiently close to the identity can be factored as $g = h_1\cdots h_n$ with $h_i \in U_i$. In other words,

$$U = U_1U_2\cdots U_n$$

is an identity neighbourhood in $\mathsf{Homeo}_0(M)$. Moreover, as $\mathsf{Homeo}_0(M)$ is connected, this means that U generates $\mathsf{Homeo}_0(M)$. The word metric ρ_U is called the *fragmentation metric* associated to the cover $\mathcal{B}$. Furthermore, as shown by E. Militon [30], any two such covers produce quasi-isometric fragmentation metrics and thus define a canonical quasimetric structure on $\mathsf{Homeo}_0(M)$.

Observe however that the definition on the fragmentation norm is not a priori intrinsic to the topological group, but rather depends on viewing $\mathsf{Homeo}_0(M)$ as a transformation group of the manifold M, that is, depends on the group $\mathsf{Homeo}_0(M)$ along with its tautological action $\mathsf{Homeo}_0(M) \curvearrowright M$.

The above examples give us concrete quasimetric structures induced by word metrics, including on groups that don't have small generating sets in any reasonable topological sense. For other specific transformation groups there may be similar constructions, but is there a way to see these as instances of a general construction that applies to all groups? The correct way of doing this is to take serious the idea that a coarse structure is somehow dual to uniform structure (without implying that there is an actual duality between these categories). We thus dualise Weil's Theorem 2.1 into a definition as follows.

Definition 2.9. [39] *The* left-coarse structure $\mathcal{E}_L$ *on a topological group G is given by*

$$\mathcal{E}_L = \bigcap_d \mathcal{E}_d,$$

where the intersection ranges over all continuous left-invariant écarts d on G.

As with uniform spaces, *metrisable* coarse spaces $(X, \mathcal{E})$, that is, so that $\mathcal{E} = \mathcal{E}_d$ for some metric or equivalently some écart d on the set X, are much simpler to understand than the general case. So let us call an écart d on X *coarsely proper* if it induces the coarse structure on X, i.e., if $\mathcal{E} = \mathcal{E}_d$.

Every coarse space $(X, \mathcal{E})$ has an associated *bornology* of bounded sets, i.e., an ideal $\mathcal{B}$ of subsets of X with $X = \bigcup_{B\in\mathcal{B}} B$. Namely, $B \subseteq X$ is said to be *bounded* if $B \times B \in \mathcal{E}$. A subset B of a topological group G is then bounded exactly when

$$\mathsf{diam}_d(B) < \infty$$

for every continuous left-invariant écart d on G. By left-invariance and continuity of the écarts defining $\mathcal{E}_L$, the bornology of bounded sets in G is furthermore stable under the operations

$$A \mapsto \mathsf{cl}(A), \quad A \mapsto A^{-1}, \quad (A, B) \mapsto A \cdot B.$$

Moreover, a continuous left-invariant écart d on G is coarsely proper provided the d-bounded sets are exactly the bounded sets of G.

Example 2.10 (Proper metrics)**.** Clearly every compact subset of a topological group is bounded. But conversely, by a theorem of R. A. Struble [40], every locally compact second countable group G admits a compatible left-invariant *proper* metric, i.e., whose closed bounded sets are all compact. It follows that the bounded sets in G are exactly the relatively compact sets and hence that d is also coarsely proper. In particular, the coarse structure on a countable discrete group is that given by any left-invariant metric whose balls are finite.

As for minimal metrics, the characterisation of bounded sets involves quantification over a large sets of écarts, so one would like a simpler criterion for boundedness and thus coarse properness too. For a Polish group G, we have a much better operative criterion. Namely, a subset B is bounded if and only if, for every identity neighbourhood V there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$B \subseteq (FV)^k,$$

which shows that the ideal of bounded sets is Borel in the Effros Borel space $\mathcal{F}(G)$ of closed subsets of G. Furthermore, we have an analogue of Birkhoff and Kakutani's characterisation of metrisable groups above.

Theorem 2.11. [39] *The following conditions are equivalent for a Polish group G,*

(1) *the coarse structure* $\mathcal{E}_L$ *is metrisable,*
(2) G *admits a compatible left-invariant coarsely proper metric,*
(3) G *is* locally bounded*, i.e., has a bounded identity neighbourhood.*

As it is straightforward to see that no identity neighbourhood in the Polish group

$$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \ldots$$

is bounded, this shows that not every Polish group has metrisable coarse structure. Nevertheless, most important transformation groups do and, in fact, they often admit a canonical compatible quasimetric structure.

To address how a quasimetric structure compatible with the coarse structure may be defined, we first define a quasiordering of continuous left-invariant écarts on G by

$$\partial \lll d \;\Leftrightarrow\; (G,d) \xrightarrow{\mathsf{id}} (G,\partial) \text{ is bornologous.}$$

Then the coarsely proper écarts are simply the maximum elements of the ordering $\lll$. Refining $\lll$, we set

$$\partial \ll d \;\Leftrightarrow\; (G,d) \xrightarrow{\mathsf{id}} (G,\partial) \text{ is Lipschitz for large distances}$$

and say that a continuous left-invariant écart is *maximal* if maximum in this ordering. Since the sum of two écarts is still an écart, these are directed orderings and hence maximal elements are automatically maximum too. Also, as any two maximal écarts are obviously quasi-isometric, when they exist they induce an inherent quasimetric structure on G identifiable exclusively from the topological group structure. Moreover, because maximal écarts are also coarsely proper, the quasimetric structure is automatically compatible with the coarse structure.

As always, we are left with three main issues, namely, (i) finding simpler operative charaterisations of maximal metrics, (ii) determine criteria for their existence and (iii) analyse concrete groups.

Proposition 2.12. *The following are equivalent for a continuous left-invariant écart* d *on a topological group,*

(1) d *is maximal,*
(2) d *is coarsely proper and* large scale geodesic*, that is, for some constant* K *and all* $x,y \in G$*, there are* $z_0 = x, z_1, \ldots, z_n = y$ *so that* $d(z_{i-1},z_i) \leqslant K$ *and*
$$\sum_{i=1}^{n} d(z_{i-1},z_i) \leqslant K \cdot d(x,y),$$
(3) d *is quasi-isometric to the word metric* ρ_B *given by a bounded generating set* $B \subseteq G$*.*

From condition (2), one easily gets that every outright geodesic metric is maximal and hence that the norm induces the quasimetric structure of the additive group $(X,+)$ of a Banach space. Since the norm metric is evidently also minimal, we see that both the locally Lipschitz and quasimetric structures on $(X,+)$ are what they should be, namely, those given by the norm.

One may also use condition (3) to given a simple criterion for when, e.g., Polish groups have maximal metrics and hence canonical quasimetric structure. But first a word of caution. Even for a Polish group, it is not true that the word metric ρ_B of every bounded generating set $B \subseteq G$ will induce the quasimetric structure. But, if ρ_B is known to be quasimetric to a compatible metric on G, then it does.

Theorem 2.13. *A Polish group G admits a maximal metric and thus a quasimetric structure if and only if G is algebraically generated by a bounded subset $B \subseteq G$. Moreover, in this case, the word metric $\rho_{\overline{B}}$ associated to $\overline{B}$ induces the quasimetric structure.*

Our examples before can now be seen as instances of this general setup and, in addition, many other groups have easily calculable quasimetric structure.

- Let Γ be a finitely generated group. Then the quasimetric structure of the discrete topological group Γ is simply that given by the word metrics.
- If G is a compactly generated locally compact second countable group, the quasimetric structure of G is that given by the word metric ρ_K where K is any compact generating set.
- If M is a closed manifold, the quasimetric structure on $\mathsf{Homeo}_0(M)$ is that given by the fragmentation metric. In particular, the fragmentation metric is intrinsic to the topological group $\mathsf{Homeo}_0(M)$ without knowledge of its tautological action on M [27].
- Let T_n be the n-regular simplicial tree for $n = 2, 3, 4, \ldots, \aleph_0$ and equip its automorphism group $\mathsf{Aut}(T_n)$ with the permutation group topology in which vertex stabilisers are declared to be open. Then, for any vertex $t \in T_n$, the orbit map
$$g \in \mathsf{Aut}(T_n) \mapsto g(t) \in T_n$$
is a quasi-isometry between $\mathsf{Aut}(T_n)$ and T_n.

Observe that, in the last example, when n is finite, $\mathsf{Aut}(T_n)$ is compactly generated locally compact. However, for $n = \aleph_0$, i.e., when the valency is denumerable, then $\mathsf{Aut}(T_n)$ is only Polish and thus cannot be compactly generated.

Of course, not every group has an inherent quasimetric structure, i.e., a maximal écart. For example, a countable, but not finitely generated, group will be such. It has a metrisable coarse structure, but any attempt at constructing a finer quasimetric structure will involve choices not dictated by the topological group structure.

With this framework in place, it is now possible develop substantial parts of geometric group theory in this larger setting; see [39] for an account. However, one must caution that there are dramatic changes from the theory of finitely generated or even locally compact groups to this more general setting. For example, if H is a closed subgroup of G, then the inclusion mapping is automatically a uniform embedding and, if G and H are locally compact second countable, then it is also a coarse embedding. On the contrary, if G and H are no longer locally compact, H is in general not coarsely embedded in G and so, as opposed to minimal metrics, a coarsely proper metric on G need not restrict to a coarsely proper metric on H. This phenomenon is similar to the fact that a finitely generated subgroup of a finitely generated group may not be quasi-isometrically embedded and leads to substantial complications and new aspects of the theory that one must contend with.

One of the many beautiful results of M. Gromov's fundamental work on geometric group theory is the fact that quasi-isometric equivalence between finitely generated groups Γ and Λ is equivalent to the groups admitting a *topological coupling*, that is, a pair
$$\Gamma \curvearrowright X \curvearrowleft \Lambda$$

of commuting proper cocompact actions by homeomorphisms on a locally compact Hausdorff space X (Theorem $0.2.C_2'$ [18]). On the one hand, this shows that one can pass from a weak metric equivalence between Γ and Λ to a more robust dynamical equivalence. On the other hand, it also provides the vantage point from which several other notions of couplings, e.g., measure theoretical, may be defined.

One direction of Gromov's theorem is rather straightforward and works in a wider generality. Namely, if $\Gamma \curvearrowright X \curvearrowleft \Lambda$ is a topological coupling, one may define a coarse equivalence $\Gamma \xrightarrow{\phi} \Lambda$ by simply requiring that, for some fixed $x \in X$ and compact set $K \subseteq X$ with $\Gamma \cdot K = X = K \cdot \Lambda$, we have

$$x \in gK\phi(g)^{-1}$$

for all $g \in \Gamma$.

For the other direction, one lets Γ and Λ act on the space Λ^Γ of functions from Γ to Λ by pre and post composing with the left shifts of the groups on themselves. Clearly the actions commute and one may simply take $X = \overline{\Gamma \cdot \phi \cdot \Lambda}$, which turns out to be locally compact.

If one tries to repeat this second construction for a coarse equivalence $G \xrightarrow{\phi} H$ between locally compact groups, one quickly realises that the action $G \curvearrowright X \subseteq H^G$ will not in general be continuous unless ϕ is uniformly continuous. Nevertheless, Gromov's theorem remains true for locally compact groups [2] and even in a much wider setting.

For this, a continuous action $G \curvearrowright X$ of a topological group on a locally compact Hausdorff space X is *coarsely proper* if, for every compact set $K \subseteq X$, the set

$$\{g \in G \mid K \cap g \cdot K \neq \emptyset\}$$

is bounded in G. Similarly, the action is *modest* if $\overline{B \cdot K}$ is compact for all bounded $B \subseteq G$ and compact $K \subseteq X$. In a second countable locally compact group, the bounded sets are relatively compact and hence all its actions are automatically modest. However, this is not the case for more general groups. Also, coarse properness is just properness for these G.

Now, as it turns out, not every group admits a coarsely proper modest cocompact action $G \curvearrowright X$ on a locally compact Hausdorff space, In fact, a Polish group G admits such an action exactly when G is coarsely equivalent to a proper metric space.Such G are said to have *bounded geometry* as it can be seen to be equivalent to G having bounded geometry as a coarse space in the sense of Roe [36].

Theorem 2.14. [39] *Two Polish groups G and H of bounded geometry are coarsely equivalent exactly when they admit a* coarse coupling*, i.e., a pair of commuting, coarsely proper, modest, cocompact, continuous actions on a locally compact Hausdorff space.*

$$\Gamma \curvearrowright X \curvearrowleft \Lambda.$$

For a prototypical example of this setup, consider the group $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of all lifts of orientation-preserving homeomorphisms h of the circle S^1 to homeomorphisms $\tilde{h}$ of $\mathbb{R}$. Then $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is given as a central extension

$$0 \to \mathbb{Z} \to \mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathsf{Homeo}_+(S^1) \to \mathsf{id}.$$

Alternatively, $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the group of homeomorphisms of $\mathbb{R}$ commuting with integral translations. Then $\mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a non-locally compact Polish group coarsely

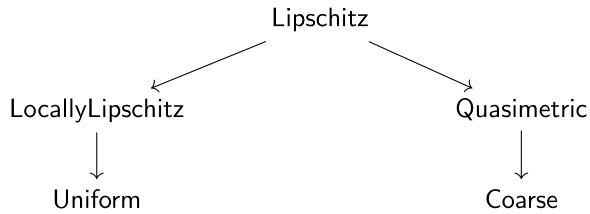

FIGURE 3. Geometric structures on topological groups

equivalent with $\mathbb{Z}$ and, in fact, the canonical actions

$$\mathbb{Z} \curvearrowright \mathbb{R} \curvearrowleft \mathsf{Homeo}_{\mathbb{Z}}(\mathbb{R})$$

amount to a topological coupling of these groups.

2.3. **Lipschitz geometry.** Having introduced the uniform and coarse structure and also discussed the conditions under which these can be further improved to provide locally Lipschitz and quasimetric structures, the last issue at hand is to determine when locally Lipschitz and quasimetric structure can be integrated. That is, suppose a topological group G has both a locally Lipschitz and a quasimetric structure. When are these two reducts of the same Lipschitz structure on G?

Proposition 2.15. *Suppose G has a minimal metric d and a maximal metric D (both compatible and left-invariant). Then G has a metric ∂ that is simultaneously minimal and maximal and any two such metrics will be Lipschitz equivalent.*

Proof. Suppose first that ∂_1 and ∂_2 are both simultaneously minimal and maximal. Then, since ∂_1 is maximal,

$$(G, \partial_1) \xrightarrow{\mathsf{id}} (G, \partial_2)$$

is Lipschitz for large distances and, since ∂_2 is minimal, it is also Lipschitz for short distances. It therefore follows that the map is Lipschitz. By symmetry, we see that the two metrics are Lipschitz equivalent.

To construct ∂ from d and D, we observe first that, since D is maximal, G must be generated by a bounded set $B \subseteq G$. Let then $r > 0$ be large enough so that B is contained in the open D-ball V of radius r centred at the identity. Then D is quasi-isometric with ρ_V and the formula

$$\partial(x, y) = \inf \Big(\sum_{i=1}^{n} d(v_i, 1) \;\Big|\; x = yv_1 \cdots v_n \;\&\; v_i \in V \Big)$$

defines a compatible left-invariant metric on G that is quasi-isometric to ρ_V and hence also to D. Moreover, if U is an identity neighbourhood so that $U^2 \subseteq V$, then d and ∂ agree on U and hence ∂ is also minimal. Thus, ∂ is both minimal and maximal. □

By Proposition 2.15, a Lipschitz structure on G, if it exists, is simply that given by any compatible left-invariant metric that is simultaneously maximal and minimal. Moreover, the existence of this is equivalent to the conjunction of existence of locally Lipschitz and quasimetric structure.

It is now time to return to the general picture of geometric categories we have constructed so far.

To sum up, every topological group G has canonical uniform and coarse structures $\mathcal{U}_L$ and $\mathcal{E}_L$. These may or may not be metrisable, depending on whether G is first countable, respectively, whether G is locally bounded (for Polish G). A locally Lipschitz structure is then that given by a minimal metric, while a quasimetric structure is that given by a maximal metric, if such exist. However, when G has both, these are integrated into a single metric that is both maximal and minimal and defines the inherent Lipschitz geometry of G. The principal examples of the later are compactly generated, second countable, locally compact groups and the additive groups $(X,+)$ of Banach spaces.

Not surprisingly, there is a tight relationship between the various geometric categories and, as for Banach spaces, there are rigidity phenomena of morphisms too. For example, by the proof of Lemma 1.12, we see that, if $G \xrightarrow{\phi} H$ is a bornologous map between topological groups with maximal metrics, then ϕ is automatically Lipschitz for large distances. In particular, every coarse equivalence between G and H is also a quasi-isometry.

Similarly, if $G \xrightarrow{\phi} H$ is a uniformly continuous map between topological groups and G has no proper open subgroups, then ϕ is bornologous. Thus, uniformly homeomorphic groups without proper open subgroups are also coarsely equivalent. As, for example, $\mathbb{Z}$ and $\mathbb{Z}^2$ are uniformly but not coarsely equivalent, this evidently fails in general.

References

[1] I. Aharoni, *Every separable metric space is Lipschitz equivalent to a subset of* c_0, Israel J. Math. 19 (1974), 284–291.

[2] Uri Bader and Christian Rosendal, *Coarse equivalence and topological couplings of locally compact groups*, Geometriae Dedicata 196 (2018), 1–9.

[3] Florent Baudier, Gilles Lancien and Thomas Schlumprecht, *The coarse geometry of Tsirelson's space and applications*, J. Am. Math. Soc. 31, (2018), no. 3, 699–717.

[4] Yoav Benyamini and Joram Lindenstrauss, *Geometric nonlinear functional analysis, Vol. 1*, AMS Colloquium Publications Vol. 48, Providence, RI, 2000.

[5] Garrett Birkhoff, *A note on topological groups*, Compositio Math. 3 (1936), 427–430.

[6] Claude Chevalley, *On a Theorem of Gleason*, Proc. Amer. Math. Soc. 2 (1951), 122–125.

[7] Yves de Cornulier and Pierre de la Harpe, *Metric geometry of locally compact groups*, EMS Tracts in Mathematics Vol. 25, European Math. Society, September 2016.

[8] R. D. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Ann. Math. (2) 93 (1971), 63–88.

[9] Per Enflo, *On the nonexistence of uniform homeomorphisms between L_p-spaces*, Ark. Mat. 8 (1969), 103–105.

[10] Per Enflo, *Uniform structures and square roots in topological groups I*, Israel J. Math. 8 (1970) no. 3, 230–252.

[11] Marian Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.

[12] Valentin Ferenczi and Christian Rosendal, *Banach spaces without minimal subspaces*, J. Funct. Anal. 257 (2009), 149–193.

[13] Tadeusz Figiel, Joram Lindenstrauss and Vitali Davidovich Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. 139 (1977), no. 1-2, 53–94.

[14] Israel M. Gelfand, *Zur Theorie der Charaktere der abelschen topologischen Gruppen*, Rec. Math. N.S. (Mat. Sbornik) 9 (51) (1941), 49–50.

[15] Julian Gevirtz, *Stability of isometries on Banach spaces*, Proc. Amer. Math. Soc. 89 (1983), 633–636.

[16] Andrew Mattei Gleason, *Square roots in locally euclidean groups*, Bull. Amer. Math. Soc. 55 (1949), 446-449.

[17] Gilles Godefroy and Nigel J. Kalton, *Lipschitz-free Banach spaces*, Studia Math., 159, (2003), 121–141.

[18] Mikhail Leonidovich Gromov, *Asymptotic invariants of infinite groups*, in "Geometric Group Theory", Vol. 2 (Sussex, 1991), London Mathematical Society Lecture Note Series, 182, Cambridge University Press, Cambridge, 1993, 1–295.

[19] Peter Manfred Gruber, *Stability of isometries*, Trans. Amer. Math. Soc. 245 (1978), 263–277.

[20] William B. Johnson, Joram Lindenstrauss, and Gideon Schechtman, *Banach spaces determined by their uniform structures*, Geom. Funct. Anal. 6 (1996), 430–470.

[21] Mikhail Iosiphovich Kadets, *Proof of topological equivalence of separable infinite-dimensional Banach spaces*, Funct. Anal. Appl. 1 (1967), 53–62.

[22] S. Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Tokyo 12 (1936), 82–84. Selected Papers, Vol. 1, Robert R. Kallman Editor, Birkhäuser (1986), 60–62.

[23] Nigel J. Kalton, *The nonlinear geometry of Banach spaces*, Rev. Mat. Complut. 21 (2008), 7–60.

[24] Nigel J. Kalton, *The uniform structure of Banach spaces*, Math. Ann. (2012) 354, 1247–1288.

[25] Stanislaw Kwapień, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. 44 (1972), 583–595.

[26] Joram Lindenstrauss, *On nonlinear projections in Banach spaces*, Michigan Math. J. 11 (1964) 263–287.

[27] Kathryn Mann and Christian Rosendal, *Large scale geometry of homeomorphism groups*, Ergodic Theory & Dynamical Systems 38 (2018), no. 7, 2748–2779.

[28] B. Maurey, G. Pisier, *Caractérisation d'une classe d'espaces de Banach par des propriétés de séries aléatoires vectorielles*, C. R. Acad. Sci. Paris Sér. 277 (1973), 687–690.

[29] S. Mazur and S. Ulam, *Sur les transformations isométriques d'espaces vectoriels normés*, C.R.A.S. Paris, 194, (1932), 946–948.

[30] Emmanuel Militon, *Distortion elements for surface homeomorphisms*, Geometry and Topology 18 (2014), 521–614.

[31] Assaf Naor, *An introduction to the Ribe program*, Japanese Journal of Mathematics, November 2012, Volume 7, Issue 2, 167–233.

[32] Assaf Naor, *Uniform nonextendability from nets*, Comptes Rendus de l'Académie des Sciences - Series I - Mathématique 353 (2015), no. 11, 991–994.

[33] Y. Raynaud, *Espaces de Banach superstables, distances stables et homéomorphismes uniformes*, Israel J. Math. 44(1) (1983), 33–52.

[34] Martin Ribe, *On uniformly homeomorphic normed spaces*, Ark. Mat. 14 (1976), no. 2, 237–244.

[35] Martin Ribe, *Existence of separable uniformly homeomorphic nonisomorphic Banach spaces*, Israel J. Math. 48, 139–147 (1984).

[36] John Roe, *Lectures on coarse geometry*, University Lecture Series, 31. American Mathematical Society, Providence, RI, 2003.

[37] Christian Rosendal, *Equivariant geometry of Banach spaces and topological groups*, Forum of Mathematics, Sigma, (2017), Vol. 5, e22, 62 pages.

[38] Christian Rosendal, *Lipschitz structure and minimal metrics on topological groups*, Arkiv för matematik, Vol. 56, No. 1 (2018), 185–206.

[39] Christian Rosendal, *Coarse geometry of topological groups*, book manuscript.

[40] R. A. Struble, *Metrics in locally compact groups*, Compositio Mathematica 28 (1974), 217–222.

[41] Dennis Sullivan, *Hyperbolic geometry and homeomorphisms*, in Geometric Topology: Proc. Topology Conf. at Athens, Ga., 1977, J.C. Cantrell, ed., Academic Press, New York, London (1979), 543–555.

[42] Terence Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics, 153. Amer- ican Mathematical Society, Providence, RI, 2014.

[43] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. 111 (1981), 247–262.

[44] V. V. Uspenskiĭ, *A universal topological group with a countable base*, Funct. Anal. and its Appl. 20 (1986), 160–161.

[45] André Weil, *Sur les Espaces à Structure Uniforme et sur la Topologie Générale*, Paris, Hermann, 1937.

Department of Mathematics, Statistics, and Computer Science (M/C 249), University of Illinois at Chicago, 851 S. Morgan St., Chicago, IL 60607-7045, USA
Email address: rosendal.math@gmail.com
URL: http://homepages.math.uic.edu/~rosendal