MA 430, SEVENTH HOMEWORK SET, DUE WEDNESDAY, NOVEMBER 4TH.

1. Exercise

Let $L = \{f, g\}$, where f and G are unary and binary function symbols respectively. Consider the following sentences

(1) $A_1\colon\ \exists x\, \exists y\ fgxy = fx$,
(2) $A_2\colon\ \forall x\, \forall y\ fgxy = fx$,
(3) $A_3\colon\ \exists y\, \forall x\ fgxy = fx$,
(4) $A_4\colon\ \forall x\, \exists y\ fgxy = fx$,
(5) $A_5\colon\ \exists x\, \forall y\ fgxy = fx$,
(6) $A_6\colon\ \forall y\, \forall x\ fgxy = fx$.

Consider the four structures whose universe is $\mathbb{N}_+$, where g is interpreted as the map $(m, n) \mapsto m + n$ and f is interpreted by respectively

a: the constant map with value 103,
b: the map which to each integer n associates the remainder after division by 4,
c: the map $n \mapsto \min(n^2 + 2, 19)$,
d: the map which to each integer n associates 1 if $n = 1$ and the smallest prime divisor of n if $n > 1$.

Decide for each of the four cases above, which of the 6 formulas $A_1, \dots, A_6$ are true in the structure.

2. Exercise

Let $L = \{P, R\}$, where P and R are unary and binary relation symbols respectively. Consider the following sentences

(1) $B_1\colon\ \exists x\, \forall y\, \exists z((Px \to Rxy)\ \wedge\ Py\ \wedge\ \neg Ryz)$,
(2) $B_2\colon\ \exists x\, \exists z((Rzx \to Rxz) \to \forall y\ Rxy)$,
(3) $B_3\colon\ \forall y\ (\exists z\, \forall v\ Rvz\ \wedge\ \forall x\ (Rxy \to \neg Rxy))$,
(4) $B_4\colon\ \exists x\, \forall y\ ((Py \to Ryx)\ \wedge\ (\forall v(Pv \to Rvy) \to Rxy))$,
(5) $B_5\colon\ \forall x\, \forall y\ ((Px\ \wedge\ Rxy) \to ((Py\ \wedge\ \neg Ryx) \to \exists z\ (\neg Rzx\ \wedge\ \neg Ryz)))$.

Consider the three L-structures defined by

a: the universe is $\mathbb{N}$, the interpretation of R is the usual order relation $\leqslant$, the interpretation of P is the set of even integers,
b: the universe is $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$), the interpretation of R is the inclusion relation $\subseteq$, the interpretation of P is the collection of all finite subsets of $\mathbb{N}$,

c: the universe is $\mathbb{R}$, the interpretation of R is the set of pairs $(a, b) \in \mathbb{R}^2$ such that $b = a^2$, the interpretation of P is the subset of rational numbers.

Decide for each of the three cases above, which of the 5 formulas $B_1, \ldots, B_5$ are true in the structure.

3. Exercise

Let $L = \{f, g\}$, where f and g are unary function symbols.

(a) Find three sentences A, B, and C such that for every L-structure $\mathcal{M} = \langle M, f^{\mathcal{M}}, g^{\mathcal{M}} \rangle$, we have

- $\mathcal{M} \models A \iff f^{\mathcal{M}} = g^{\mathcal{M}}$ and $f^{\mathcal{M}}$ is a constant map,
- $\mathcal{M} \models B \iff \mathrm{Im}(f^{\mathcal{M}}) \subseteq \mathrm{Im}(g^{\mathcal{M}})$,
- $\mathcal{M} \models C \iff \mathrm{Im}(f^{\mathcal{M}}) \cap \mathrm{Im}(g^{\mathcal{M}})$ is a singleton.

Consider the following L-sentences:

(1) $E_1\colon\ \forall x\ fx = gx$,
(2) $E_2\colon\ \forall x\ \forall y\ fx = gy$,
(3) $E_3\colon\ \forall x\ \exists y\ fx = gy$,
(4) $E_4\colon\ \exists x\ \forall y\ fx = gy$,
(5) $E_5\colon\ \exists x\ \exists y\ fx = gy$.

(b) Construct structures satisfying each of the following six formulas:

$$E_1 \wedge \neg E_2 \qquad E_2 \qquad \neg E_1 \wedge E_3$$
$$\neg E_1 \wedge E_4 \qquad \neg E_3 \wedge \neg E_4 \wedge E_5 \qquad \neg E_5.$$