1. Exercise

g is addition

Structure (a) : $f \equiv 103$

A_1 : $\exists x \exists y$ $103 = 103$ ✓

A_2 : $\forall x \forall y$ $103 = 103$ ✓

⋮

A_6 — — — " — ✓

Structure (b) f: remainder mod 4

A_1 : $\exists x \exists y$ $x + y = x$ mod 4 ✓

A_2 : $\forall x \forall y$ $x + y = x$ mod 4 %

A_3 $\exists y \forall x$ — " — ✓

A_4 $\forall x \exists y$ — " — ✓

A_5 $\exists x \forall y$ — " — %

A_6 $\forall y \forall x$ — " — %

Structure (c) $f : n \mapsto \min(n^2 + 2, 19)$

A_1 : $\exists x \exists y$ $\min((x+y)^2 + 2, 19) = \min(x^2 + 2, 19)$ ✓

A_2 : $\forall x \forall y$ %

A_3 : $\exists y \forall x$ % A_5 : $\exists x \forall y$ ✓

A_4 : $\forall x \exists y$ % A_6 : $\forall y \forall x$ %

Structure (d): $f(u) = \begin{cases} 1 & \text{for } u = 1 \\ \text{minimal prime divisor of } u & \text{otherwise} \end{cases}$

A_1 : $\exists x \exists y \ \ f(x+y) = f(x)$ $\checkmark$ ($x = y = 2$: $f(x+y) = f(x) = 2$)

$\forall x \forall y$ [illegible]/0 $\qquad \forall y \forall x$ [illegible]/0

$\exists y \forall x$ 0/1

$\forall x \exists y$ 0/0

$\exists x \forall y$ [illegible]/1

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2. Exercise

Using the equivalence $\exists x\,(A \to B) \equiv \forall x\, A \to \exists x\, B$
we see

$$B_1 \equiv \exists x \forall y\, (Px \to Rxy) \wedge \forall y\, Py \wedge \forall y \exists z\, \neg Ryz$$

$$B_2 \equiv \forall x \forall z\, (Rzx \to Rxz) \to \exists x \forall y\, Rxy$$

$$B_3 \equiv \exists z \forall v\, Rvz \wedge \forall y \forall x\, (Rxy \to \neg Ryx)$$

$$B_4 \equiv \exists x \left(\forall y\, (Py \to Ryx) \wedge \forall y\, \left(\forall v\, (Pv \to Rvy) \to Rxy\right)\right)$$

$$B_5 \equiv \forall x \forall y \left((Px \wedge Py \wedge Rxy \wedge \neg Ryx) \to \exists z (\neg Rzx \wedge \neg Ryz)\right)$$

Structure (a) : Universe $\mathbb{N}$, $R = \leq$, $P =$ even numbers :

B_1 %, B_2 ✓, B_3 %, B_4 %, B_5 ✓

Structure (b) : Universe $\mathcal{P}(\mathbb{N})$, $R = \subseteq$, $P =$ Finite sets

B_1 %, B_2 ✓, B_3 %, B_4 ✓, B_5 %

Structure (c) : Universe $\mathbb{R}$, $aRb \iff b = a^2$, $P = \mathbb{Q}$

B_1 %, B_2 ✓, B_3 %, B_4 %, B_5 ✓

3. Exercise :

(a)

A : $\forall x \forall y \; fx = gy$

B : $\forall x \exists y \; fx = gy$

C : $\exists x \forall y \forall z \; (fy = gz \rightarrow fy = z)$

(b)

E_1 $\forall x \; fx = gx$ $(f = g)$

E_2 $\forall x \forall y \; fx = gy$ $(f = g \equiv \text{const})$

E_3 $\forall x \exists y \; fx = gy$ $(\text{rg}(f) \subseteq \text{rg}(g))$

E_4 $\exists x \forall y \; fx = gy$ $(g \equiv f(x) \text{ for some } x)$

E_5 $\exists x \exists y \; fx = gy$

Universe $\mathbb{N}$

Models of

$E_1 \wedge \neg E_2$: $f(x) = g(x) = x+1$

E_2 : $f(x) = g(x) \equiv 0$

$\neg E_1 \wedge E_3$: $f(x) = x+2$, $g(x) = x+1$

$\neg E_1 \wedge E_4$: $f(x) = x$, $g(x) \equiv 0$

$\neg E_3 \wedge \neg E_4 \wedge E_5$: $f(x) = x$, $g(x) = x+1$

$\neg E_5$: $f(x) = 2x$, $g(x) = 2x+1$