Conceptual Understanding in Chapter 1 of Math 410

1. Define what it means for c to be an upper bound of the set S.
2. Define set S that has an upper bound, and T that has no upper bound.
3. Define sup S and inf S.
4. State carefully the Completeness Axiom (also called the Least Upper Bound Axiom).
5. Let S be the set of rationals in $[0, 1)$. Does $\sup S$ exist? If so, find it. Does $\inf S$ exist. If so, find it.
6. Suppose that $\sup S$ and $\inf S$ exist for a set S. Must $\inf S \leq \sup S$? Explain your answer.
7. Suppose we wish to prove that statement P implies statement Q. Tell how one begins a proof by contradiction.
8. State the three parts to a proof that uses the Law of Induction.
9. Define what it means for a set S to be inductive.
10. Let S be the set of all positive integers that are squares (i.e., 1, 4, 9, 16, ...). Is S inductive? Explain why, or why not.
11. State carefully the Archimedean Property.
12. Tell why the Archimedean Property is equivalent to the following statement: For any positive $\epsilon > 0$, there is a natural number n such that $1/n < \epsilon$.
13. State the Triangle Inequality.
14. Write down the general Binomial Formula, and the Binomial Formula for $n = 4$.
15. Define what it means for a set S to be dense in a set T.
16. Is the set Q of rational numbers dense in R?

Conceptual Understanding in Chapter 2 of Math 410

1. Sequences

   (a) Define carefully what a sequence is. What is an index of a sequence. Must there be infinitely many indices for a given sequence?

   (b) What are the possible numbers for the initial index of a sequence?

   (c) Define carefully what it means for $\{a_n\}$ to converge.

   (d) What are two ways in which a sequence $\{a_n\}$ can diverge?

   (e) Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences, with $a_n \leq b_n$ for all n in N. Suppose that $\{b_n\}$ converges. Under what conditions will $\{a_n\}$ automatically converge? Explain your answer.

   (f) Suppose that $\{a_n\}$ and $\{b_n\}$ both converge. Under what conditions does $\{a_n/b_n\}$ converge?

   (g) Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences, and that $\lim_{n\to\infty} a_n = 0$. Under what conditions must $\lim_{n\to\infty} a_n b_n$ converge?

   (h) Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences, and that $\{a_n + b_n\}$ converges. Give an example for which $\lim_{n\to\infty}(a_n + b_n) \neq \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

   (i) Under what conditions are we guaranteed that $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

   (j) Suppose that $\{a_n^2\}$ converges. Give an example for which $\{a_n\}$ diverges.

   (k) Suppose that $\{|a_n|\}$ converges. Give an example for which $\{a_n\}$ diverges.

   (l) Is every convergent sequence bounded? Explain your answer.

   (m) is every bounded sequence convergent? Explain your answer.

   (n) Is every real number the limit of rational numbers? Explain why, or why not.

   (o) Define carefully what is meant by a sequence being monotonically increasing. Give an example of a sequence $\{a_n\}$ that is monotonically increasing, and a sequence $\{b_n\}$ that is *not* monotonically increasing.

   (p) Is the product of monotone sequences automatically monotone? Explain, or give a counterexample.

   (q) State carefully the ***Monotone Convergence Theorem.

2. Dense sets

   (a) Define what it means for a set A to be dense in a set B.

   (b) Explain why each of the following sets is, or is not, dense in the reals R: the set J of integers, the set Q of rational numbers, the set I of irrational numbers.

   (c) Is it true that every convergent sequence of rational numbers has a rational limit? Explain why, or why not.

3. Closed sets

   (a) Define what it means for a set A to be a closed set in R.

   (b) Let A be the set of all $1/n$, where n is a positive integer. Determine if A is closed in R, giving reasons.

(c) Explain why the set of irrational numbers A is not closed in R. What is the smallest closed set B that contains all irrational numbers in $[0, 1]$.

(d) State carefully the ***Nested Interval Theorem, and draw a picture to show the idea of the theorem.

4. Subsequence

(a) Give a careful definition of a subsequence of $\{a_n\}$.

(b) Can a subsequence have only finitely many indices?

(c) If a sequence $\{a_n\}$ converges to L, and $\{a_{n_k}\}$ is a subsequence, must the subsequence converge? And if it must converge, must it converge to L? Explain why, or why not.

(d) Must every sequence in $[0, 3]$ have a convergent subsequence? Explain why, or give a counter-example.

(e) Must every subsequence of a given bounded sequence be bounded? Explain why, or give a counter-example.

(f) If a given sequence $\{a_n\}$ has a convergent subsequence, must $\{a_n\}$ converge? Explain why, or give a counter-example.

(g) If a given sequence $\{a_n\}$ is monotone and has a convergent subsequence, must $\{a_n\}$ converge? Explain why, or give a counter-example.

(h) Define carefully what is meant by a set S being sequentially compact.

(i) Is every closed interval sequentially compact? Explain why, or give a counter-example.

(j) Is every bounded interval sequentially compact? Explain why, or give a counter-example.

(k) What is a peak index for a given sequence $\{a_n\}$?

(l) If a given sequence $\{a_n\}$ has only finitely many peak indices, what can you say about $\{a_n\}$ and the existence of a certain type of subsequence?

(m) If a given sequence $\{a_n\}$ has infinitely many peak indices, what can you say about $\{a_n\}$ and the existence of a certain type of subsequence?

(n) Can a sequence be a subsequence of itself? Explain why, or why not.

Conceptual Understanding in Chapter 3 of Math 410

1. Continuity of a function.

   (a) Define carefully what it means for $f : D \to R$ to be continuous at x_0. Explain graphically what the sequence definition of continuity at x_0 means. Likewise, explain graphically what the $\epsilon - \delta$ definition of continuity at x_0 means.

   (b) Must a function f be defined at x_0 in order to be continuous at x_0? Explain why, or why not.

   (c) Explain carefully what it means for the function f to be a continuous function.

   (d) Let $g(x) = 1/x$. Is g a continuous function? Explain your answer.

   (e) Let $h(x) = \sqrt{x}$. Is h continuous at $x = 0$? Explain your answer.

   (f) Let $k : N \to R$. Is k a continuous function? Explain your answer.

   (g) Are sequences continuous functions? Explain your answer.

   (h) Let $p(x) = x$ for $0 < x < 1$ and $p(x) = x + 2$ for $1 < x < 2$. Is p continuous? Explain.

   (i) Suppose that g is continuous on the interval $[a, b]$. Must g be bounded?

   (j) Suppose that $f : D \to R$ is a continuous function. Must $|f|$ be a continuous function? Explain.

   (k) Let $f : [-1, 1] \to R$ be defined by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Is f a continuous function? Explain your answer.

2. Extreme values of a function

   (a) What is the difference between a maximum value of a function f and a maximizer of f?

   (b) Can a function f have more than one maximum value? Can a function g have more than one maximizer? Explain your answers.

   (c) ***Extreme Value Theorem: State it carefully.

   (d) Can a function $g : (1, 3) \to R$ have a maximum value? Can $h : R \to R$ have a maximum value? Explain your answers.

   (e) Write down a function $f : (-1, 1) \to R$ that is continuous but has no extreme values.

3. ***Intermediate Value Theorem: State it carefully.

   (a) Give an example of a function $f : [a, b] \to R$ that is not continuous, and does not satisfy the conclusion of the Intermediate Value Theorem.

   (b) Given the statement of the Intermediate Value Theorem, why is $x_0 = a$ not allowed?

   (c) Let g be a polynomial of odd order. Use the Intermediate Value Theorem to show that g has a zero.

   (d) Let h is a polynomial of even order. Under what conditions would h have a zero?

   (e) If $f : [a, b] \to R$ is continuous, then must the range of f be an interval? Explain your answer.

   (f) If $g : (a, b) \to R$ is continuous, then must the range of g be an interval? Explain your answer.

4. Uniform continuity

   (a) Give the definition of $f : [a, b] \to R$ being uniformly continuous on $[a, b]$.

(b) Show by an example that if $f : [a, b] \to R$ is uniformly continuous, then the sequences (u_n) and (v_n) need not themselves converge in order to have $|u_n - v_n| \to 0$ and $|f(u_n) - f(v_n)| \to 0$

(c) Suppose that $f : [a, b] \to R$ is continuous. Then must f be uniformly continuous on $[a, b]$? Explain your answer.

(d) Suppose that $f : (a, b) \to R$ is continuous. Then must f be uniformly continuous on (a, b)? Explain your answer.

(e) Can a function g be uniformly continuous at a single point? Explain your answer.

5. Monotone and strictly monotone functions

(a) Is a constant function monotonic? Is a constant function strictly monotonic? Explain your answers.

(b) What is the difference between a strictly monotonic function and a one-to-one function?

Is a monotonic function automatically one-to-one? Is a strictly monotonic function automatically one-to-one? Explain your answers, giving examples where relevant.

(c) What is the difference between a monotonically decreasing function and a strictly monotonically decreasing function?

(d) If a function $f : [a, b] \to R$ is continuous and monotonically decreasing, what can you say about the range of f?

(e) Suppose that $g : [a, b] \to R$ is monotonically increasing, and $g[a, b] = [c, d]$. Must g be continuous? Explain your answer.

(f) If f is monotonically increasing, then does f^{-1} automatically exist? If f is strictly monotonically increasing, then does f^{-1} automatically exist? Explain your answers, giving examples where relevant.

(g) Suppose that $f : [a, b] \to R$ is monotonically decreasing and f^{-1} exists. Then must f be strictly monotonically decreasing? Explain your answer.

(h) Suppose that $f : [a, b] \to R$ is monotonically decreasing and f^{-1} exists. Then must the range of f be an interval? Must f^{-1} also be monotonically decreasing? Explain your answers, giving examples where relevant.

(i) Suppose that f has an inverse. If f is continuous, must f^{-1} be continuous? Explain your answer, giving examples where relevant.

Conceptual Understanding in Chapter 4 of Math 410

1. Elements of sets

   (a) Define what it means for x_0 in a set D to have a neighborhood in D.

   (b) How are a point interior to a set D and an isolated point of D related?

2. Derivative of a function

   (a) Define carefully what it means for f to have a derivative at x_0.

   (b) What does it mean for f to be a differentiable function?

   (c) For $f'(x_0)$ to exist, must f be defined in a neighborhood of x_0? Explain your answer.

   (d) Let $g(x) = x^{3/2}$. Does $g'(0)$ exist? Explain your answer.

   (e) Prove that if $f'(x_0)$ exists, then f is continuous at x_0.

   (f) Give an example of a function g such that g is continuous at z_0 but $g'(z_0)$ does not exist.

   (g) Suppose that $g(a) = c$ and $g'(a) = r$. Write down an equation of the line L tangent to the graph of g at (a, c).

   (h) Can a line L tangent to the graph of g at (a, c) cross, or touch, the graph of g at another point on the graph of g? Explain your answer.

   (i) State carefully the Product Rule and the Quotient Rule for derivatives.

   (j) State carefully the Chain Rule for derivatives.

   (k) Show that the following two formulas for $f'(x_0)$ are equivalent:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

   (l) Suppose that f is a differentiable function, that f has an inverse, and that $f(a) = c$. State carefully the additional hypotheses that are needed in order that $(f^{-1})'(c)$ exists, and give a formula for $(f^{-1})'(c)$.

   (m) Suppose that f has an inverse, and that $f'(x_0)$ exists. What is the relationship graphically between $f'(x_0)$ and the appropriate derivative of (f^{-1}) (providing that the latter exists)?

   (n) Find a function $g : R \to R$ that is strictly monotone but such that $g'(0) = 0$. Can g have infinitely many values of a such that $g'(a) = 0$? Explain your answer.

   (o) Let $g(x) = x^2$. Determine if for each $c \neq 0$ there is a line L tangent to the graph of g such that L passes through the point $(c, 0)$ on the x axis. Support your answer.

   (p) Let f be continuous on $[a, b]$, differentiable on (a, b), and such that $f(a) = f(b) = 0$. Prove that if $f(c) > 0$ for some c in (a, b), then there exist x_1 and x_2 in (a, b) such that $f'(x_1) > 0 > f'(x_2)$.

   (q) Suppose that f has a bounded derivative on R. Determine whether f must be uniformly continuous on R, giving reasons.

3. ***The Mean Value Theorem

   (a) State carefully the Mean Value Theorem.

   (b) In what way is Rolle's Theorem a special case of the Mean Value Theorem?

(c) In the statement of the Mean Value Theorem, an assumption is that f is continuous on the closed interval $[a, b]$. Suppose that f were continuous on (a, b) and also differentiable on (a, b). Give an example to show that the conclusion would not necessarily be valid.

(d) Let $f : R \to R$ be differentiable. Show that there is exactly one function g such that $f'(x) = g'(x)$ for all x in R, and such that $g(0) = \pi$.

(e) Suppose that $f : R \to R$ is differentiable, and assume that $f'(x) = 0$ has exactly k solutions, where $k \geq 1$. Determine the maximum number of solutions to the equation $f(x) = 0$.

(f) Is it possible for a function f to be increasing on $[0, 1]$ and simultaneously have an infinite number of values of x in $(0, 1)$ such that $g'(x) = 0$? Explain your answer.

(g) Suppose that g is differentiable on an open interval I, and assume that g has exactly two local maximizers, x_0 and x_1, with $x_0 < x_1$. Must g have a local minimizer in the interval (x_0, x_1)? Explain your answer.

(h) What is the maximum number of local minimizers a 4th degree polynomial can have? Support your answer.

(i) How does the Cauchy Mean Value Theorem differ from the Mean Value Theorem?

(j) Which form of l'Hopital's Rule is a by-product of the Cauchy Mean Value Theorem?

(k) State the 2nd Derivative Test. Draw the graph of a function f that has a local maximum value $f(x_0)$, and show that the hypotheses and the conclusion of the 2nd Derivative Test are consistent with the graph of f near $(x_0, f(x_0))$.

Conceptual Understanding in Chapter 6 of Math 410

1. Preparing for the Integral

   (a) Suppose that $a < b$. Can a partition P of $[a, b]$ have only one point? Alternatively, can a partition P of $[a, b]$ have infinitely many distinct points? Explain your answers.

   (b) If f is defined on $[a, b]$ but is not bounded above on $[a, b]$, is it possible for the lower and upper Darboux sums $L(f, P)$ and $U(f, P)$ to exist? Explain your answer.

   (c) Under what conditions on f must $L(f, P) \neq U(f, P)$?

   (d) Suppose that P is a partition of $[a, b]$. Is P a refinement of itself? Explain your answer.

   (e) Let P and Q be partitions of $[a, b]$. Describe the process for finding a common refinement.

   (f) Let P and Q be partitions of $[a, b]$. Tell why $L(f, P) \leq U(f, Q)$.

   (g) Describe the relationship between $L(f, P)$ and $\int_{-a}^{b} f$. Analogously, describe the relationship between $U(f, P)$ and $\int_{a}^{-b} f$.

   (h) Let $f : [a, b] \to R$ be bounded. Show that the lower integral and the upper integral automatically exist.

   (i) Give a condition under which $\int_{-a}^{b} f \neq \int_{a}^{-b} f$, but f is bounded.

   (j) Let $f : [0, 1] \to R$, with $f(x) = 0$ if x is rational, and $f(x) = 1$ if x is irrational. Let P be an arbitrary partition of $[0, 1]$. Find $L(f, P), U(f, P)$, $\int_{-a}^{b} f$, and $\int_{a}^{-b} f$.

2. Archimedes-Riemann Theorem and Related Items

   (a) State the Archimedes-Riemann Theorem, and tell what kinds of functions were shown to be integrable by means of the theorem.

   (b) To show that $\int_{a}^{b} f$ exists by the Archimedes-Riemann Theorem, could one just show that for two sequences (P_n) and (Q_n) of partitions of $[a, b]$, one has $\lim_{n\to\infty} [U(f, P_n) - L(f, Q_n)] = 0$? Explain your answer.

   (c) Define what is meant by the gap of a partition P of $[a, b]$.

   (d) Why is it generally easier to prove that a function is integrable on an interval $[a, b]$ if the partitions are regular?

   (e) Suppose $f : [a, b] \to R$ is continuous on (a, b). Then is f automatically integrable? Explain your answer.

   (f) Suppose $f : [a, b] \to R$ is continuous on (a, b) and f is bounded on $[a, b]$. Is f automatically integrable? Explain your answer.

   (g) Suppose that $g : [a, b] \to R$ is a step function. Explain why the function is automatically integrable.

   (h) Find a function $g : [a, b] \to R$ that is bounded but not integrable.

3. Integrals and Area

   (a) Under what conditions does $\int_a^b f$ represent the area A of the region between the graph of f and the x-axis between $x = a$ and $x = b$?
   (b) What do the linearity and the monotonicity results in Chapter 6 tell about properties of area?
   (c) If $f : [a, b] \to R$ is integrable, then describe $L_(f, P)$ and $U_(f, P)$ in terms of properties of area.

4. The Fundamental Theorems of Calculus and Related Items

   (a) Suppose that $f : [a, b] \to R$ is continuous. Does f automatically have an antiderivative? If so, write down a formula for an antiderivative.
   (b) Suppose that $f : [a, b] \to R$ is integrable, and $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of F. If G is another antiderivative of f on $[a, b]$, is it automatic that $\int_a^b f(x)\,dx = G(b) - G(a)$? Explain your answer.
   (c) Let $f : [a, b] \to R$ be continuous and $\int_a^b f(x) = 0$. Must $f(x_0) = 0$ for some x_0 in $[a, b]$? Explain your answer.
   (d) Let $f : [a, b] \to R$ be integrable, and $f(x) \geq 0$ for all x in $[a, b]$. Must $\int_a^b f \geq 0$? Explain your answer.
   (e) How is the Mean Value Theorem for Integrals like the Mean Value Theorem?
   (f) Let $f : [a, b] \to R$ be continuous, and let $f \geq 0$ on $[a, b]$. Give an interpretation of the Mean Value Theorem for Integrals in terms of the area of a particular rectangle.
   (g) Let $f : [a, b] \to R$. Show that the Mean Value Theorem for Integrals does not hold if we replace the assumption that f is continuous with the assumption that f is merely integrable.
   (h) Evaluate $\frac{d}{dx} \int_{x^2}^{e^{2x}} x^4 \sin(t^2)\,dt$.
   (i) It is known that we cannot find a simple formula for an antiderivative of $g(x) = \sqrt{1 + x^4}$, for $0 \leq x \leq 2$. Does this mean that $\int_0^2 \sqrt{1 + x^4}\,dx$ does not exist? Explain your answer.

Conceptual Understanding in Chapter 7 of Math 410

1. Why is "integration by parts" so called?
2. If we wish to use integration by parts on $\int f(x)g'(x)\,dx$, what are the properties of $f(x)$ and $g'(x)$ that generally help with the integration by parts process?
3. To get the following integrals into the form $\int u\,dv$ for integration by parts, what would you choose for u, and for dv?

$$\int x^2 e^{4x}\,dx,\ \int \ln x\,dx,\ \int \sin^9 t\cos^3 t\,dt,\ \int \tan^5 t\sec^4 t\,dt,\ \int \tan^5 t\sec^5 t\,dt$$

4. If one is to evaluate $\int x^2\sqrt{1+x^3}\,dx$ by substituting $u = 1 + x^3$, then what does $du =$?
5. Determine a non-zero number k such that we can easily evaluate $\int x^k(\ln(x^4))\,dx$.
6. What does the Trapezoidal Rule have to do with trapezoids?
7. For the Trapezoidal Rule with n subintervals of $[a, b]$ to approximate $\int_a^b f(x)\,dx$, what are restrictions on the lengths of the subintervals of $[a, b]$?
8. For $\int_a^b f(x)\,dx$, why is the Trapezoidal Rule with n subintervals $= \frac{b-a}{2n}[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)]$
9. Describe how the Trapezoidal Rule error E_n^T formula is related to the integral $\int_a^b f(x)\,dx$.
10. $E_n^T \leq \frac{M_T}{12n^2}(b-a)^3$. What does M represent, and why does $\frac{M_T}{12n^2}(b-a)^3 \to 0$ as $n \to \infty$?
11. Explain why the Trapezoidal Rule gives the exact value of $\int_a^b f(x)\,dx$ if f is a linear function.
12. For Simpson's Rule, why must the number of subintervals of $[a, b]$ be an even integer?
13. For $\int_a^b f(x)\,dx$, Simpson's Rule with n (n even!) subintervals is $= \frac{b-a}{2n}[f(a) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4(f(x_{n-1}) + f(b)]$. Why the coefficients 1, 4, 2, 4, 2, ..., 2, 4, 1?
14. The Simpson Rule error is: $E_n^S \leq \frac{M_S}{180n^4}(b-a)^5$. What does M_S represent? For large n, why is the Simpson Rule apt to be a better approximation than the Trapezoidal Rule?
15. Explain why Simpson's Rule gives the exact value of $\int_a^b f(x)\,dx$ if f is a degree 3 polynomial.

Conceptual Understanding in Chapter 8 of Math 410

1. What is the order of contact for a function f and its nth Taylor polynomial? Explain your answer.

2. Among the most important common non-polynomial functions in calculus are: e^x, $\sin x$, $\cos x$, and $\frac{1}{1-x}$. For each of these functions, write down the nth Taylor polynomial about 0.

3. Let $g(x) = \sqrt{x}$. Does g have any Taylor polynomials about $x = 0$? Explain your answer.

4. Let $f : I \to R$ have $n+1$ derivatives on I, with I open and x_0 in I. By the Lagrange Remainder Theorem, the nth Taylor remainder $r_n(x)$ for x in I is given by $\frac{f^{(n+1)}(x_c)}{(n+1)!}(x-x_0)^{n+1}$. What is x_c, and how does it relate to the numbers x and x_0?

5. Let $f : I \to R$ have $n+1$ derivatives on I, with I open and x_0 in I. Let p_n and r_n denote the nth Taylor polynomial and the nth Taylor remainder function, respectively. How do $f(x)$, $p_n(x)$, and $r_n(x)$ relate to one another?

6. Let $f : I \to R$ have derivatives of all orders on the open interval I, with x_0 in I. Define the Taylor series expansion of f about x_0, and define the radius of convergence R for the Taylor series.

7. Define a function g whose Taylor series expansion about 0 has the given radius of convergence R:
   (a) $R = 0$ (b) $R = 2$ (c) $R = \sqrt{3}$ (d) $R = \infty$

8. Let $f : I \to R$ have derivatives of all orders, and let x_0 be in I. Give a nontrivial assumption on the derivatives of f at x in I such that $\lim_{n\to\infty} r_n(x) = 0$ for all x in I.

9. If the assumption on the derivatives of f at x in the preceding item is met, then how are $f(x)$, $\lim_{n\to\infty} p_n(x)$, and $\lim_{n\to\infty} r_n(x)$ related?

10. Give an example of a function $f : R \to R$ with derivatives of all orders at 0, such that $f(x) \neq$ the Taylor series of f about 0.

Conceptual Understanding in Chapter 9 of Math 410

1. Series

   (a) What is the definition of a Cauchy sequence?

   (b) Which appears easier to prove: That a Cauchy sequence must converge, or that a convergent sequence is necessarily a Cauchy sequence? Give reasons for your answer.

   (c) What is a geometric series, and what are the conditions for a geometric series to converge?

   (d) What are the two equivalent statements for the Comparison Test?

   (e) How are convergent series and absolutely convergent series related?

   (f) State carefully the Alternating Series Test.

   (g) How are the Integral Test and the p-Test related?

   (h) How does the proof of the Ratio Test rely on geometric series?

2. Pointwise and Uniform Convergence of Functions

   (a) Let $f_n : D \to R$ for all $n \geq 1$, and let $f : D \to R$. Define carefully what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ to converge pointwise to f.

   (b) Let $f_n : D \to R$ for all $n \geq 1$, and let $f : D \to R$. Define carefully what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ to converge uniformly to f.

   (c) Let $f_n : D \to R$ for all $n \geq 1$, and let $f : D \to R$. Show that if the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, then the sequence converges pointwise to f, but that the converse is false.

   (d) Let $f_n : D \to R$ for all $n \geq 1$, and let $f : D \to R$. Suppose the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Which of the properties of continuity, integrability, and differentiability for each f_n is inherited by f? Explain your answer.

   (e) Let $f_n : D \to R$ for all $n \geq 1$, and let $f : D \to R$. Suppose the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f. Which of the properties of continuity, integrability, and differentiability for each f_n is inherited by f? Explain your answer.

3. (a) Power Series

   (b) Define power series expansion.

   (c) The domain of convergence D of a power series is the interval of convergence I for the power series. How is D (and thus I) related to the radius of convergence R?

   (d) Find a power series with $R = 0$, and another with $R = \infty$, and another with $R = 4$.

   (e) Given $f(x) = \sum_{k=0}^{\infty} a_k x^k$, with $R > 0$. Find the power series for $f'(x)$, and the radius of convergence R_1 of that power series.