University of Illinois at Chicago

Interactive Notes For
Real Analysis

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Preface

These notes are all about the Real Numbers and Calculus. We start from scratch with definitions and a set of nine axioms. Then, using basic notions of sets and logical reasoning, we derive what we need to know about real numbers in order to advance through a rigorous development of the theorems of Calculus.

In Chapter 0 we review the basic ideas of mathematics and logical reasoning needed to complete the study. Like Euclid's Basic Notions, these are the things about sets and logic that we hold to be self-evident and natural for gluing together formal arguments of proof. This chapter can be covered separately at the beginning of a course or referred to throughout on an 'as needed' basis. It contains all the common definitions and notation that will be used throughout the course.

Students already think about real numbers in different ways: decimal representation, number line, fractions and solutions to equations, like square roots. They are familiar with special real numbers, with infinite, non-repeating decimals, like π and e. All these ways of representing real numbers will be investigated throughout this axiomatic approach to the development of real numbers. The Axioms for Real Numbers come in three parts:

The Field Axioms (Section 1.1) postulate basic algebraic properties of number: commutative and associative properties, the existence of identities and inverses.

The Order Axioms (Section 1.2) postulate the existence of positive numbers. Consequences of include the existence of integers and rational numbers.

The Completeness Axiom (Section 1.3) postulates the existence of least upper bound for bounded sets of real numbers. Consequences of completeness include infinite decimals are real numbers and that there are no 'gaps' in the number line.

The completeness of the real numbers paves the way for develop the concept of limit, Chapter 2, which in turn allows us to establish the foundational theorems of calculus establishing function properties of continuity, differentiation and integration, Chapters 4 and 5.

Goals

1. Prove the Fundamental Theorem of Calculus starting from just nine axioms that describe the real numbers.

2. Become proficient with reading and writing the types of proofs used in the development of Calculus, in particular proofs that use multiple quantifiers.

3. Read and repeat proofs of the important theorems of Real Analysis:

   - *The Nested Interval Theorem*
   - *The Bolzano-Weierstrass Theorem*
   - *The Intermediate Value Theorem*
   - *The Mean Value Theorem*
   - *The Fundamental Theorem of Calculus*

4. Develop a library of the examples of functions, sequences and sets to help explain the fundamental concepts of analysis.

Two Exercises to get started.

True or False 1

Which of the following statements are true? Explain your answer.

a) $0.\overline{9} > 1$

b) $0.\overline{9} < 1$

c) $0.\overline{9} = 1$

Calculation 1

Using your calculator only for addition, subtraction, multiplication and division, approximate $\sqrt{56}$. Make your answer accurate to within 0.001 of the exact answer. Write a procedure and explain why it works.

Contents

Chapter 0

Basic Notions

0.0 Getting Started

Biggles to secretary: Now, when I've got these antlers on - when I've got these antlers on I am dictating and when I take them off (takes them off) I am not dictating.
– from "Biggles Dictates a Letter," Monty Python's Flying Circus.

About moose antlers Υ: Many things in this book are already understood (or maybe we just think we understand them) and we don't want to forget them completely. At the same time we develop the real number system with a minimal set of concepts to guide us, we want to be able to use our intuition and ideas already mastered to guide our way and help us understand. We do want to keep straight where we are in this game. That's where moose antlers come in. This is how it works: when moose antlers are on, we can use what we already know to think about examples and proofs. When they are off we only think about the axioms and theorems that we have proven so far.

Look for the Υ moose antlers throughout the book. At those points feel free to use what mathematical knowledge and intuition you have to answer the questions. Otherwise, what you have at your disposal is the nine axioms and any previous theorems we have derived from those axioms using the basic notions of sets and logic that are summarized in this chapter.

0.1 Sets

0.1.1 Common Sets

Υ Some of all of these sets will be familiar to you from previous mathematical experiences. Throughout this book, we will be starting from scratch and defining each of them. References are provided below. They are all listed here to establish common notation. You may have used different notation for some of these sets and you may have other common sets you'd like to include. Do not hesitate to make your concerns known!

$\mathbb{R}$ represents the set of all real numbers. This set is the main interest and star of this course. And as in all good books the character will be developed slowly and carefully throughout the course. In the beginning, we assume a few things about how the elements in this set behave under the operations of addition and multiplication. This is quite abstract – we don't have any idea what the elements (which we will call numbers) of this set really are or even if such a set of things exists in any "real" (You can decide if this pun is intended or not) sense. From the axioms we will derive enough information to set up the familiar models for real numbers are – principally, decimal representation and the number line. We will also be able to conclude that any other system that satisfies the same axioms is essentially the same as the real number system we describe.

$\mathbb{R}^+$ represents the set of positive real numbers. Defining characteristics of this set will be established in Section 1.2

$\mathbb{R}^{\geq} = \mathbb{R}^+ \cup \{0\}$ represents the set of non-negative real numbers.

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of ordered pairs of real numbers - also called the Cartesian plane. In this book it is mostly used in reference to functions that map $\mathbb{R}$ to $\mathbb{R}$. In subsequent study of real analysis, $\mathbb{R}^n$ - ordered n-tuples of real numbers - take more central roles.

$\mathbb{N}$ and $\mathbb{Z}^+$ both represent the set of positive integers. It is a subset of the real numbers and we will later establish the characteristics of this set from the axioms of $\mathbb{R}$. Also called the set of Natural numbers. Very often the characteristics of these sets are establish by The Peano Axioms. The real numbers are then constructed from the integers. This is not the approach in this book. See Section 1.2.2.

$\mathbb{N}_0$ or $\mathbb{Z}^{\geq}$ both denote the set of non-negative integers.

$\mathbb{Z}$ represents the set of all integers. From our Υantler-less point of view we know nothing about this set. We will establish defining characteristics that will agree school-based ideas of what integers are.

$\mathbb{Q}$ represents the set of all rational numbers. They can be defined after we clarify the notion of devision and have defined the integers.

$\mathbb{Q}^+$ represents the set of all positive rational numbers.

$\varnothing$ represents the empty set.

0.1.2 Set Notation

Υ Moose antlers are tricky in this section. The book will assume you know (antlers on or not) about sets and functions and that you understand the set notation described in this chapter. It's part of the basic notions you'll need to proceed. However, all the examples of sets below require Υ moose antlers to understand. And you'll want the antlers on to come up with other examples.

We discuss three different ways to denote a set.

1. *By list.* This works perfectly for small finite sets, like $\{3, 36, 17\}$. It is also used for infinite sets that can be listed. For example, $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ or $\{1, 6, 11, 16, \cdots\}$. Describing a set this way requires that everyone knows what rule is being used to generate the numbers.

   Example 0.1 The set $\{2, 3, 5, \cdots\}$ might be the prime numbers or it might be the Fibonacci numbers or it might be the integers of the form $2^n + 1$.

2. *By condition.*

   Example 0.2 $\{x : x \text{ is a prime number}\}$, read "the set of all x such that x is a prime number."

3. *Constructively by giving a formula that describes the elements of the set.* For example, $\{n^2 : n \in \mathbb{Z}\}$, the set of perfect square numbers. Note that you need to describe the set of all possible values for each variable in the formula. Note that an element of the set may be described more than once but this does not change the set. That is, $\{n^2 : n \in \mathbb{Z}\} = \{n^2 : n \in \mathbb{Z}^{\geq}\}$.

   Example 0.3 $\{\sin \frac{n\pi}{2} : \text{n is an integer}\} = \{0, 1, -1\}$

Exercise 0.1 Υ Is there a simpler description of $\{x^2 : x \in \mathbb{R}\}$?

Exercise 0.2 Υ Let S be the set of all odd positive integers. Describe this set in each of the three ways listed above.

Exercise 0.3 Υ Find numbers a, b, c so that the formula, $an^2 + bn + c$, for $n \in \mathbb{Z}^{\geq}$, describes a set like the one indicated in Example 0.1

0.1.3 Operations on Sets

We can also construct sets from other sets. This book assumes you are familiar with the union and intersection of a collection of sets and the complement of set with respect to another one. We write the complement of X with respect to Y as $Y \setminus X = \{y \in Y : y \notin X\}$.

0.2 Logic

0.2.1 Logical Statements

Throughout this book we will be proving theorems about real numbers. Theorems are statements that are either true or false and are stated in the form 'If p, then q' and notated $p \implies q$, where p and q are also statements. p is called the hypothesis (or antecedent) and q is the conclusion (or consequence) of the theorem. The proof of the theorem proceeds from the assumed fact that p is true and goes through a series of logically valid statements until one can conclude q.

Sometimes it is easier to show that the negation of the statement of the theorem is false in order to prove the theorem true. Therefore it is good to understand how to negate a statement. $\sim p$ denotes the negation of p. Please keep in mind that either p is true or $\sim p$ is true, but not both. Here is a table of different related statements and their negations. We will use these names for related statements throughout the book.

	statement	negation of statement
	$p \implies q$	p and $\sim q$
converse	$q \implies p$	q and $\sim p$
contrapositive	$\sim q \implies \sim p$	$\sim q$ and p

A statement and its contrapositive always have the same truth value. A statement and its converse may have different values.

Exercise 0.4 How do the following two statements fit into the table.

a) $\sim p$ or q

b) p or $\sim q$

Sometimes the hypothesis of a theorem is not explicitly stated but it is always there – if for no other purpose than to establish what sets the variables in the statement belong to. Throughout the book you are encouraged to explicitly state the hypothesis and the conclusion of theorems.

True or False 2

Which of the following statements is true? Explain. Modify the false statement to make a true statement.

a) If $x \in [2, 4]$, then $x^2 \in [4, 16]$.

b) If $x^2 \in [4, 16]$, then $x \in [2, 4]$.

0.2.2 Quantifiers

Our logical statements will almost always contain variables and those variables may be 'quantified.' One way to think about it is that the statement is describing a set by a condition. The quantifiers tell you 'how many' numbers are in the set. There are two different quantifiers:

1. FOR ALL Consider the following two statements and convince yourself that they mean the same thing:

$$\text{For all } x \in \mathbb{R}, x^2 \geq 0 \tag{0.1}$$
$$\{x \in \mathbb{R} : x^2 \geq 0\} = \mathbb{R} \tag{0.2}$$

2. THERE EXISTS Consider the following two statements and convince yourself that they mean the same thing:

$$\text{There exists } x \in \mathbb{R}, x^2 = 9 \tag{0.3}$$
$$\{x \in \mathbb{R} : x^2 = 9\} \neq \varnothing \tag{0.4}$$

Exercise 0.5 Write the negation of each of the above statements.

How to prove statements that contain quantifiers is a main concern in later discussions about real numbers.

0.3 Functions

No ϒmooseantlers needed for this section. Everything is defined in terms of sets and using basic logic!

0.3.1 Definitions, Notation and Examples

Definition A *function, f,* is a set of order pairs in $\mathbb{R} \times \mathbb{R}$ with the property that for each first coordinate x, there is a unique second coordinate, y, such that (x, y) is in the function. Because the y is unique for a given x, we can unambiguously write $y = f(x)$, the customary functional notation. The *domain of f* is the set of all first coordinates in f and the *target of f* is a set that contains all second coordinates in f. In these notes the target will always be a subset $\mathbb{R}$. Thus the function is the set, $\{(x, f(x)) : x \in D\}$. Sometimes this set is called the *graph* of f, but the graph is not a separate object from the function, as we have defined it.

Here is a long list of definitions and notation conventions that we will use throughout the book. It is assumed they are mostly familiar to the reader:

Notation If D is the domain of a function f, we write $f : D \to \mathbb{R}$

Definition The following definitions apply to the function, $f : D \to \mathbb{R}$. These notions should be familiar as they are fundamental to any study of mathematics.

- For any $S \subset D$, we say the *image* of S under f, and write $f(S)$, to mean $\{f(x) : x \in S\}$. The *image* of f is $f(D)$.
- For any $T \subset f(D)$, we say the *pre-image* of T under f, and write $f^{-1}(T)$, to mean $\{x : f(x) \in T\}$.
- We say that f is $1-1$ or f is an *injection,* if for all $x, y \in D$
$$f(x) = f(y) \implies x = y,$$
- If $E \subset \mathbb{R}$ is the target of f and if $f(D) = E$, we say that f is *onto* $E \subset \mathbb{R}$, In this case, we may also say that f is a surjection.
- We say that f is *bijection*, if it is both an injection and surjection. In this case, we also say that f shows a $1-1$ correspondence between D and E.

Example 0.4 The pre-image of $[.25, 1]$ under the function $x \to x^2$ is $[-1, -.5] \cup [.5, 1]$

Exercise 0.6 Give an example of each of the following. Include the domain as part of the description. Sketch the graph of the function.

a) a function that is not $1-1$ but is *onto*

b) a function whose pre-image of some set T is $\mathbb{R}^+$

c) a function whose image is $\mathbb{R}^+$

Figure 0.1: Graph of $s_n = \dfrac{1}{n}$

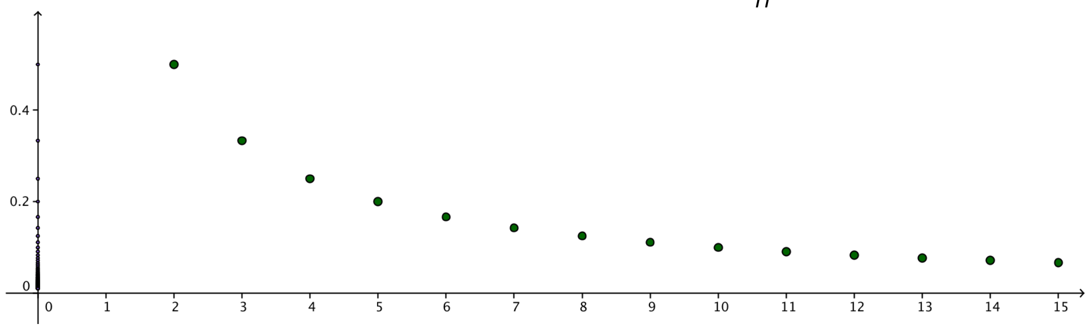
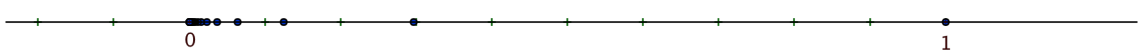


Figure 0.2: The image of $s_n = \dfrac{8}{n^3}$.



0.3.2 Sequences are functions $\mathbb{Z}^+ \to \mathbb{R}$ or $\mathbb{Z}^{\geq} \to \mathbb{R}$

Definition A function whose domain is $\mathbb{Z}^+$ or $\mathbb{Z}^{\geq}$ is called a *sequence*.

Notation A sequence is most usually denoted with subscript notation rather than standard function notation, that is we write s_n rather than $s(n)$.

Example 0.5 The graph in Figure 0.1 shows part of the graph of a sequence that maps $\mathbb{Z}^+ \to \mathbb{R}$ and is given by the formula, $s_n = \dfrac{1}{n}$. In addition, the first 100 numbers in the image of the sequence on the $y - axis$.

Example 0.6 Another way to picture a sequence is to plot the image on a number line, as shown in Figure 0.2. The downside is that the order of the sequence is not explicitly given. Here the image of the sequence, $s_n = \dfrac{8}{n^3}$, is shown on a horizontal number line. The order of the sequence values is not shown on this picture. You need to see the formula, as well, to understand that the values are being listed in order from right to left. The values in the image bunch up at zero to become indistinguishable from each other and from 0. The picture is insightful, but imprecise.

Notation Because of the ordering of the natural numbers, a sequence can be given by listing the first few values without reference to the domain or a formula, as in

$$3, \frac{3}{4}, \frac{1}{3}, \frac{3}{25}, \frac{1}{12} \cdots \tag{0.5}$$

This type of notation can be convenient but it never tells the whole story. How does the sequence continue past the values given? The finite sequence may suggest a pattern but one can't be sure without more information. We don't know whether the domain starts at 0 or 1, but a formula could be adjusted to fit either case. NOTE: It would be wrong to include braces $\{\}$ around the sequence because that would indicate a *set*. It would be how to denote the *image* of the sequence.

Exercise 0.7 Find a possible formula that would generate the sequence in Example 0.5. Sketch the graph of this sequence.

0.4 True or False

The following True or False problems explore different logical statements using quantifiers in a variety of ways.

True or False 3

Which of the following statements are true? Explain. Change $\mathbb{Z}$ to $\mathbb{R}$ and redo.

a) There exists an $x \in \mathbb{Z}$, such that x is odd.

b) For all $x \in \mathbb{Z}$, x is even.

c) There exists an $x \in \mathbb{Z}$, such that $2x$ is odd.

d) For all $x \in \mathbb{Z}$, $2x$ is even.

True or False 4

(From Morgan) Which of the following statements are true? Explain.

a) For all $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$ such that $y > x^2$.

b) There exists an $y \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, $y > x^2$.

c) There exists an $y \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, $y < x^2$.

d) For all $a, b, c \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $ax^2 + bx + c = 0$.

True or False 5

Which of the following statements are true? Explain.

a) There exists a real number, x, such that $x^2 = 9$.

b) There exists a unique real number, x, such that $x^2 = 9$.

c) There exists a unique positive real number, x, such that $x^2 = 9$.

The following True or False problem concerns the notion of pre-image.

True or False 6

Which of the following statements true? Prove or give a counterexample. Consider conditions of f that would make the statements True.

a) $f(X \cap Y) = f(X) \cap f(Y)$

b) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

c) $f(f^{-1}(Y)) = Y$.

d) $f^{-1}(f(Y)) = Y$.

Exercise 0.8 In some cases, it may be easier to determine if the negation of a statement is true or false. If you haven't already, write the negation of each statement in the True or False problems.

Chapter 1

The Real Number System

We begin by supposing the existence of a set, $\mathbb{R}$, whose elements we call *real numbers* We suppose we know a few things about numbers. We know when two of them are equal, section 1.0.1. We know that we can add or multiple two of them and get an unique number, section 1.0.2. We know some conventions for how to write expressions involving adding and multiplying, section 1.0.3. Most importantly we postulate nine *Axioms*, the defining characteristics of real numbers. These include the *Field Axioms*, Section 1.1; the *Order Axioms*, Section 1.2; and the *Completeness Axiom*, Section 1.3.

We do not assume that we can represent real numbers as decimals. Nor how to represent real numbers on a number line. However, our Υ intuition using these two models for real numbers can guide our thinking.

1.0 Definitions and Basic Notions from Algebra

1.0.1 Equality

Basic Notion 1 EQUALITY OF REAL NUMBERS IS AN EQUIVALENCE RELATION. *The following properties apply for all real number x and y:*

Reflexive $x = x$
Symmetric *if* $x = y$, *then* $y = x$
Transitive *if* $x = y$ *and* $y = z$, *then* $x = z$

1.0.2 Addition and Multiplication

We assume the existence of a set of two *binary* operations on the 'numbers' in this set. Basic notions about equality apply. Both addition and multiplication produce a unique, answer, meaning that adding a number a and multiplying by a number m are both functions.

Basic Notion 2 UNIQUENESS OF ADDITION *For all* $a, x, y \in \mathbb{R}$.

$$x = y \implies a + x = a + y.$$

Basic Notion 3 UNIQUENESS OF MULTIPLICATION *For all* $m, x, y \in \mathbb{R}$,

$$x = y \implies m \cdot x = m \cdot y.$$

It is often useful, and some people prefer, to consider addition and multiplication as functions. That is, for every real number a, there is a function, $s_a : \mathbb{R} \to \mathbb{R}$, given by

$$s_a(x) = x + a,$$

and, for every real number m, there is a function, $t_m : \mathbb{R} \to \mathbb{R}$, given by

$$t_m(x) = m \cdot x.$$

The UNIQUENESS OF ADDITION AND MULTIPLICATION says that these functions are indeed functions, i.e. there is only one value for each element in the domain.

The uniqueness of these operations is used in our preliminary work when doing things like adding the same number to both sides of an equation.

1.0.3 Expressions

Binary means that the operation works on only two numbers at a time, so expressions like $a + b + c$ aren't meaningful until we know more about what rules apply. However, we can include parentheses in expressions and so legitimately know what to do. The expression, $a + (b + c)$, is meaningful: first add b to c, then add a to the result. This use of parentheses is assumed familiar to the student of this book. (Υ Once we know the associative and commutative rules, $a + b + c$ is not an ambiguous expression.)

Basic to working with equations with variables and real numbers is being able to 'substitute' equal expressions for each other. This is how we can build more and more complicated, and thus interesting, expressions. This is however, not an easy concept to formalize. Here is a stab at it:

Basic Notion 4 SUBSTITUTION *If u is a real number such that $u = E(x, y, z, \cdots)$, where $E(x, y, z, \cdots)$ is any legitimate expression for a real number involving other real numbers $x, y, z, \cdots$, then u may be interchanged for $E(x, y, z, \cdots)$ in any other expression without changing the value of that expression. NOTE: transitivity, from Basic Notion 1, is an elementary example of substitution.*

Υ Eventually, we want the substitution principle to apply to more sophisticated expressions like $\sin(a+b)$ or re^{-3c} or $\lim_{x\to p} f(x)$, so we use the (admittedly) imprecise language, 'legitimate expression'. Once one learns that these expressions, or others, represent real numbers, we are free to use substitution on such expressions: If $u = \sin(x)$, then $\sin^2(x) + 2\sin(x) + 1 = u^2 + 2u + 1$

1.1 The Field Axioms

There is a set $\mathbb{R}$ of *real numbers* with an addition ($+$) and a multiplication ($\cdot$) operator, that satisfy the following properties:

Axiom 1 COMMUTATIVE LAWS. *For all real numbers, a and b,*

FOR ADDITION $a + b = b + a$
FOR MULTIPLICATION $a \cdot b = b \cdot a$

Axiom 2 ASSOCIATIVE LAWS. *For all real numbers, a, b and c,*

FOR ADDITION $a + (b + c) = (a + b) + c$
FOR MULTIPLICATION $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Axiom 3 DISTRIBUTIVE LAW. *For all real numbers, a, b and c,*

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Axiom 4 EXISTENCE OF IDENTITY ELEMENTS.

FOR ADDITION *There is a real number,* 0*, such that,* $a + 0 = a$*, for all* $a \in \mathbb{R}$
FOR MULTIPLICATION *There is a real number,* 1*, such that,* $1 \cdot a = a$*, for all* $a \in \mathbb{R}$
FURTHERMORE, $0 \neq 1$

Axiom 5 EXISTENCE OF INVERSES.

FOR ADDITION *For all* $a \in \mathbb{R}$*, there is an* $x \in \mathbb{R}$*, such that* $a + x = 0$*.*
FOR MULTIPLICATION *For all* $a \in \mathbb{R}$*,* $a \neq 0$*, there is an* $x \in \mathbb{R}$*, such that* $a \cdot x = 1$*.*

1.1.0 Consequences of the Field Axioms

In this section we state and prove many facts about real numbers. We call these facts theorems and, occasionally, corollaries or lemmas. Theorems are not always stated explicitly in the form $p \implies q$, but they can be stated that way. Often the only hypothesis is that the elements being discussed are real numbers, i.e. we are always assuming that there is a set of real numbers that satisfy the basic axioms as well as all theorems we prove.

Exercise 1.1 Provide proofs for Theorems 1.0 - 1.15. Some of the proofs are provided.

Theorem 1.0 *For all* $a \in \mathbb{R}$, $0 + a = a$ *and* $1 \cdot a = a$

Theorem 1.1 CANCELLATION LAW FOR ADDITION *For all* $a, b, c \in \mathbb{R}$,

$$\text{if } a + b = a + c \text{ , then } b = c.$$

Proof of Theorem. Let e be a real number from **Axiom 5** such that $a + e = 0$. By **Axiom 1**, it is also true that $e + a = 0$. *Note: here we use what is sometimes called a two column proof: the left side is a valid conclusion following from the previous statements and the right side is the justification or* warrant *for that conclusion.*

$a + b = a + c$	given, the hypothesis
$e + (a + b) = e + (a + c)$	uniqueness of addition, add e to both sides
$(e + a) + b = (e + a) + c$	associative law for addition, **Axiom 2**
$0 + b = 0 + c$	by substitution, as stated above: $e + a = 0$
$b = c$	Theorem 1.0

□

Theorem 1.2 *The number,* 0*, the additive identity of* **Axiom 4***, is unique.*

Proof. Note: the strategy used to show that a number is unique is to assume there are two numbers that satisfy the given condition and then show that they are equal. Assume there exists another real number $0'$ such that $a + 0' = a$ for all $a \in \mathbb{R}$, then for all $a \in \mathbb{R}$, we have that $a + 0 = a + 0'$. By the CANCELLATION LAW FOR ADDITION, Theorem 1.1, we can cancel the a's to get, $0 = 0'$. This shows that 0 is unique. □

Theorem 1.3 EXISTENCE AND UNIQUENESS OF SUBTRACTION *For all* $a, b \in \mathbb{R}$, *there is a unique solution,* $x \in \mathbb{R}$, *to the equation* $a + x = b$.

Notation The unique solution to the equation, $a + x = b$, is denoted by $b - a$. The unique solution to the equation, $a + x = 0$, $0 - a$, is written simply $-a$. We call it the *additive inverse* of a or the *negative* of a.

Proof. Let e be a real number from **Axiom 5** such that $a + e = 0$ and let $x = e + b$. Now consider

$a + x = a + (e + b)$	substituting our definition of x
$= (a + e) + b$	associative law for addition, **Axiom 2**
$= 0 + b$	substituting 0 for $a + e$
$= b$	Theorem 1.0

To show uniqueness, assume that there is another real number, y, such that $a+y = b$. Then $a+x = a+y$. So by the CANCELLATION LAW FOR ADDITION, Theorem 1.1, $x = y$. □

Theorem 1.4 *For all* $a \in \mathbb{R}$,

$$-(-a) = a.$$

Proof. Given any number $a \in \mathbb{R}$,

$-a + a = a + (-a)$	commutative law for addition, **Axiom 1**
$a + (-a) = 0$	definition of $-a$ from Theorem 1.3.
$-a + a = 0$	transitive property of equality

So a is the negative of $-a$. The negative of $-a$ is written, $-(-a)$, as seen in the note after Theorem 1.3. □

Theorem 1.5 MULTIPLICATION DISTRIBUTES ACROSS SUBTRACTION *For all* $a, b, c \in \mathbb{R}$,

$$a(b - c) = ab - ac.$$

Proof.

$a(b - c) + ac = a((b - c) + c)$	distributive law, **Axiom 3**
$a((b - c) + c) = ab$	definition $b - c$ from Theorem 1.3
$a(b - c) + ac = ab$	transitive property of equality
$ac + a(b - c) = ab$	commutative law for addition, **Axiom 1**
$a(b - c) = ab - ac$	EXISTENCE AND UNIQUENESS OF SUBTRACTION, Theorem 1.3

□

Theorem 1.6 *For all $a \in \mathbb{R}$,*

$$0 \cdot a = a \cdot 0 = 0.$$

Proof. This is a special case of Theorem 1.5, where $b = c$. □

Theorem 1.7 CANCELLATION LAW FOR MULTIPLICATION *For all $a, b, c \in \mathbb{R}$,*

$$ab = ac \text{ and } a \neq 0 \implies b = c.$$

Proof. EFS □

Theorem 1.8 *The number, 1, of* **Axiom 4** *is unique.*

Proof. EFS □

Theorem 1.9 EXISTENCE AND UNIQUENESS OF DIVISION *For all a, $b \in \mathbb{R}$, $a \neq 0$, there is a unique solution, $x \in \mathbb{R}$, to the equation $a \cdot x = b$.*

Notation The unique solution to the equation $a \cdot x = b$ is written $\frac{b}{a}$. The unique solution to the equation $a \cdot x = 1$, $\frac{1}{a}$, is also written as a^{-1}. It is the multiplicative inverse of a and also callied the *reciprocal* of a.

Proof. EFS □

Theorem 1.10 THERE ARE NO ZERO DIVISORS *For all $a, b \in \mathbb{R}$,*

$$a \cdot b = 0 \implies a = 0 \text{ or } b = 0.$$

Proof. EFS □

Example 1.1 Modular arithmetic is an example where there are zero divisors. Because $2 \cdot 4 \equiv 0 \mod 8$ and neither is $\equiv 0 \mod 8$ we call both 2 and 4 *zero divisors mod* 8.

Theorem 1.11 *For all $a \in \mathbb{R}$, $a \neq 0$,*

$$(a^{-1})^{-1} = a \text{ or } \frac{1}{\frac{1}{a}} = a.$$

Proof. EFS □

Theorem 1.12 *For all $a, b \in \mathbb{R}$, $b \neq 0$,*

$$\frac{a}{b} = a \cdot b^{-1}.$$

Proof. EFS □

Theorem 1.13 ADDITION OF FRACTIONS *For all* $a, b, c, d \in \mathbb{R}$, $b \neq 0$ *and* $d \neq 0$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Proof. EFS □

Theorem 1.14 MULTIPLICATION OF FRACTIONS *For all* $a, b, c, d \in \mathbb{R}$, $b \neq 0$ *and* $d \neq 0$,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Proof. EFS □

Corollary 1.14.1 *For all* $a, b \in \mathbb{R}$, $a \neq 0$ *and* $b \neq 0$,

$$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b} \text{ or } (a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$$

Proof. EFS □

NOTE: Sometimes corollary 1.14.1 is proven first as a lemma and is then used to prove theorem 1.14.

Theorem 1.15 EXISTENCE AND UNIQUENESS OF SOLUTION TO LINEAR EQUATIONS *For all* a, b, $c \in \mathbb{R}$, $a \neq 0$, *there is a unique solution,* $x \in \mathbb{R}$, *to the equation*

$$a \cdot x + b = c.$$

Proof. EFS □

Exercise 1.2 Prove using the Axioms 1 through 5 and the Theorems 1.0 - 1.15.

For all $a, b, c, d \in \mathbb{R}$.

a) $-0 = 0$; $1^{-1} = 1$

b) $(-a) \cdot (-b) = a \cdot b$

c) $-(a + b) = -a - b$

d) $-(a - b) = -a + b$

e) $(a - b) + (b - c) = a - c$

f) $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$

g) $-(\frac{a}{b}) = \frac{-a}{b} = \frac{a}{-b}$

1.1.1 Comments on the Field Axioms

Addition vs multiplication Except for the Distributive Law, **Axiom 3**, and the part of **Axiom 5** where 0 is excluded from having a multiplicative inverse, the axioms are symmetric in addition and multiplication. **Axiom 3** says that 'multiplication distributes across addition.' What would happen if the opposite were true that 'addition distributes across multiplication?'

Exercise 1.3 Write down a rule that would say that addition distributes across multiplication. Prove that it cannot be true if the Field Axioms are true.

Exercise 1.4 Something to contemplate: Why is it necessary to exclude a multiplicative inverse for 0?

Associativity and commutativity Without parentheses, we do not know how to resolve an expression like $a + b + c$ or $a \cdot b \cdot c$ The Associative Law, **Axiom 2**, tells us that the two ways to add parentheses will give you the same answer. This is worth pondering with specific numbers (Υ required). Consider the expression, $3+4+5$. We can resolve to either $7 + 5$ or $3 + 9$. We get the same thing, 12, either way.

Exercise 1.5 How many ways can you add parentheses to $2+3+4+5$ to get a different way to sum the numbers? Let $c_n =$ the number of ways to put parentheses on an addition string with n numbers. What is c_n? (These are well-known as the Carmichael Numbers.)

1.1.2 Examples of Fields

A set, together with a well-defined addition and multiplication, is called a *Field* if the Field Axioms (Section 1.1) are all satisfied.

Example 1.2 Υ Think of some other number systems, such as

a) $\mathbb{Q}$, the rational numbers.

b) $\mathbb{R}^2$, the plane of ordered pairs of real numbers.

c) $\mathbb{C}$, the complex numbers.

d) $\mathbb{Z}/\mathbb{Z}n$, integers $\mod n$

e) $\mathbb{Q}[x]$, polynomials with rational coefficients

f) $\mathbb{Q}(x)$, rational functions (ratios of functions $\in \mathbb{Q}[x]$).

Does each system have a well-defined addition and multiplication? Which satisfy the Field Axioms?

1.2 The Order Axioms

ϒSome geometric concerns One useful way to represent real numbers is on a number line. To do this we need rules to decide how to place the numbers. When is one number to the right or to the left of another number? Put another way, when is one real number 'larger' than another? Or even, what does 'larger' mean? The standard approach is to first decide which ones are 'larger' than 0, section 1.2. So we can declare that 'b is larger than a' whenever '$b - a$ is larger than 0.' The Order Axioms, then, are closely related to notions of distance and length, section 1.2.4. More surprisingly, perhaps, is that the Order Axioms allow us to think about integers, section 1.2.2. Then rational numbers can be defined, section 1.2.3. In this section we see how these concepts are developed from axioms. There exists a subset, $\mathbb{R}^+ \subset \mathbb{R}$, with the following properties:

Axiom 6 *If a and b are in $\mathbb{R}^+$, then $a + b \in \mathbb{R}^+$ and $a \cdot b \in \mathbb{R}^+$.*

Axiom 7 *If $a \neq 0$, then either $a \in \mathbb{R}^+$ or $-a \in \mathbb{R}^+$ but not both.*

Axiom 8 $0 \notin \mathbb{R}^+$

1.2.1 Consequences of the Order Axioms

Definition We say that a real number, x, is a *positive* number whenever $x \in \mathbb{R}^+$. We say that a real number, x, is a *negative* number whenever $-x \in \mathbb{R}^+$.

Notation If $b - a$ is a positive number, we write $a < b$ or $b > a$. In this case, we say 'a is less than b' or 'b is greater than a.'

An immediate and important consequence of the order axioms is:

Theorem 1.16 1 *is a positive number.*

Proof. By **Axiom 7** if $1 \notin \mathbb{R}^+$, then $-1 \in \mathbb{R}^+$ which would mean that

$$(-1) \cdot (-1) = 1 \in \mathbb{R}^+.$$

Which shows not only that $1 \in \mathbb{R}^+$ but also that $-1 \notin \mathbb{R}^+$. □

Exercise 1.6 Provide proofs for Theorems 1.17 - 1.21. Some of the proofs are provided.

Theorem 1.17 LAW OF TRICHOTOMY *For all $a, b, c \in \mathbb{R}$, exactly one of the following are true:*

$$a = b,\ a < b,\ b < a$$

NOTE: In the special case, when one of a and b is zero the Law of Trichotomy says that a real number is exclusively positive, negative, or zero.

Proof. EFS □

Theorem 1.18 TRANSITIVITY *For all* $a, b, c \in \mathbb{R}$,

$$a < b \text{ and } b < c \implies a < c$$

Proof. EFS □

NOTE: 'Less than' ($<$) forms a relation between numbers. We have just shown that it is a transitive relation. However, it is neither symmetric nor reflective so it is not an equivalence relation like '$=$'. See **Basic Notion 1**

Exercise 1.7 There are three variations of TRANSITIVITY when '$\leq$' replaces '$<$' in one or the other or both of spots in the hypothesis. State each one, providing the strongest conclusion in each case. Prove at least one of your statements. Use TRANSITIVITY , Theorem 1.18 rather than repeating proofs.

Exercise 1.8

Notation There are many different varieties of *intervals.* Write each one of the following using set notation:

$(a, b) = \{x \in \mathbb{R} : a < x$ and $x < b\}$. Sometimes abbreviated: $a < x < b$

$[a, b]$

$(a, b]$

$[a, b)$

(a, ∞)

$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$.

(∞, b)

$(\infty, b]$

Exercise 1.9 How would you notate the set, $\{x : a < x$ or $x < b\}$? How does it vary with whether or not $a < b$ or $b < a$?

Theorem 1.19 ADDITION PRESERVES ORDER *If* a *is a real number, then*

$$x < y \implies a + x < a + y$$

Proof. EFS □

Theorem 1.20 *If* $a < b$ *and* $c < d$*, then* $a + c < b + d$

Proof. EFS □

Multiplication preserves order only when the multiplication factor is positive:

Theorem 1.21 *If $m > 0$, then multiplication by m preserves order, that is*

$$x < y \implies m \cdot x < m \cdot y.$$

If $m < 0$, then multiplication by m reverses order, that is

$$x < y \implies m \cdot x > m \cdot y.$$

Proof. EFS □

Exercise 1.10 Here are some more basic facts about order. You may want to prove them in a different 'order.'

a) For all $a \in \mathbb{R}$, if $a \neq 0$, then $a^2 > 0$.

b) If $a < b$, then $-b < -a$.

c) If $a > 0$, then $a^{-1} > 0$.

d) If $0 < a < b$, then $a^{-1} > b^{-1} > 0$.

e) The sum of two negative numbers is negative.

f) The product of two negative numbers is positive.

g) The product of a negative number and a positive number is negative.

h) For all $a, b \in \mathbb{R}$, $a^2 + b^2 = 0 \iff a = b = 0$.

True or False 7

If true, prove the statement. If false, restate to make a true fact and prove it.

a) $x^2 > x$.

b) If w_1, $w_2 > 0$ and $w_1 + w_2 = 1$, then

$$a < b \implies a < w_1 a + w_2 b < b$$

c) If $a < b$ and $c < d$, then $ac < bd$.

Definition We say a function, $f : D \to \mathbb{R}$, is *increasing* whenever, for all $x, y \in D$,

$$x \leq y \implies f(x) \leq f(y).$$

We say a function is *strictly increasing* whenever, for all $x, y \in D$,

$$x < y \implies f(x) < f(y).$$

We say a function is *decreasing* whenever, for all $x, y \in D$,

$$x \leq y \implies f(y) \leq f(x).$$

We say a function is *strictly decreasing* whenever, for all $x, y \in D$,

$$x < y \implies f(y) < f(x).$$

Example 1.3 The function $s_a(x) = x + a$ is increasing by Theorem 1.19.

Exercise 1.11 When is multiplication by m an increasing function? When is it decreasing?

Exercise 1.12 Prove that the function, $f(x) = x^2$, is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$.

Exercise 1.13 Prove that the function, $f(x) = \frac{1}{x}$, is decreasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

Exercise 1.14 Prove that the function, $f(x) = \frac{x+1}{x-1}$, is decreasing on $(1, \infty)$.

1.2.2 Integers

ϒMathematical induction. We have mentioned earlier that another way to develop the real number system is to start with integers and the Peano Axioms then construct the rational and irrational numbers. With our current endeavor, however, we still have nothing to say about integers. We do know 1 and so could define integers: $2 = 1 + 1$ and $3 = 2 + 1$ and so on. Of course, it is the 'so on' that leaves us with less precision than we like for a usable definition. But we have the idea of an inductive process and, in this section, we see how to make induction part of our development. We start with a definition:

Definition We say that a subset of $\mathbb{R}$ is an *Inductive Set* whenever both of the following conditions hold:

- $1 \in S$
- If $n \in S$, then $n + 1 \in S$

Example 1.4 The set of positive real numbers is an inductive set. $1 \in \mathbb{R}^+$ by Theorem 1.16 . The second condition follows because the sum of two positive real numbers is a positive real number by **Axiom 6.**

Exercise 1.15 What is the largest inductive set you can think of? What is the smallest?

Definition We say that a real number is a *positive integer* if it is contained in every inductive set.

Notation We denote the set of all positive integers as $\mathbb{Z}^+$.

$\mathbb{Z}^+$ is the *smallest inductive set* in the sense that it is contained in every other one.

Theorem 1.22 *The positive integers are positive real numbers.*

Proof. This is because the set of positive real numbers is an inductive set, so every positive integer is contained in it. □

Definition The *negative integers* are $\{-n : n \in \mathbb{Z}^+\}$, the negative positive integers, denoted by $\mathbb{Z}^-$. The *integers* are the positive integers together with the negative integers and 0, denoted by $\mathbb{Z} = \mathbb{Z}^+ \cup \{0\} \cup \mathbb{Z}^-$.

Exercise 1.16 Prove: There is no integer in the open interval $(0, 1)$.

Mathematical Induction.

To establish the algebraic structure of the integers there is some work to do and Mathematical Induction will be a major tool. The following theorem establishes the legitimacy of the induction procedure.

Theorem 1.23 MATHEMATICAL INDUCTION *Let S be a set of positive integers that is an inductive set, i.e. satisfies the following two conditions:*

- $1 \in S$
- *If $n \in S$, then $n + 1 \in S$*

then $S = \mathbb{Z}^+$.

Proof. By definition, S is an induction set so $\mathbb{Z}^+ \subset S$. By hypothesis, $S \subset \mathbb{Z}^+$. So $S = \mathbb{Z}^+$. □

This theorem is the basis for proof by mathematical induction: To prove a fact by mathematical induction, first restate the fact as a statement about a subset of positive integers. For example, define a set of positive integers, S, such that $n \in S$ if and only if some property, $P(n)$, is true. Then show that S is an inductive set by

- Showing a 'base case,' $P(1)$ is true ($1 \in S$)
- Showing an inductive step: If $P(n)$ is true ($n \in S$), then $P(n+1)$ is true ($n+1 \in S$).

Finally, apply Theorem 1.23 to conclude that S is all positive integers, so $P(n)$ is true for all $n \in \mathbb{Z}^+$. Υ By all means use methods you have used before and are comfortable with, but do understand how the process fits into the grand scheme of things.

The first theorems we will prove with induction establish the algebraic structure of the integers.

The integers form a commutative ring

Since the integers are a subset of the real numbers, they satisfy of all the field axioms, except for the existence of multiplicative inverses, which is not ring axiom. So we only need to show closure:

Theorem 1.24 ALGEBRAIC PROPERTIES OF INTEGERS

1. *The sum of two integers is an integer.*
2. *The product of two integers is an integer.*
3. *The negative of an integer is an integer.*

Outline of proof for sums. First, fix a positive integer m. Use induction to show that $\{n \in \mathbb{Z}^+ : m + n \in \mathbb{Z}\} = \mathbb{Z}^+$ and $\{n \in \mathbb{Z}^+ : -n + m \in \mathbb{Z}\} = \mathbb{Z}^+$. Finally, show that the sum of two negative integers is an integer without another induction proof. $\square$

Exercise 1.17 Prove: The only two integers that have a multiplicative inverse are 1 and -1.

The Well-Ordering Principle

When establishing integers from axioms, MATHEMATICAL INDUCTION is sometimes used as an axiom. Sometimes the following theorem is used instead and this theorem will be useful for us later. In any case, it can be proved with mathematical induction.

Theorem 1.25 THE WELL-ORDERING PRINCIPLE *Every non-empty set of positive integers contains a smallest integer.*

Proof. Let W be a subset of the positive integers that does not contain a smallest element. We will show that $W = \varnothing$. Let $S = \{k \in \mathbb{Z}^+ : [1, k] \cap W = \varnothing\}$. We will show that S is an inductive set.

- $1 \in S$ because if it were not in S it would be in W and it would be the smallest element in W.
- Assume $n \in S$. This means that no $k \leq n$ are in W. Now if $n+1 \in W$ it would be the smallest integer in W, but W does not have a smallest element so $[1, n+1] \cap W = \varnothing$. In other words, $n+1 \in S$.

Therefore S is an inductive set of positive integers, so must be all of them. In other words W is empty. We conclude that any non-empty set of positive integers must contain a smallest element. □

NOTE: Do not confuse the above theorem with THE WELL-ORDERING THEOREM, a theorem dependent on the Axiom of Choice. The Axiom of Choice is often included in the axioms for set theory despite certain bizarre behavior such as the Banach-Tarski Paradox. These considerations become of more interest in the the study of Lebesgue integration and will not come up for us in this course.

Exercise 1.18 Use the THE WELL-ORDERING PRINCIPLE to prove that there is no positive integer M such that $2^k < M$ for all $k \in \mathbb{Z}^+$.

Exercise 1.19 You may have noticed that our proof of THE WELL-ORDERING PRINCIPLE, Theorem 1.25, could be simplified by using *Strong Induction.* Find a good statement of strong induction and prove it using MATHEMATICAL INDUCTION, Theorem 1.23. Proceed to rewrite the proof of THE WELL-ORDERING PRINCIPLE, Theorem 1.25, using strong induction.

Practice with Induction.

Exercise 1.20 For practice with mathematical induction, prove the following two theorems. We will need both of them later in the course.

Theorem 1.26 *For all positive integers n,* $\sum_{k=0}^{n} k^2 = \frac{1}{6} \cdot n \cdot (n+1) \cdot (2n+1)$

Proof. EFS □

Theorem 1.27 BERNOULLI'S INEQUALITY *For any positive real number, x, and for all positive integers n,* $(1+x)^n \geq 1 + n \cdot x$

Proof. EFS □

Exercise 1.21 Bernoulli's Inequality is useful for many things. Use it to prove that

$$\frac{1}{2^n} < \frac{1}{n}, \text{ for all positive integers, } n.$$

Inductive Definitions

This way to define integers is an example of using an *inductive definition*. Defining a sequence *by recursion* is another. The first few elements in the sequence are given and the rest are defined in terms of the previous ones. For example we define a sequence, s_n by recursion (or inductively) by first defining $s_0 = 1$ and then, once we have defined s_k for $k < n$, define $s_n = n + 4$. Now we understand the sequence to be $1, 5, 9, 13 \ldots$. A direct or, closed form, description of the sequence is simply $s_n = 1 + 4 \cdot n$. Later, we will see examples of defining sequences of sets recursively when using the NESTED INTERVAL THEOREM.

Exercise 1.22 List the first 10 numbers in the following sequence, given by a recursive formula.

a) $x_0 = 1$ and $x_n = x_{n-1} + 5$.

b) $x_0 = 1$, $x_1 = 1$ and $x_n = x_{n-2} + x_{n-1}$.

Exercise 1.23 Identify a pattern for the following sequences. Write a recursive formula and a closed formula to describe each one.

a) $6, 18, 54, 162, \cdots$

b) $2.0, 0.2, 0.02, 0.002 \cdots$

Exercise 1.24 Write a recursive formula to capture and continue this sequence:
$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\ \cdots$

Exercise 1.25 ϒ Make a formal inductive definition that captures the procedure you established for approximating $\sqrt{56}$, accurate to within $(0.1)^n$.

1.2.3 Rational Numbers

Definition We say that a real number, r, is a *rational number* whenever there exist integers n and m such that

$$r = \frac{n}{m}$$

We denote the set of all rational numbers by $\mathbb{Q}$.

Exercise 1.26 Prove the following two theorems to establish the structure of the rational numbers.

Theorem 1.28 ALGEBRAIC PROPERTIES OF $\mathbb{Q}$

1. 0 *and* 1 *are rational numbers.*
2. *The sum of two rational numbers is a rational number.*
3. *The product of two rational numbers is a rational number.*
4. *The negative of a rational number is a rational number.*
5. *The multiplicative inverse a rational number is a rational number.*

Proof. EFS □

Theorem 1.29 $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$ *satisfies the Order Axioms.*

Proof. EFS □

Exercise 1.27 Use the previous two theorems to support the claim: the rational numbers form an Ordered Field.

Exercise 1.28 The following exercise about even and odd integers gives enough ammunition to show that the square root of 2 cannot be rational. Think of a good definition for an even integer and an accompanying good definition for an odd integer. From your definitions, prove the following things (not necessarily in this order; find the order that works well for your definitions):

a) The sum of two even integers is even; the sum of two odd integers is even; the sum of an even integer and an odd integer is odd.

b) Zero is an even; 1 is an odd; every integer is either odd or even, but not both.

c) x is odd iff $-x$ is odd; x is even iff $-x$ is even.

d) x is odd iff $x+1$ is even; x is even iff $x+1$ is odd.

e) Every integer (positive or negative or zero) is either odd or even.

f) The product of two odd integers is odd; the product of an even integer and any integer is even.

g) For any integer n, $2n$ is an even integer and $2n+1$ is an odd integer.

h) If n^2 is an even integer, then n is also an even integer.

i) Every integer can be expressed as a power of 2 times an odd integer.

j) Every rational number can be written as $\frac{m}{n}$ where both m and n are integers and at least one of them is odd.

If you cannot prove these things from your definitions, you will need to change your definitions so that you can. We can now prove the following theorem.

Theorem 1.30 *There is no rational number, s, such that $s^2 = 2$.*

Proof. Assume not, so there is a rational number $s = \frac{n}{m}$ and $s^2 = 2$. By Exercise 1.28 j) we can assume that one of n or m is odd. Now $s^2 = 2 = \frac{n^2}{m^2}$ or $2m^2 = n^2$. n^2 is even so n is even by Exercise 1.28 h). Let k be the integer such that $n = 2k$. Then $2m^2 = 4k^2$ and $m^2 = 2k^2$. Now Exercise 1.28 j) says m is even, in contradiction to the assumption that one of m and n is odd. So there can be no rational number whose square is 2. □

NOTE: Nothing is being said about whether or not the square root of 2 actually exists, only that, if it does exist, it cannot be a rational number. Because the rational numbers satisfy all of our axioms so far, we know we need more to be able to say that there is a real number, s, such that $s^2 = 2$. The final axiom is THE COMPLETENESS AXIOM which provides what is needed and we will get there soon.

1.2.4 Distance, absolute value and the Triangle Inequality

Discussion of the number line

It's easy to figure out a way to mark real numbers on a line using compass and straightedge constructions from Euclidean geometry. First, label one point on the line 0 and another 1. By convention, 1 is to the right of 0. Set the compass to the length of the segment determined by these two points. Use this setting to mark off 2, 3 and so on. By going the other way, mark off -1, -2, -3, $\cdots$ Rational numbers can be included by employing similar triangles. Addition is defined by copying a segment next to another segment. Multiplication can be defined by similar triangle constructions. But this does not get all of the real numbers because we know we can construct $\sqrt{2}$ – it is the diagonal of a square which has sides of length 1. However, there are many other real numbers that can not be constructed. Eventually we will be able to locate numbers by using THE COMPLETENESS AXIOM. We do not intend to development the number line rigorously but we do use it to enhance our intuition. In particular, we are about to introduce the idea of the distance between to real numbers.

Absolute Value and Distance

Definition The *absolute value of x*, written as $|x|$, defines a function. The value of the function is given in parts:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

Theorem 1.31 *Four obvious facts about any real number, x:*

1. *If $x > 0$, then $x = |x|$*
2. *If $x < 0$, then $x = -|x|$*
3. *$|x| = 0 \iff x = 0$.*
4. *If $x \neq 0$, then $|x| > 0$.*

Proof. EFS □

Definition We say that the *distance between a and b* is $|b - a|$. We say that the *length* of an interval is the distance between its endpoints.

It can be useful to think about absolute value in the more intuitive ideas of distance. For example, Theorem 1.31 can be restated: The distance between any real number and zero is positive, unless that number is zero - in which case, the distance is zero.

Theorem 1.32 *The distance between two numbers is zero if and only if the two numbers are equal.*

Proof. $|a - b| = 0 \iff a - b = 0 \iff a = b$. □

The following, while obvious, is useful because it can eliminate cases when proving the THE TRIANGLE INEQUALITY, Theorem 1.2.4, which in turn is very useful throughout analysis.

Lemma 1.33 $-|x| \leq x \leq |x|$

Proof. If x is positive, it is the right endpoint of the interval $[-|x|, |x|]$. If x is negative, it is the left endpoint. If $x = 0$, then $-|x| = x = |x|$. In any case, $-|x| \leq x \leq |x|$. □

Exercise 1.29 Provide proofs for Theorems 1.34 - 1.36. State the theorems in terms of distances when possible. Some of the proofs are provided.

Theorem 1.34 $|x| < |b| \iff -|b| < x < |b|$

Proof. (Distance formulation of theorem: if x is closer to zero than b, then x is in the interval, $[-|b|, |b|\,]$)

$\Longrightarrow$ Using Lemma 1.33 and order facts from Exercise 1.10, we have

$$-|b| < -|x| \leq x \leq |x| < |b|$$

Showing that $-|b| < x < |b|$.

$\Longleftarrow$ Starting with $-|b| < x < |b|$ and using the definition of absolute value, we have two cases:

1. if $x \geq 0$, then $|x| = x < |b|$.
2. if $x \leq 0$, then $|x| = -x < -(-|b|) = |b|$.

In either case, we have $|x| < |b|$.

□

Theorem 1.35 $|x| \geq |b| \iff x \leq -|b|$ *or* $x \geq |b|$

Proof. The statement of this theorem is the contrapositive of Theorem 1.34 and hence true. □

The Triangle Inequality

THE TRIANGLE INEQUALITY and related facts will be used repeated when we discuss limits.

Theorem 1.36 THE TRIANGLE INEQUALITY *in four versions.*

The first version is the most commonly used in analysis:

1. THE TRIANGLE INEQUALITY *For any two real number, x & y,*

$$|x + y| \leq |x| + |y|$$

2. DISTANCE FORM OF THE TRIANGLE INEQUALITY *For any three real numbers, x, y & z,*

$$|x - y| \leq |x - z| + |z - y|$$

Distance formulation: the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides.

3. BACKWARDS TRIANGLE INEQUALITY *For any two real number, x & y,*

$$|x - y| \geq |x| - |y|$$

Distance formulation: the distance between two numbers is greater than the difference between the absolute values of the numbers.

4. *For any two real numbers, x & y,*

$$|x - y| \geq ||x| - |y||$$

Distance formulation: the distance between two points is greater than or equal to the distance between the absolute values of the numbers.

Proof. The last three versions follow from the first by judicious choice of variables.

1. The following makes use of the fact that $|x| + |y| = ||x| + |y||$ when using Theorem 1.34.

$$\begin{aligned} -|x| \leq x &\leq |x| && \text{by Lemma 1.33} \\ -|y| \leq y &\leq |y| && \text{by Lemma 1.33} \\ -(|x| + |y|) \leq x{+}y &\leq |x| + |y| && \text{add the two inequalities} \\ |x + y| &\leq |x| + |y| && \text{by Theorem 1.34} \end{aligned}$$

2. Let $\overline{x} = x - z$ and $\overline{y} = z - y$ and apply 1. to $\overline{x} + \overline{y}$.

3. EFS

4. EFS

□

Exercise 1.30 Prove: If $a < c < d < b$ then the distance between c and d is less than the distance between a and b.

Exercise 1.31 Use THE TRIANGLE INEQUALITY, Theorem 1.36, to prove these two other versions.

a) $|x - y| \leq |x| + |y|$

b) $|x + y| \geq |x| - |y|$

Exercise 1.32 Here are some more basic facts about absolute value. Prove them directly from the definition.

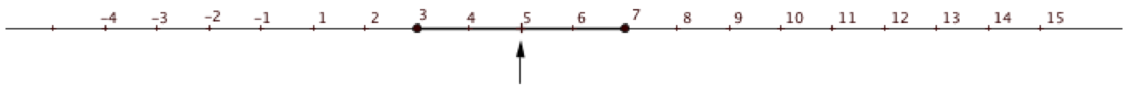
a) $|-x| = |x|$

b) $|y - x| = |x - y|$

c) $|x^2| = |x|^2$

d) If $x^2 = c$, then $|x|^2 = c$

e) $|x \cdot y| = |x| \cdot |y|$



f) $|x^{-1}| = |x|^{-1}$

True or False 8

Which of the following statements are true? Try stating and graphing each one as a fact about distances.

a) $x < 5 \implies |x| < 5$

b) $|x| < 5 \implies x < 5$

c) $|x - 5| < 2 \implies 3 < x < 7$

d) $|1 + 3x| \leq 1 \implies x \geq -\frac{2}{3}$

e) There are no real numbers, x, such that $|x - 1| = |x - 2|$

f) For every $x > 0$, there is a $y > 0$ such that $|2x + y| = 5$

g) $|a - x| < \epsilon \iff x \in (a - \epsilon, a + \epsilon)$

Example 1.5 Using absolute value is often a convenient way to define intervals. Confirm that

$$[3, 7] = \{x : |x - 5| \leq 2|\}$$

In general, if $a \leq b$, then

$$[a, b] = \{x : |x - m| \leq \frac{d}{2}\}, \text{ where } m = \frac{a + b}{2} \text{ and } d = b - a.$$

Exercise 1.33 Graph the set determined by each inequality on a number line. Explain your conclusion.

a) $|2x - 4| < 5$

b) $|2x - 4| \geq 5$

Exercise 1.34 Prove: If x is in the interval (a, b) then the distance between x and the midpoint of the interval is less than half the length of the interval.

Exercise 1.35 If the distance between two integers is less than 1, the integers are equal.

1.2.5 Bounded and unbounded sets

Definition We say a set $S \subset \mathbb{R}$ is *bounded* whenever there exists a positive real number, M, such that $|s| \leq M$ for all $s \in S$.

Definition We say a set $S \subset \mathbb{R}$ is *bounded above* whenever there exists a real number M such that $s \leq M$ for all $s \in S$.

Definition We say a set $S \subset \mathbb{R}$ is *bounded below* whenever there exists a real number m such that $s \geq m$ for all $s \in S$.

Both of the following theorems describe common techniques used in proofs about boundedness.

Theorem 1.37 *A set $S \subset \mathbb{R}$ is bounded if and only if it is bounded above and bounded below.*

Proof. EFS □

Theorem 1.38 *A $S \subset \mathbb{R}$ is bounded above if and only if the set,*

$$-S = \{x \in \mathbb{R} : -x \in S\},$$

is bounded below.

Proof. $\Longrightarrow$ Given a set, $S \subset \mathbb{R}$, that is bounded above by M we claim that $m = -M$ is a lower bound for $-S$. To see this, let $x \in -S$ so that $-x \in S$. We know that $-x < M$ so that from Exercise 1.10 b), we conclude that $x > -M$ or $x > m$. Because this is true for any $x \in -S$ we know that m is a lower bound for $-S$ and $-S$ is bounded below.

$\Longleftarrow$Conversely, if the set $-S$ is bounded below by m then $M = -m$ is an upper bound for $-(-S) = S$. □

Theorem 1.39 *The real numbers are not bounded*

HINT: State the negation of the definition of boundedness.

$$\text{For all } M \in \mathbb{R}^+, \text{ there exists } r \in \mathbb{R} \text{ such that } |r| > M.$$

Proof. Given $M \in \mathbb{R}^+$, let $r = 2M$. □

Exercise 1.36 Give a lower bound and an upper bound for each set.

a) $\{x \in \mathbb{R} : 1 \leq x^2 \leq 4\}$

b) $\{x^2 : 1 \leq x \leq 4\}$

Exercise 1.37 Υ Give an example of a function, $f : D \subset \mathbb{R} \to \mathbb{R}$, and sets $S \subset D$ and $T \subset \mathbb{R}$ for each of the following:

a) S is bounded but $f(S)$ is not.

b) T is bounded but $f^{-1}(T)$ is not.

c) $f^{-1}(f(S))$ is bounded but S is not.

d) $f(f^{-1}(T))$ is not bounded but T is bounded.

Definition We say that $\mathcal{L}$ is a *least upper bound* of a set $S \subset \mathbb{R}$ whenever both of the following conditions hold:

1. $\mathcal{L}$ is an upper bound of S
2. if u is an upper bound of S, then $\mathcal{L} \leq u$.

If $\mathcal{L}$ is a *least upper bound* of a set S and if $\mathcal{L} \in S$, we call $\mathcal{L}$ the *maximum* of S.

Definition Write out a definition for *greatest lower bound*, $\mathcal{G}$, and *minimum* of S.

Theorem 1.40 *Any two least upper bounds for a non-empty set, S, are equal. Any two greatest lower bounds for a non-empty set, S, are equal.*

Proof. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ both be least upper bounds for S. Without loss of generality, assume that $\mathcal{L}_1 \leq \mathcal{L}_2$ Since $\mathcal{L}_2$ is a least upper bound, it must be less than or equal to $\mathcal{L}_1$, which is a upper bound of S. Hence, $\mathcal{L}_2 \leq \mathcal{L}_1 \leq \mathcal{L}_2$. By trichotomy, $\mathcal{L}_1 = \mathcal{L}_2$. A similar proof works for the greatest lower bounds. □

Notation As usual uniqueness allows us to name the least upper bound and the greatest lower bound of a set, should they exist. We use the abbreviation $\sup S$ and say, *supremum of S*, for the least upper bound of set S. Similarly we use the abbreviation $\inf S$ and say, *infinum of S*, for the greatest lower bound of S. In the case when $\sup S \in S$, we also call it the *maximum* of S. If $\inf S \in S$, we call it the *minimum* of S

Example 1.6 The greatest lower bound of the open interval, $(5, 10)$, is 5

Proof. There are two steps in the proof of this fact:

1. 5 is a lower bound: By definition of the open interval, $5 < x$ for all $x \in (5, 10)$.

2. 5 is greater than any other lower bound: Suppose h is a lower bound greater than 5, so $10 > h > 5$. Consider $m = \frac{5+h}{2}$, the average of 5 and h. We know $5 < m < h < 10$. Since $m \in (5, 10)$, h is not a lower bound for $(5, 10)$. Since any number greater than 5 is not a lower bound, 5 must be the greatest one.

□

Exercise 1.38 For any two real numbers, $a < b$, the least upper bound of the interval (a, b) is b.

Exercise 1.39 Let N be an integer and let $S = \{s \in \mathbb{R} : s^2 \leq N\}$. Find a rational number that is a upper bound of S. Prove your assertion.

Definition We say that a set is *finite* whenever there exists a $1-1$ correspondence between the set and the set of all positive integers less than or equal to n, for some positive integer n. The *order* of the set, or the number of elements in the set, is n. [A $1-1$ correspondence is a bijection.]

Theorem 1.41 *A finite set has a maximum and a minimum element.*

Proof. HINT: Use induction on the number of elements in the set. □

Example 1.7 Υ What is the $\inf\{\frac{1}{n} : n \in \mathbb{Z}^+\}$?

Example 1.8 Υ Does $\mathbb{Z}^+$ have an upper bound?

Definition We say a function is *bounded (bounded above) (bounded below)* whenever the image of the function is bounded (bounded above) (bounded below). The definition includes the possibility that the function is a sequence.

Exercise 1.40 Give three examples of functions that are bounded.

Exercise 1.41 Give three examples of functions that are not bounded.

1.3 The Completeness Axiom

True or False 9

Which of the following statements are true? Explain.

a) There exists a least positive real number.

b) For all positive numbers ϵ, there exists a positive integer, N, such that $\dfrac{1}{N} < \epsilon$.

The answers to the last question is 'Yes' (see Theorem 1.51), but we can't prove it yet. We need another axiom:

Axiom 9 THE COMPLETENESS AXIOM *A non-empty set of real numbers that is bounded above has a least upper bound.*

1.3.1 Consequences of the Completeness Axiom

Clearly, there is an analogous fact for lower bounds, but it need not be stated as part of the axiom. Instead it can be proved from the axiom. The technique is a standard good trick to know.

Theorem 1.42 EXISTENCE OF GREATEST LOWER BOUND *A non-empty set of real numbers that is bounded below has a greatest lower bound.*

Outline of Proof: Given a set $S \subset \mathbb{R}$ that is non-empty and bounded below, let $T = \{x : -x \in S\}$. Because T is bounded above (Theorem 1.38)and non-empty (because S is non-empty), we know that $\sup T$ exists.

We claim that $\inf S = -\sup T$ and show it in two steps:

First, $-\sup T$ is a lower bound for S: Because $\sup T$ is an upper bound for T we have

$$t < \sup T, \text{ for all } t \in T, \text{ and so}$$

$$-t > -\sup T, \text{ for all } t \in T$$

Now if $s \in S \iff -s \in T$, so substituting $-s$ for t, we have

$$-(-s) = s > -\sup T, \text{ for all } s \in S.$$

So $-\sup T$ is a lower bound for S.

Second, Suppose $-\sup T$ is not the greatest lower bound for S, then there is another lower bound, $h > -\sup T$. But $-h$ would be an upper bound for T and hence $-h \geq \sup S$ (because $\sup S$ is the least upper bound). But that means $h \leq -\sup S$. This contradiction shows that $-\sup T$ □

A word about ϵ The greek letter, 'epsilon,' ϵ, is often used in situations where the interesting part is numbers getting arbitrary small. What we mean by *arbitrarily small* is that the inf of the set of positive ϵ's we are considering is 0. We use ϵ to stand in for 'error,' which we like to be small.

Exercise 1.42 The following three lemmas are simple but provide principles which can be used in proving the next few theorems. State an dprove the counterparts for greatest lower bounds.

Lemma 1.43 *If a set, S, has a least upper bound, $\sup S$, and if $r < \sup S$, then there exists $s \in S$ such that $r < s < \sup S$.*

Proof. If $r < \sup S$, then r can not be an upper bound for S so there must exists some number in S that is greater than r. □

Lemma 1.44 *If* $A \subset B \subset \mathbb{R}$*, then* $\sup A \leq \sup B$.

Proof. Any upper bound for B is also an upper bound for A, so the least upper bound for B, as an upper bound for A, is greater than or equal to the least upper bound for A. □

Lemma 1.45 *If* $u \in S$ *and* u *is an upper bound of* S*, then* $u = \sup S$.

Theorem 1.46 *Let* S *be a non-empty set of real numbers that is bounded above.*

$$\text{For all } \epsilon > 0, \text{ there exists } x \in S \text{ such that } \sup S - \epsilon < x.$$

NOTE: This is what we mean when we say that there are numbers in S get *arbitrarily close to* $\sup S$.

Outline of Proof. If $\epsilon > 0$, $\sup S - \epsilon$ cannot be a upper bound. Draw a numberline picture to help explain the situation. □

Exercise 1.43 State and prove the analogous theorem for the greatest lower bound of a set that is bounded below.

Theorem 1.47 *Let* S *be a non-empty set of real numbers that is bounded above. Let* U *be the set of all upper bounds for* S*, that is,*

$$U = \{u \in \mathbb{R} : u \geq s \text{ for all } s \in S\},$$

then $\inf U = \sup S$.

Outline of Proof. Any $u \in U$ is an upper bound for S, so $u \geq \sup S$, the least upper bound. So $\sup S$ is a lower bound for U. Any number greater than $\sup S$, is an upper bound for S and hence $\in U$. So $\sup S$ is the greatest lower bound. Draw a numberline picture to help explain the situation. □

Exercise 1.44 State and prove an analogous theorem for a set of real numbers that is bounded below.

The following theorem is a forerunner to THE NESTED INTERVAL THEOREM which we will be using extensively fro the rest of the course.

Theorem 1.48 *Given two non-empty subsets,* $A \subset \mathbb{R}$ *and* $B \subset \mathbb{R}$*, such that every element in* A *is a lower bound for* B *and every element in* B *is an upper bound for* A*, there exists a real number between the two sets. That is, there exists a real number,* r*, such that for all* $a \in A$ *and* $b \in B$*,* $a \leq r \leq b$ *In fact,*

$$\sup A \leq r \leq \inf B$$

Outline of Proof: Apply **Axiom 9** to argue that A has a least upper bound and that B has a greatest lower bound. Show that $\sup A$ is a lower bound for B, and hence that $\sup A \leq \inf B$. Then r could be any number in between the two. Draw a number line picture to illustrate the proof. □

Exercise 1.45 Prove: If $S \subset \mathbb{R}^{\geq}$ and if S is bounded below, then $\inf\{x^2 : x \in S\} = (\inf S)^2$.

1.3.2 The Nested Interval Theorem

Theorem 1.49 THE NESTED INTERVAL THEOREM *The intersection of a sequence of non-empty, closed, nested intervals is not empty. Furthermore, if the least upper bound of the left endpoints is equal to the greatest lower bound of the right endpoints, there is only one point in the intersection.*

NOTE: Notate the sequence by $I_n = [a_n, b_n]$, for $n > 0$. Convince yourself of the following and draw a numberline picture to help explain the theorem.

1. That each interval, I_n, is not empty is equivalent to saying that for all n, $a_n \leq b_n$,
2. That each interval is closed means that the endpoints are contained in the interval and that's why we used the closed brackets to denote the intervals.
3. That the sequence is *nested* means that, for all $n > 0$, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ or that $a_n \leq a_{n+1}$ and that $b_{n+1} \leq b_n$.

Proof of The Nested Interval Theorem 1.49. Let $A = \{a_n : n > 0\}$ and let $B = \{b_n : n > 0\}$. Since the intervals are non-empty and nested, $a_n \leq b_m$ for all positive integers n and m. Theorem 1.48 applies to the sets, A and B, so there exists a real number, r, such that $a_n \leq r \leq b_n$ for all n. Since r is in all the intervals, it is also in the intersection. And since it is also true that $\sup a_n \leq r \leq \inf b_n$, if $\sup a_n = \inf b_n$, then any point in the intersection must be equal to both. □

Exercise 1.46 Find a sequence of non-empty, nested intervals whose intersection is empty.

1.3.3 Archimedes Principle

The integers are not bounded

Theorem 1.50 *The set of positive integers is not bounded above.*

Proof. We will prove this theorem by contradiction: assume the set of positive integers is bounded. By **Axiom 9** there would be a least upper bound. Let L be this least upper bound. Then $L-1$, being less than L, is not an upper bound for the positive integers. Let N be a positive integer greater than $L-1$. We have $L-1 \leq N \implies L \leq N+1$. Since L is an upper bound for the set of positive integers and $N+1$ is a positive integer, we also have that $L \geq N+1$. Together this means that $L = N+1$. But then $N+2$ is an integer great than L so L couldn't be a upper bound. □

Exercise 1.47 Use Theorem 1.50 to show that the integers are not bounded.

Theorem 1.51 ARCHIMEDES PRINCIPLE *For all real numbers* $\epsilon > 0$, *there exists a positive integer* N *such that* $\frac{1}{N} < \epsilon$.

Proof. The negation of this statement is that there exists a positive real number ϵ, such that for all positive integers, N, $\frac{1}{N} \geq \epsilon$. But this says that $N \leq \frac{1}{\epsilon}$ for all N, or that the integers are bounded by $\frac{1}{\epsilon}$. This is false so the Archimedes Principle must be true. □

This easy restatement of Archimedes Principle is the first of many squeeze theorems.

Theorem 1.52 SQUEEZE THEOREM 1 *If* $0 \leq h < \frac{1}{n}$ *for all* $n \in \mathbb{Z}^+$, *then* $h = 0$.

Proof. Assume h satisfies the hypotheses but is not 0. By ARCHIMEDES PRINCIPLE, Theorem 1.51 , there exist $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < h$. □

Example 1.9 The following is the converse of Theorem 1.46:

Theorem 1.53 *Let* S *be a non-empty set of real numbers that is bounded above. If there is real number,* s, *such that*

$$\textit{For all } \epsilon > 0, \textit{ there exists } x \in S \textit{ such that } s - \epsilon < x.$$

Then $s = \sup S$

Proof. Since S is non-empty and bounded above $\sup S$ exists. If $s > \sup S$, let ϵ be half the distance between s and $\sup S$. So that $s - \epsilon$ is strictly larger than $\sup S$ and hence a strict upper bound of S so there is no $x \in S$ such that $s - \epsilon < x$.
On the other hand, if $s < \sup S$, then, there is $x_n \in S$ such that $s - \frac{1}{n} < x_n$. But $x_n < \sup S$ so we can conclude that $s < \sup S + \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ or $0 < s - \sup S < \frac{1}{n}$. By SQUEEZE THEOREM 1 , $s = \sup S$. □

Theorem 1.54 GENERALIZED ARCHIMEDES PRINCIPLE *For all real numbers* x *and* $d > 0$, *there exists a positive integer* N *such that* $x < N \cdot d$.

Proof. Use Archimedes Principle to prove this theorem. Note that Archimedes Principle, in turn, follows from this theorem. □

Exercise 1.48 Show that Archimedes Principle holds for rational numbers. That is, prove the following theorem without using the Archimedes Principle.

Theorem 1.55 *For all rational numbers, $r > 0$, there exists a positive integer, N, such that $0 < \frac{1}{N} < r$.*

Proof. □

Exercise 1.49 The following theorem follows from Archimedes Principle using Exercise 1.21. We use it in the next (optional) section and in later work. Prove it.

Theorem 1.56 *For all $B, h > 0$, there exists an integer $n > 0$ such that $\dfrac{B}{2^n} < h$*

Proof. EFS □

1.3.4 Optional – Nested Interval Theorem and Archimedes prove the Completeness Axiom

Theorem 1.57 *The Nested Interval Theorem and Archimedes Principle imply the Completeness Axiom*

Proof. Let S be a non-empty set with an upper bound. Construct a sequence of nested closed intervals, $[a_n, b_n]$, by bisection, so that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, $a_n \in S$, b_n an upper bound for S, and $|b_n - a_n| \leq \frac{b_0 - a_0}{2^n}$.

- Let a_0 be a point in S (S is non-empty!) and let b_0 be a upper bound for S If $a_0 = b_0$ this is the least upper bound and we are done, so assume $a_0 < b_0$.
- Assume $[a_n, b_n]$ is defined as required. Let m be the mid-point of $[a_n, b_n]$. There are two cases:

  1. If m is an upper bound for S, let $a_{n+1} = a_n$ and $b_{n+1} = m$.
  2. If m is not an upper bound for S then there exists $a_{n+1} \in S$ that is greater than or equal to m. Note that $a_{n+1} \leq b_n$ because b_n is an upper bound for S. Let $b_{n+1} = b_n$.

  If $a_{n+1} = b_{n+1}$ this is the least upper bound and we are done, so assume $a_{n+1} < b_{n+1}$. Notice that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ and $|b_{n+1} - a_{n+1}| \leq \frac{1}{2}|b_n - a_n| \leq \frac{1}{2}\frac{b_0 - a_0}{2^n} = \frac{b_0 - a_0}{2^{n+1}}$, so the new interval satisfies the requirements.

By THE NESTED INTERVAL THEOREM, all these intervals contain a common point, b. By ARCHIMEDES PRINCIPLE, b is the only common point: For, if a is another one then $0 < |b-a| < \frac{b_0-a_0}{2^n} < \frac{1}{n}$ for all n, this implies $b = a$ by SQUEEZE THEOREM 1.

Claim: b is an upper bound for S. Proof: Suppose not. then there is some $a \in S$ with $a > b$. By ARCHIMEDES PRINCIPLE, in the form of Theorem 1.56, for some n, $|a-b| > \frac{|b_0-a_0|}{2^n} = |b_n - a_n|$ so (removing parentheses) $a - b > b_n - a_n > b_n - b$ ($a_n < b$) or $a > b_n$. This shows that a must be an upper bound. a must be the least upper bound since it is in S. So $a \leq b_n$ all n. a is in all the intervals so $a = b$.

Supposes b is not the least upper bound, there there exists an upper bound for S, a, with $a < b \leq b_n$, for all n. Since a is an upper bound for S, $a_n \leq a$ for all n so a is in all the intervals. $a = b$ again.

So b is the least upper bound for S. □

1.3.5 Rational numbers are dense in $\mathbb{R}$

Definition We say that a subset $D \subset \mathbb{R}$ is *dense in* $\mathbb{R}$, whenever every open interval of $\mathbb{R}$ contains an element of D.

Exercise 1.50 The following sequence of theorems can be used to prove that the rational numbers are dense in $\mathbb{R}$. Prove Theorems 1.59 - 1.63.

Theorem 1.58 *Any non-empty set of integers that is bounded below has a minimum number.*

Proof. EFS □

Theorem 1.59 *For every real number, r, there exists a unique integer, n, such that $r < n \leq r + 1$.*

Proof. HINT: For $r > 0$, use well-ordering to find smallest positive integer greater than or equal to r. This will be n. □

Theorem 1.60 *For every real number, r, there exists a unique integer, m, such that $m \leq r < m + 1$.*

Proof. EFS □

Theorem 1.61 *If $b > a + 1$, then the open interval, (a, b) contains an integers.*

Proof. EFS □

Theorem 1.62 $\mathbb{Q}$ IS DENSE IN $\mathbb{R}$ *Every non-empty open interval contains a rational number.*

Proof. EFS □

Theorem 1.63 *Given any real number, α, there exists a rational number arbitrarily close to α. That is, given $\epsilon > 0$, there exists a rational number, r, such that $|\alpha - r| < \epsilon$*

Proof. Let r be a rational number in the interval, $(\alpha - \epsilon, \alpha + \epsilon)$. □

1.3.6 Optional – An Alternative Definition of Interval

This section is a mini-lesson on how making good, mathematical definitions can simplify understanding and proving. The problem with our current definition of interval is that there are too many parts to it. We have to worry about open and closed endpoints as well as unbounded intervals. A simplified definition may make it easier to prove things about intervals. The following two theorems exploit a condition that is simple to use. Reminder: our current definition says that an interval is one of the following:

$$(a, b) \text{ or } [a, b] \text{ or } (a, b] \text{ or } [a, b) \text{ or } (a, \infty) \text{ or } [a, \infty) \text{ or } (\infty, b) \text{ or } (\infty, b]$$

Exercise 1.51 Write all cases for the proof of the following Theorem, 1.64. The proof is straight forward but tedious because each type of interval must be dealt with separately.

Theorem 1.64 *If S is an interval, then*

$$a, b \in S \text{ and } a < c < b \implies c \in S. \tag{1.1}$$

Proof. EFS □

What if we used the condition 1.1 from the theorem as the definition of interval? Would we get the same sets are intervals?

Theorem 1.65 *If a set $S \subset \mathbb{R}$ satisfies condition 1.1, then S is an interval.*

Proof. First, suppose S is bounded. Let z be any point strictly between $a = \inf S$ and $b = \sup S$. There is a point, $x \in S$ greater than z (otherwise z would be a upper bound, less than b, the least upper bound) and a point $y \in S$ less than z (otherwise z would be a lower bound, greater than a, the greatest lower bound). By the condition, $z \in S$. Now $(a, b) \subset S \subset [a, b]$, so S is an interval by our previous definition.

Second, suppose that S is unbounded above, but not below. Let z be any point greater than $a = \inf S$. There is a point, $x \in S$ greater than z (otherwise z would be a upper bound for S.) and a point $y \in S$ less than z (otherwise z would be a lower bound, greater than a, the greatest lower bound). By the condition, $z \in S$. Now $(a, \infty) \subset S \subset [a, \infty)$ so S is an interval by our previous definition.

The other two cases are similar.

□

Notice that this proof depends on the completeness axiom. In the rational numbers, there are sets that satisfy condition (1.1) but are not intervals in our original definition.

Exercise 1.52 Show that $\{r \in \mathbb{Q} : r^2 > 2\}$ satisfies condition (1.1), but it cannot be written as $(a, +\infty)$ for any $a \in \mathbb{Q}$.

Chapter 2

Limits of Sequences

Calculus Student: $\lim_{n\to\infty} s_n = 0$ means the s_n are getting closer and closer to zero but never gets there.

Instructor: *ARGHHHHH!*

Exercise 2.1 Think of a better response for the instructor. In particular, provide a counterexample: find a sequence of numbers that 'are getting closer and closer to zero' but aren't really getting close at all. What about the 'never gets there' part? Should it be necessary that sequence values are never equal to its limit?

2.1 Definition and examples

We are going to discuss what it means for a sequence to converge in three stages:

First, we define what it means for a sequence to converge to zero

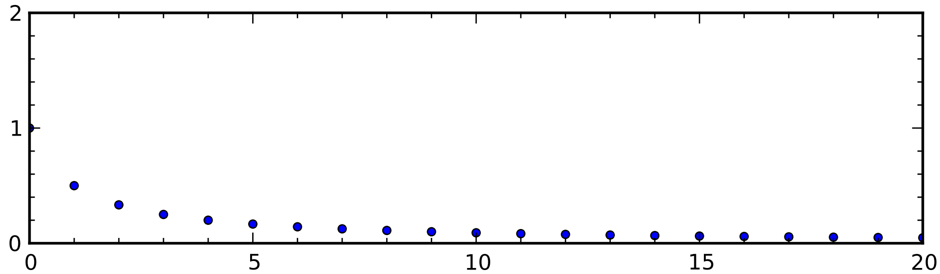
Then we define what it means for sequence to converge to an arbitrary real number.

Finally, we discuss the various ways a sequence may diverge (not converge).

In between we will apply what we learn to further our understanding of real numbers and to develop tools that are useful for proving the important theorems of Calculus.

Recall that a sequence is a function whose domain is $\mathbb{Z}^+$ or $\mathbb{Z}^{\geq}$. A sequence is most usually denoted with subscript notation rather than standard function notation, that is we write s_n rather than $s(n)$. See Section 0.3.2 for more about definitions and notations used in describing sequences.

Figure 2.1: $s_n = \dfrac{1}{n}$.



2.1.1 Sequences converging to zero.

Definition We say that the sequence s_n *converges to* 0 whenever the following hold:

For all $\epsilon > 0$*, there exists a real number,* N*, such that*

$$n > N \implies |s_n| < \epsilon.$$

Notation To state that s_n converges to 0 we write $\lim_{n\to\infty} s_n = 0$ or $s_n \to 0$.

Example 2.1 $\lim_{n\to\infty} \dfrac{1}{n} = 0$. See the graph in Figure 2.1.

Proof. Given any $\epsilon > 0$, use Archimedes Principle, *Theorem* 1.51, to find an N, such that $\frac{1}{N} < \epsilon$. Note that, if $n > N$, then $\frac{1}{n} < \frac{1}{N}$ (Exercise 1.10 d). Now, if $n > N$, we have

$$|s_n| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

In short:

$$n > N \implies |s_n| < \epsilon,$$

so we have shown that $\lim_{n\to\infty} \dfrac{1}{n} = 0$. □

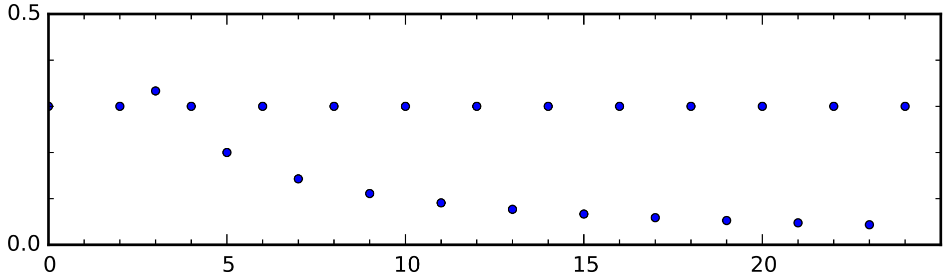
Example 2.2 If $s_n = 0$, for all n, then $\lim_{n\to\infty} s_n = 0$

Proof. Given any $\epsilon > 0$, let N be any number. Then we have

$$n > N \implies |s_n| = 0 < \epsilon,$$

because that's true for any n. □

Figure 2.2: Some values approach 0, but others don't.



Example 2.3 Why isn't the following a good definition?

" $\lim_{n\to\infty} s_n = 0$ means

For all $\epsilon > 0$*, there exists a positive integer,* N*, such that* $|s_N| < \epsilon$."

The problem is we want the sequence to get arbitrarily close to zero and to stay close. Consider the sequence:

$$s_n = \begin{cases} \frac{1}{n}, & \text{if } n \text{ is odd} \\ 0.3, & \text{otherwise.} \end{cases}$$

For any ϵ there is always an odd n with s_n less than ϵ but there there are also many even n's with values far from zero. The '$n > N$'example is an important part of the definition. See the graph in Figure 2.2.

Exercise 2.2 Prove that $\lim_{n\to\infty} \frac{3}{n} = 0$

Exercise 2.3 Prove that $\lim_{n\to\infty} \frac{1}{n^2} = 0$

Exercise 2.4 Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ See Figure 2.3.

Exercise 2.5 Prove that $\lim_{n\to\infty} \frac{1}{n(n-1)} = 0.$

It is good to understand examples when the definition of converging to zero does not apply, as in the following example.

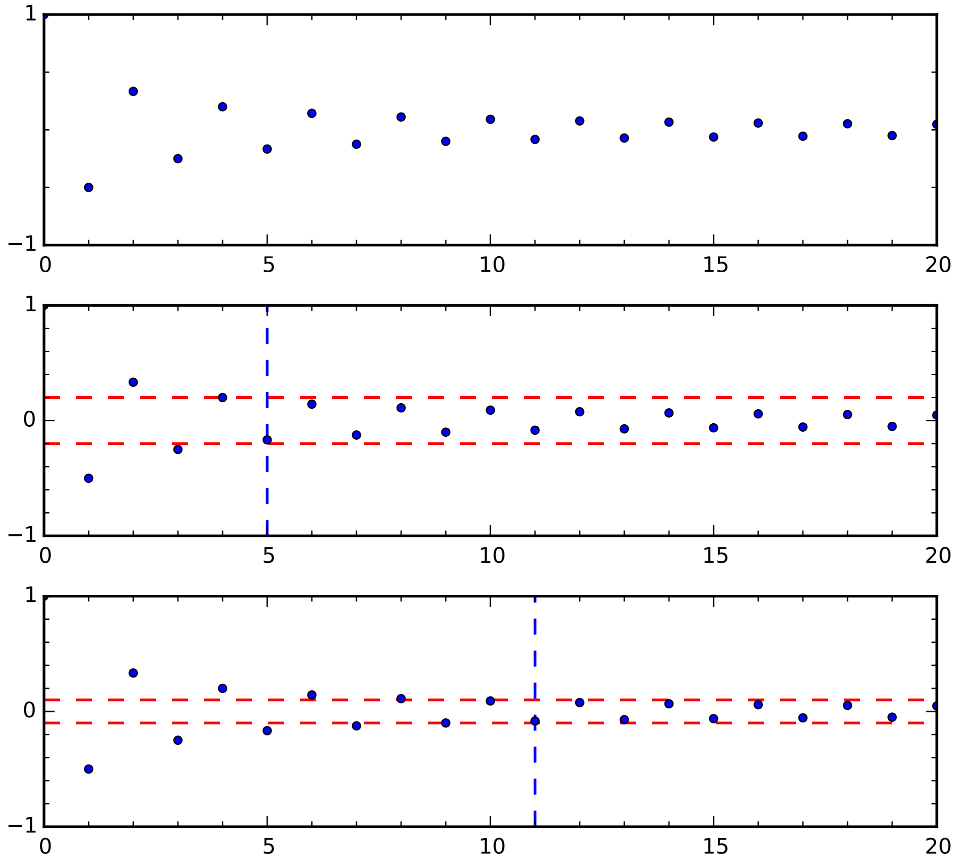
Example 2.4 Prove that the sequence, $s_n = \frac{n+1}{n+2}$ does not converge to 0.

Proof. We must show that there exists a positive real number, ϵ, such that for all real numbers, N, it's possible to have $n > N$ and $|s_n| > \epsilon$. $\epsilon = 0.5$ will do. We can see that

$$\frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{2} \geq \frac{1}{2}.$$

So, in fact, any $n > N$ works for any N to give that $|s_n| > \epsilon$. □

Figure 2.3: Picking N for smaller and smaller ϵ for the sequence $s_n = \frac{(-1)^n}{n}$.



The above are good exercises but problems like these will be easier to prove – that is, no epsilons nor multiple quantifiers will be needed – once we have some theorems. For example:

Exercise 2.6 Use the following theorem to provide another proof of *Exercise* 2.4.

Theorem 2.1 *For any real-valued sequence, s_n:*

$$s_n \to 0 \iff |s_n| \to 0 \iff -s_n \to 0$$

Proof. Every implications follows because $|s_n| = ||s_n|| = | - s_n|$ □

Theorem 2.2 *If $\lim_{n\to\infty} a_n = 0$, then the sequence, a_n, is bounded. That is, there exists a real number, $M > 0$ such that $|a_n| < M$ for all n.*

Proof. Since $a_n \to 0$, there exists $N \in \mathbb{R}^+$ such that $n > N \implies |a_n| < 1$. Here we use the definition of converging to 0 with $\epsilon = 1$. (NOTE: We could use any positive number in place of 1.) Let B be a bound for the finite set $\{a_n : n \leq N\}$. This set is bounded by Theorem 1.41. Let $M = \max\{B, 1\}$ Hence any a_n is bounded by M because it is either in the finite set ($n \leq N$) and bounded by B or it is bounded by 1, because $n > N$. □

Theorem 2.3 ALGEBRAIC PROPERTIES OF LIMITS 1

Given three sequences, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$ and a real number, c, then:

1. $\lim_{n\to\infty} a_n + b_n = 0$
2. $\lim_{n\to\infty} c \cdot a_n = 0$.
3. $\lim_{n\to\infty} a_n \cdot b_n = 0$.

Proof. 1. Let $\epsilon > 0$ be given. Because the sequences converge to 0, we can find N_1 such that

$$n > N_1 \implies |a_n| < \frac{\epsilon}{2}$$

and we can find N_2 such that

$$n > N_2 \implies |b_n| < \frac{\epsilon}{2}$$

Note that $|a_n + b_n| \leq |a_n| + |b_n|$ by the THE TRIANGLE INEQUALITY, Theorem 1.2.4. Let $N = \max\{N_1, N_2\}$, so that any $n > N$ is larger than both N_1 and N_2. Then

$$n > N \implies |a_n + b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

so we have shown that $\lim_{n\to\infty} a_n + b_n = 0$ *NOTE: The method of finding the common N from two others is often shortcut with the following words:* Find N sufficiently large so that both $|a_n| < \frac{\epsilon}{2}$ and $|b_n| < \frac{\epsilon}{2}$. *It is assumed the reader understands the process.*

2. If $c = 0$, then $c \cdot a_n = 0$ for all n and converges to 0. So assume $c \neq 0$. Let $\epsilon > 0$ be given. Because $a_n \to 0$, we can find N such that

$$n > N \implies |a_n| < \frac{\epsilon}{|c|}$$

Note that $|c \cdot a_n| = |c| \cdot |a_n| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon$. So we have shown that $c \cdot a_n \to 0$.

3. HINT: Make use of the fact that a_n is bounded and mimic the previous proof.

□

Exercise 2.7 Prove the following theorem:

Theorem 2.4 *If c_n is bounded and $a_n \to 0$, then $c_n \cdot a_n \to 0$*

The following theorem is the first in a series of 'squeeze' theorems, among the most useful tools we have at our disposal.

Theorem 2.5 SQUEEZE THEOREM *If $a_n \to 0$ and $b_n \to 0$ and $a_n \leq c_n \leq b_n$, for all $n \in \mathbb{Z}^+$, then $\lim_{n\to\infty} c_n = 0$.*

Proof. Given $\epsilon > 0$, let N be large enough so that whenever $n > N$, then both $|b_n| < \epsilon$ and $|a_n| < \epsilon$. Now, for any $n > N$, if $c_n > 0$, we have $|c_n| \leq |b_n| < \epsilon$. or if $c_n < 0$, then $|c_n| = -c_n \leq -a_n = |a_n| < \epsilon$. So, for all $n > N$ we have $|c_n| < \epsilon$. We have shown that $c_n \to 0$. □

True or False 10

Which of the following statements are true? If false, modify the hypothesis to make a true statement. In either case, prove the true statement.

a) $\lim_{n\to\infty} \frac{n^2+n}{n^3} \to 0$

b) For all $r \in \mathbb{R}$, $\lim_{n\to\infty} \frac{1}{n+r} \to 0$.

c) For any integer, m, $\lim_{n\to\infty} \frac{1}{n^m} = 0$

d) For $r \in R$, $r^n \to 0$.

Exercise 2.8 One way to modify the last **True or False**, part d), is given in the following theorem. Use BERNOULLI'S INEQUALITY Theorem 1.27 to prove the theorem.

Theorem 2.6 *If* $0 \leq r < 1$*, then* $r^n \to 0$

Proof. EFS □

2.1.2 Sequences that converge to arbitrary limit

Definition We say that s_n converges whenever there exists a real number, s, such that $|s - s_n| \to 0$. In this case, we say that s_n converges to s, and write

$$\lim_{n\to\infty} s_n = s \text{ or } s_n \to s$$

Example 2.5 $\lim_{n\to\infty} \frac{n+1}{n+2} = 1$, because $1 - \frac{n+1}{n+2} = 1 - (1 - \frac{1}{n+2}) = \frac{1}{n+2} \to 0$, as shown in **True or False**.

Exercise 2.9 Show that $s_n \to 0$ means the same thing for both definitions: converging to 0 and converging to an arbitrary limit that happens to be 0.

Theorem 2.7 UNIQUENESS OF LIMIT *If* $a_n \to a$ *and* $a_n \to b$*, then* $a = b$.

Proof. Use the triangle inequality to see that $0 \leq |a - b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b|$. Apply THE SQUEEZE THEOREM (Theorem 2.5.): the left-most term is the constant sequence, 0, the right-most term is the sum of two sequences that converge to 0, so also converges to 0, by ALGEBRAIC PROPERTIES OF LIMITS, Theorem 2.3. Hence the middle term (which is a constant sequence) also converges to 0. So $|a - b| = 0 \implies a = b$. □

Exercise 2.10 Prove: If $a_n = c$, for all n, then $\lim_{n\to\infty} a_n = c$

Theorem 2.8 *If* $\lim_{n\to\infty} a_n = a$*, then the sequence,* a_n*, is bounded.*

Proof. EFS Consider using Theorem 2.2. □

Theorem 2.9 *If* $\lim_{n\to\infty} a_n = a$ *and if* $a_n \neq 0$ *and* $a \neq 0$*, then the sequence,* a_n*, is bounded away from* 0*. That is, there exists a positive number* B*, such that* $|a_n| > B$*, for all* n*.*

Proof. *(Draw a numberline picture to help see this proof.)*To find such a bound, B, first note that there is $N > 0$ such that $|a_n - a| < |\frac{a}{2}|$ for all positive integers $n > N$. (Using $\epsilon = |\frac{a}{2}|$ in the definition of limit.) For those n,

$$|a_n| \geq |a| - |a_n - a| > |a| - |\frac{a}{2}| = |\frac{a}{2}|.$$

Now let $\overline{B} = \min\{|a_n| : n \leq N\}$. This set has a minimum value because it is a finite set. (Theorem 1.41) Of course, $\overline{B} > 0$ because none of the $a_n = 0$. Finally, let $B = \min\{\overline{B}, |\frac{a}{2}|\}$. So $|a_n| > B$, for all n. □

Theorem 2.10 ALGEBRAIC PROPERTIES OF LIMITS 2

Given two sequences, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then:

1. $\lim_{n\to\infty} a_n + b_n = a + b$
2. $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$
3. *If $a_n, a \neq 0$, then* $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

Proofs. For all the proofs make use of all the theorems we have about sequences that converge to zero.

1. EFS
2. HINT: Use this trick $|a_n \cdot b_n - a \cdot b| = |a_n \cdot b_n - a_n \cdot b + a_n \cdot b - a \cdot b|$, the triangle inequality and the boundedness of a converging sequence.
3. Theorem 2.9 applies to this sequence, let B be that positive number such that $|a_n| > B$, for all n. Consider the inequality

$$0 < |\frac{1}{a_n} - \frac{1}{a}| = \frac{|a - a_n|}{|a_n \cdot a|} < \frac{|a_n - a|}{B \cdot |a|}$$

Since $|a_n - a| \to 0$, we can apply ALGEBRAIC PROPERTIES OF LIMITS 1,Theorem 2.3 and THE SQUEEZE THEOREM, Theorem 2.5, to conclude that $|\frac{1}{a_n} - \frac{1}{a}| \to 0$ and hence $|\frac{1}{a_n} \to \frac{1}{a}|$.

□

LIMITS OF RATIOS An important concern of calculus is what happens to the ratio of two limits when both the numerator and denominator converge to 0. If the denominator converges to zero, but the numerator is bounded away from zero, then the ratio will be unbounded and not converge. See more in Section 2.1.5.

Exercise 2.11 Give examples of two sequences, $a_n \to 0$ and $b_n \to 0$, such that

a. $\frac{a_n}{b_n} \to 0$

b. $\dfrac{a_n}{b_n} \to c$, where c is a positive real number.

c. $\dfrac{a_n}{b_n}$ does not converge.

Theorem 2.11 ORDER PROPERTIES OF LIMITS

For real sequences, a_n, b_n, c_n and real numbers, a and c.

1. *If $a_n > c$ for all $n \in \mathbb{Z}^+$ and $a_n \to a$, then $a \geq c$*
2. *If $a_n \leq c \leq b_n$ for all n and $|a_n - b_n| \to 0$, then $a_n \to c$ and $b_n \to c$.*
3. THE SQUEEZE THEOREM *If $a_n \to c$ and $b_n \to c$ and $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{Z}^+$, then $c_n \to c$*

Proof. Confirm each statement and explain how it proves the corresponding part of the theorem. Drawing a numberline picture of the situation may help.

1. Suppose $c - a > 0$, find N such that $n > N \implies |a - a_n| < c - a$.
2. For all n, $0 \leq |a_n - c| \leq |b_n - a_n|$
3. $0 \leq |c - c_n| \leq |c - a_n| + |a_n - c_n| \leq |c - a_n| + |a_n - b_n|$

□

Exercise 2.12 Counterexample for Theorem 2.11 2. Find two sequences a_n and b_n such that $|a_n - b_n| \to 0$, but neither sequence converges. Is it possible that one sequence could converge but the other does not?

Theorem 2.12 THE TAILEND THEOREM *If $a_n \to a$ and if $b_n = a_{n+m}$ for some fixed, positive integer m, then*

$$b_n \to a.$$

(A tailend of a sequence is a special case of a subsequence, see Section 2.1.4.)

Proof. EFS □

Exercise 2.13 Prove: If $a_n \to c$ and $b_n \to c$, then $|a_n - b_n| \to 0$

Exercise 2.14 Conjecture what the limit might be and prove your result.

$$s_n = \frac{3n^2 + 2n + 1}{n^2 + 1}$$

Exercise 2.15 Prove that, if $c \neq 0$, then

$$\lim_{n\to\infty} \frac{a \cdot n + b}{c \cdot n + d} = \frac{a}{c}.$$

Exercise 2.16 Prove: If $a_n \to a$, $b_n \to b$, and $a_n < b_n$ for all n, then $a \leq b$.

Exercise 2.17 Prove: If S is a bounded set, then there exists a sequence of points, $s_n \in S$ such that $s_n \to \sup S$.

Exercise 2.18 Prove: If $a_n \to a$ then $a_n^2 \to a^2$

Monotone sequences

Definition We say a sequence is *monotone* whenever it is an increasing sequence or a decreasing sequence.

Theorem 2.13 MONOTONE CONVERGENCE THEOREM *Every bounded, monotone sequence converges to a real number.*

Proof. Let s_n be a bounded, increasing sequence. Let $s = \sup\{s_n\}$ which exists because s_n is bounded above. We claim that $s_n \to s$. Given $\epsilon > 0$, use Theorem 1.46 to find $x = s_N \in \{s_n\}$ such that $s - \epsilon < s_N$. Now if $n > N$, we know $s_n > s_N$ because the sequence is increasing, so

$$|s_n - s| = s - s_n < s - s_N < \epsilon.$$

We conclude that $s_n \to s$.

If t_n is a bounded, decreasing sequence, then $s_n = -t_n$ is bounded and increasing. Since $s_n \to s$, for some s, we known that $t_n = -s$. □

Best Nested Interval Theorem

Theorem 2.14 BEST NESTED INTERVAL THEOREM *There exists one and only one real number, x, in the intersection of a sequence of non-empty, closed, nested intervals if the lengths of the intervals converge to 0. Furthermore, the sequence of right endpoints and the sequence of left endpoints both converge to x.*

Proof. Denote the intervals by $[a_n, b_n]$. Because they are nested we know that a_n is increasing and b_n is decreasing so, by MONOTONE CONVERGENCE THEOREM, there are real numbers a and b such that $a_n \to a$ and $b_n \to b$. Let c be any real number in the intersection of all the intervals. Then $a_n \leq c \leq b_n$ and since $|b_n - a_n| \to 0$ we have by the ORDER PROPERTIES OF LIMITS (Theorem 2.11 *2.*), $c = a = b$. □

Rational Approximations to Real Numbers

We have not yet shown that there are real numbers other than rational numbers. However, if there is one, the following method indicates that you can approximate it by rational numbers; that is, there is a sequence of rational numbers that converge it. The method of bisection used here is a well-used tool of analysis.

Example 2.6 Let r be any non-rational real number. Find a sequence of rational numbers that converge to r.

Using the method of bisection. There are other ways to show the existence of such a sequence. The advantage to this method is that it gives a way to construct approximations of the given real number.

First, find rational numbers a_0 and b_0 such that $a_0 < r < b_0$. By Theorem 1.60, they can be consecutive integers, so $|b_0 - a_0| = 1$. Recursively define a sequence of non-empty, closed, nested intervals $[a_n, b_n]$ such that each interval contains r and $|b_n - a_n| = \frac{1}{2_n}$. We already have the base case, $[a_0, b_0]$. Assume $[a_n, b_n]$ has been defined as required. Let m be the midpoint of $[a_n, b_n]$. Since m is a rational number(why?), $m \neq r$, so there are two cases to consider:

1. If $m < r$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
2. If $r < m$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

In either case $r \in [a_{n+1}, b_{n+1}]$ and $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. The length of this interval is half the length of the previous interval $= \frac{1}{2} \cdot \frac{1}{2^n} = \frac{1}{2^{n+1}}$, so the lengths of the intervals converge to 0. So we have the intervals as required and they satisfy the BEST NESTED INTERVAL THEOREM (Theorem 2.14). That theorem tells us that both a_n and b_n converge to the unique number that is in all the intervals. But r is in all the intervals. So both $a_n \to r$ and $b_n \to r$. Either sequence will do to prove the theorem. □

2.1.3 Application: Existence of square roots

The proof given here of the existence of square roots is by construction. There are other ways to prove the existence of square roots – the advantage to this method is that it gives a way to calculate approximations to $\sqrt{2}$.

Example 2.7 There exists a unique positive real number, s, such that $s^2 = 2$.

Proof by bisection. We will show that there is a nested sequence of closed intervals, $I_n = [a_n, b_n]$, such that ${a_n}^2 \leq 2 \leq {b_n}^2$ and $|b_n - a_n| = \frac{1}{2^n}$. By the BEST NESTED INTERVAL THEOREM, there is a unique number, s, in all of the intervals. We will show that $s^2 = 2$

using a squeeze argument.
We define the intervals inductively: Base case: Let $a_0 = 1$ and $b_0 = 2$, so

$$a_0{}^2 = 1 \leq 2 \leq 4 = b_0{}^2, \text{ and } |b_0 - a_0| = 1 = \frac{1}{2^0}.$$

Assume a_n and b_n have been defined as desired, that is

$$a_n{}^2 \leq 2 \leq b_n{}^2, \text{ and } |b_n - a_n| = \frac{1}{2^n}.$$

Proceed inductively to define the next interval, $I_{n+1} = [a_{n+1}, b_{n+1}]$. Let m be the midpoint of the interval $[a_n, b_n]$. We know that $m^2 \neq 2$ because we know the square root can not be a rational number. (Why do we claim m is rational?) So there are only two cases to consider:

1. If $m^2 < 2$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
2. If $2 < m^2$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

Since m is the midpoint of the previous interval the length of I_{n+1} is half the length of I_n, so $|b_{n+1} - a_{n+1}| = \frac{1}{2} \cdot \frac{1}{2^n} = \frac{1}{2^{n+1}}$
Let s be the unique number in all the intervals, $[a_n, b_n]$. We know $a_n \to s$ and $b_n \to s$. So we also know that $a_n^2 \to s^2$, $b_n^2 \to s^2$. (See Exercise 2.18) Now consider the image, under the squaring function of all those intervals. The intervals $[a_n^2, b_n^2]$ are closed, non-empty ($a_n^2 < b_n^2$ because $a_n < b_n$); and nested (a_n^2 is increasing because a_n is increasing, and b_n^2 is decreasing because b_n is decreasing). These all follow because the function $x \to x^2$ is increasing on positive numbers, see Exercise 1.12. Furthermore, $|b_n^2 - a_n^2| \to 0$. The BEST NESTED INTERVAL THEOREM applies and we conclude there is a unique real number in all the intervals and that the endpoints converge to that number. 2 is in all the intervals and $a_n \to s^2$, so $s^2 = 2$, by uniqueness of limits.
To see why there is only one positive solution to the equation $x^2 = 2$, we will use Theorem 1.10, THERE ARE NO ZERO DIVISORS. Let $\sqrt{2}$ be the positive solution found above. Note, through the distributive law and some simplification, that

$$(x - \sqrt{2})(x + \sqrt{2}) = x^2 + x\sqrt{2} - x\sqrt{2} - (\sqrt{2})^2 = x^2 - 2$$

Now suppose $x^2 = 2$, then

$$\begin{aligned} x^2 - 2 &= 0 && \text{by subtracting 2} \\ (x - \sqrt{2})(x + \sqrt{2}) &= 0 && \text{as seen in the note} \end{aligned}$$

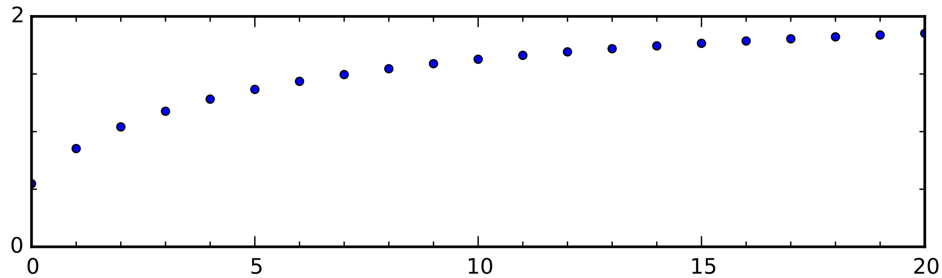
We conclude by Theorem 1.10 that $x = \sqrt{2}$ or $x = -\sqrt{2}$ are the only two solutions to $x^2 = 2$. Only $\sqrt{2}$ is positive. □

Theorem 2.15 *For all $A > 0$, there exists a unique positive real number, s, such that $s^2 = A$.*

Proof. HINT: What in the proof for Example 2.7 depends on the choice $A = 2$? □

Definition We use the symbol $\sqrt{A}$ to denote the unique number such that $(\sqrt{A})^2 = A$.

Figure 2.4: $s_n = \dfrac{\sqrt{5n+3}}{\sqrt{n+10}}$



Now that we know that every positive number has a unique positive square root we are free to use square roots in our other work.

Exercise 2.19 Conjecture what the limit might be and prove your result.

$$s_n = \frac{\sqrt{5n+3}}{\sqrt{n+10}}$$

Exercise 2.20 Use Theorem 2.13 to show that $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$.

2.1.4 Subsequences

Definition We call sequence, s_{n_k}, whose values are a subset of the values of s_n, a *subsequence* of s_n, whenever the sequence n_k is strictly increasing. (We will assume that k is indexed on $\mathbb{Z}^{\geq}$, i.e. n_0 is the first value.)

Example 2.8 If $n_k = 2k$, then the subsequence is every other element, starting at 0, of the sequence.

Exercise 2.21 Prove: If s_{n_k} is a subsequence of s_n, then $n_k \geq k$. Note that this is true for any increasing sequence of positive integers n_k.

Exercise 2.22 Show how a 'tailend' of a sequence, as discussed in Theorem 2.12, is a subsequence of that sequence.

Theorem 2.16 *If $s_n \to s$, then any subsequence of s_n also converges to s.*

Proof. Let s_{n_k} be a subsequence of s_n. Given $\epsilon > 0$, find N so that $n > N \implies |s_n - s| < \epsilon$. Such N exists because $s_n \to s$. Now consider the subsequence: If $k > N$, then $n_k > N$ (by Exercise 2.21), so $|s_{n_k} - s| < \epsilon$. $\square$

Example 2.9 The sequence is Exercise 2.23 has many subsequences, each of which converges to one of five different numbers. For example, $s_{5k+2} \to \sin \frac{4\pi}{5}$.

2.1.5 Divergent Sequences

Definition A sequence is said to *diverge* if there is no real number L such that the sequence converges to L.

To show that a sequence, s_n, converges we would first conjecture a possible limit, L, and then prove $s_n \to L$. To show that the sequence does not converge is perhaps harder because we have to show it doesn't converge for all possible values L. And we would need to prove the negation of the statement, $s_n \to L$, for all values of L. Here are both statements:

$s_n \to L$, means

$$\text{For all } \epsilon > 0, \text{ there exists } N > 0, \text{ such that}$$
$$n > N \implies |s_n - L| < \epsilon.$$

$s_n \not\to L$, means

$$\text{There exists an } \epsilon > 0, \text{ such that for all } N > 0$$
$$\text{there is an } n > N \text{ with } |s_n - L| \geq \epsilon.$$

Fortunately there is an easier way to show that a sequence diverges by observing subsequence behavior, using Theorem 2.16.

Example 2.10 The sequence,

$$s_n = \begin{cases} 1, & n, \text{ odd} \\ -1, & n, \text{ even} \end{cases}$$

diverges, i.e. does not converge to any real number L.

Proof. The subsequence given by $n_k = 2k + 1$ is a constant sequence: $s_{n_k} = 1$ for all k. This subsequence converges to 1. The subsequence given by $n_k = 2k$ converges to -1. If the sequence converged, both subsequences would have to converge to the same number, by Theorem 2.16. So the sequence does not converge. □

Exercise 2.23 ϒ Explain why the following sequence, $s_n = \sin(\frac{2\pi n)}{5}$, diverges. The graph of this sequence is shown in Figure 2.5.

Example 2.11 Another ϒ example of a *divergent* sequence is $s_n = \sin(n)$. That this sequence does not converge seems a correct conclusion considering the graph, shown in Figure 2.6. Proving that goes beyond the scope of our present discussion.

Another way a sequence may fail to converge is if it is unbounded. We consider separately the case when the limit appears to be infinite as in the sequence, $s_n = n$.

Figure 2.5: $s_n = \sin(\frac{2\pi n}{5})$.

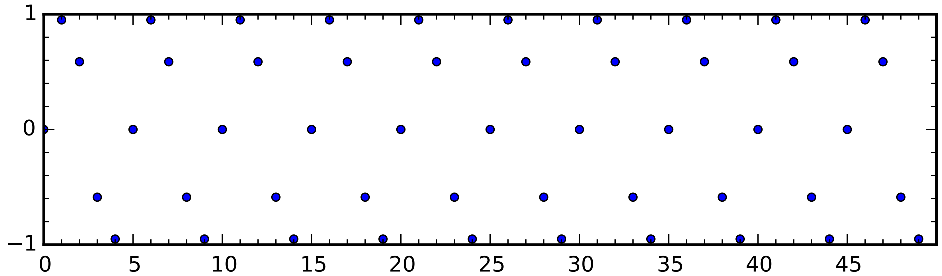
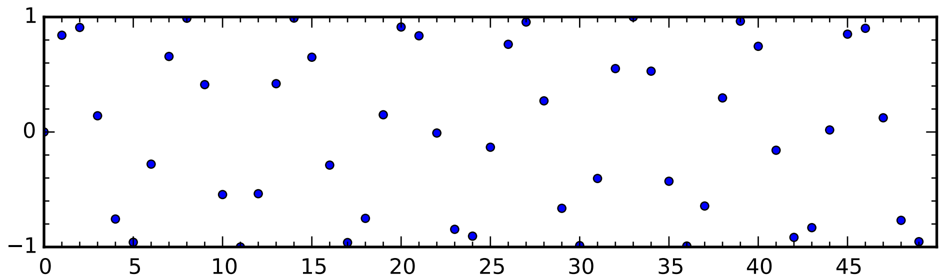


Figure 2.6: $s_n = \sin(n)$.



Definition We say that s_n *diverges to infinity* and we write, $\lim_{n\to\infty} s_n = +\infty$, whenever, for all $M > 0$, there exists $N > 0$ such that

$$n > N \implies s_n \geq M.$$

Example 2.12 $\lim_{n\to\infty} n^2 = +\infty$: If M is any positive real number, let $N = M$. Then, if $n > N$, we have that $n^2 > n > N = M$ or simply that $n^2 > M$. (EFS: Explain the step $n^2 > n$)

Exercise 2.24 Give a sequence that is unbounded but does not diverge to $+\infty$

Theorem 2.17 *If $s_n \to +\infty$, then any subsequence of s_n also diverges to $+\infty$.*

Proof. EFS □

Exercise 2.25 Prove the following theorem.

Theorem 2.18 ALGEBRAIC PROPERTIES OF DIVERGENT LIMITS

1. $a_n \to +\infty$ *and* $b_n \to b$ *or* $b_n \to +\infty \implies (a_n + b_n) \to +\infty$.
2. $a_n \to +\infty$ *and* $b_n \to b > 0$ *or* $b_n \to +\infty \implies (a_n \cdot b_n) \to +\infty$.
3. $a_n \neq 0,\ a_n \to +\infty \implies \dfrac{1}{a_n} \to 0$

4. *If $a_n > 0$ for all n, then $a_n \to 0 \implies \frac{1}{a_n} \to +\infty$*

Exercise 2.26 Prove the following theorem.

Theorem 2.19 *If $r > 1$, then $r^n \to +\infty$*

Proof. □

Exercise 2.27 LIMITS OF RATIOS Give examples of two sequences, $a_n \to +\infty$ and $b_n \to +\infty$, such that

a. $\dfrac{a_n}{b_n} \to +\infty$

b. $\dfrac{a_n}{b_n} \to 0$

c. $\dfrac{a_n}{b_n} \to c$, where c is a positive real number.

Exercise 2.28 Provide a definition and theorems about diverging to $-\infty$

a. What would it mean for the limit of a sequence to be $-\infty$?

b. If it is not part of your definition prove: $\lim_{n\to\infty} s_n = -\infty \iff \lim_{n\to\infty} -s_n = +\infty$?

c. If it is not part of your definition prove: If $\lim_{n\to\infty} s_n = -\infty$ then, for all $M < 0$, there exists $N > 0$ such that

$$n > N \implies s_n \leq M.$$

d. Give an example of a sequence that diverges to $-\infty$ and prove that it does.

e. State and prove statements of ALGEBRAIC PROPERTIES OF DIVERGENT LIMITS for sequences that diverge to $-\infty$

f. If $s_n \to -\infty$, then any subsequence of s_n also diverges to $-\infty$.

Exercise 2.29 Give a sequence $s_n \to 0$ where $\dfrac{1}{s_n}$ does not diverge $+\infty$ and does not diverge to $-\infty$.

Theorem 2.20 *If p is a polynomial that is not constant, then either*

$$\lim_{n\to\infty} p(n) = +\infty \text{ or } \lim_{n\to\infty} p(n) = -\infty$$

Proof. HINT: Use induction on the degree of the polynomial. □

2.2 Limits and Sets

2.2.1 Limit Points and Boundary Points

We have already seen that there is a sequence in a set S that converges to inf S and another that converges to sup S. In this section, we investigate other characteristics of sets and points that would guarantee the existence of a sequence of elements within the set that converge to the point.

Definition We say the a point, p, is a *limit point of a set*, S, whenever every open interval about p contains an infinite number of points in S. In particular, it contains a point in S that is not equal to p.

NOTE: The point p need not be in S.

Definition We say the a point, p, is a *limit point of a sequence*, s_n, whenever every open interval about p contains an infinite number of s_n.

Example 2.13 The limit points of the image of s_n may be different than the limit points of s_n. Consider $s_n = (-1)^n$. The image of s_n is $\{, -1, 1\}$, a set that has no limit points. However, both -1 and 1 are limit points of the sequence because they each appear an infinite number of times in the sequence.

Theorem 2.21 *A real number p is a limit point of a set S, if and only if there exists sequence of points in $S \setminus \{p\}$ that converge to p.*

Proof. $\Longrightarrow$ Suppose p is a limit point of S. For each $n \in \mathbb{Z}^+$, let

$$s_n \in S \cap (p - \frac{1}{n}, p + \frac{1}{n}) \setminus \{p\}.$$

Such a point exists because p is a limit point. We claim that $s_n \to p$. For we know, for all n,

$$0 < |p - s_n| < \frac{1}{n}$$

From the squeeze theorem, we conclude that $|p - s_n| \to 0$ or $s_n \to p$.

$\Longleftarrow$Let $s_n \in S \setminus \{p\}$ converge to p. Let (a, b) be an open interval containing p. Consider any positive $\epsilon < \min(b - p, p - a)$, so that $p \in (p - e, p + \epsilon) \subset (a, b)$. By the convergences of s_n, there exists N such that $n > N \implies |p - s_n| < \epsilon$. These s_n's are an infinite number of values of the sequences that are in $(a, b) \setminus \{p\}$. If there were only a finite number of numbers from S in this sequence, we would have a subsequence of s_n that converges to some other number which cannot happen. So the s_n for $n > N$ are an infinite number of points in (a, b), as required to show that p is a limit point of the set S. $\square$

Theorem 2.22 *A real number s is a limit point of a sequence s_n if and only if there exists a subsequence of s_n that converge to s.*

Proof. EFS □

Definition We say the a point, p, is an *boundary point* of a set, S, whenever every open interval containing p contains points in both S and $\mathbb{R} \setminus S$.

Example 2.14 Every point in a finite set is a boundary point of the set. Every point in a finite set is a boundary point of the complement of set.

True or False 11

Which of the following statements are true? If false, modify the statement to be true. Explain.

a) The endpoints of an interval are boundary points of the interval.

b) Every point in an interval is a boundary point of the interval.

c) Every point of an interval is a limit point of the interval.

d) The inf S is a limit point of S.

e) The inf S is a boundary point of S.

f) The maximum value of a S is a boundary point of S.

Example 2.15 Give an example of each of the following and explain.

a) A set and a point that is a boundary point but not a limit point of the set.

b) A set and a point that is a limit point but not a boundary point.

c) A set and a point that is neither a limit point nor a boundary point of the set.

d) A set and a point that is both a boundary point and a limit point of the set.

2.2.2 Open and Closed Sets

Definition We say a set is *open* whenever it contains none of its boundary points.

Definition We say a set is *closed* whenever it contains all of its boundary points.

Example 2.16 Open intervals are open sets because the only boundary points of an interval are the endpoints and neither are contained in the open interval.

Example 2.17 Closed intervals are closed sets be because the only boundary points of an interval are the endpoints and both are contained in the closed interval.

Theorem 2.23 *The following are equivalent*

1. *S is an open set*
2. *Every $s \in S$ is contained in an open interval that is completely contained in S.*
3. *$\mathbb{R} \setminus S$ is closed.*

Proof. □

Theorem 2.24 *The following are equivalent*

1. *S is an closed set*
2. *S contains all of its limit points.*
3. *$\mathbb{R} \setminus S$ is open.*

Proof. □

Exercise 2.30 Is $\mathbb{R}$ open or closed? Is $\varnothing$ open or closed?

Exercise 2.31 $\{x : x^2 \leq 57\}$ is a closed set.

True or False 12

Which of the following statements are true? If false, modify the statement to be true. Explain.

a) An open set never contains a maximum.

b) A closed set always contains a maximum.

Theorem 2.25 *Union and Intersection properties*

1. *The intersection of a collection of closed sets is closed.*
2. *The union of a collection of open sets is open.*
3. *The intersection of a finite collection of open sets is open.*
4. *The union of a finite collection of closed sets is closed.*

Proof. HINTS:

1. Use the 'contains all limit points' criteria for closed sets.
2. Use the 'there is an open interval about any point' criteria for open sets.

□

2.2.3 Optional – Connected sets

Suppose that there were no real number, s, such that $s^2 = 56$. Consider the two sets $U = \{s : s^2 < 56\}$ and $V = \{s : s^2 > 56\}$. Then an interval like $[7, 8]$ could be divided into two distinct parts by the disjoint sets, U and V. There would be a 'hole' in the numberline. This leads to the definition of connected that says a connected set cannot be covered by two distinct open sets. That intervals are connected is a way of understanding the completeness axiom and investigating sets that may have more complicated structure than intervals.

Definition We say a set, S, is *connected* if it *is not* contained in the union of two *disjoint* non-empty, open sets.

Example 2.18 Finite sets are not connected

Here we present another nice application of the Nested Interval Theorem.

Theorem 2.26 *A connected set is a, possibly infinite, interval.*

Proof. Hint: this will be easiest to handle using the criteria for intervals investigated in Section 1.3.6. Let C be a connected subset of $\mathbb{R}$. Suppose $a, b \in C$ and $a < b$. Suppose there is an x such that $a < x < b$. If $x \notin C$, then let $U = (-\infty, x)$ and $V = (x, \infty)$. U and V are open intervals and together they cover C if x is not in C. This contradicts that C is connected. So we conclude $x \in C$. WE have shown that C satisfies the condition 1.1 that defines an interval. So C is an interval. □

The more interesting part is that every interval is connected.

Theorem 2.27 *Any interval is connected.*

Proof. Again, we will use the criteria 1.1. Let I be an interval and assume it is not connected. Let U and V be two disjoint open sets such that $I \in U \cup V$. Since both $\mathbb{R} \setminus U$ and $\mathbb{R} \setminus V$ are closed this means that I is also covered by two distinct closed sets. We will use this and the fact that closed sets contain all of their limit points later in the proof.

Since neither U nor V are empty, pick $u_0 \in U$ and $v_0 \in V$ and assume, without loss of generality, that $u_0 < v_0$. Define inductively a sequence of closed, nested intervals, $[u_n, v_n]$ with $u_n \in I \cap U$ and $v_n \in I \cap V$ and length, $|v_n - u_n| = \frac{|v_0 - u_0|}{2^n}$. The base case, $[u_0, v_0]$, satisfies the conditions. Assume $[u_n, v_n]$ has been defined. Let m be the midpoint of the interval $[u_n, v_n]$. So,

$$u_0 < m < v_0$$

Now $u_n, v_n \in I$ and so $m \in I$ by 1.1. There are two cases:

1. If $m \in U$, let $u_{n+1} = m$ and $v_{n+1} = v_n$.

2. If $m \in V$, let $u_{n+1} = u_n$ and $v_{n+1} = m$.

In either case, then $u_{n+1} \in I \cap U$ and $v_{n+1} \in I \cap V$. and $[u_{n+1}, v_{n+1}] \subset [u_n, v_n]$. So we have a closed interval of the required form. Since m is the midpoint, we known that

$$|v_{n+1} - u_{n+1}| = \frac{1}{2}|v_n - u_n| = \frac{1}{2} \cdot \frac{v_0 - u_0}{2^n} = \frac{v_0 - u_0}{2^{n+1}}$$

These intervals are nested, closed and non-empty so we can applied the Best Nested Interval Theorem to say there is a point, $x \in [u_n, v_n]$, for all n, and that $u_n \to x$ and $v_n \to x$. Since x is in all the intervals, it is between two points of I and so is in I.

Now x is a limit point of $\mathbb{R} \setminus U$, a closed set, so it must be in $\mathbb{R} \setminus U$. That is, x is not in U. But x is also a limit point of $\mathbb{R} \setminus V$, another closed set, so x is not in V. x is in neither V nor U, but it is a point in I so U and V cannot cover the interval as originally supposed. Therefore, the interval is connected. □

2.3 The Bolzano-Weierstrass Theorem and Cauchy Sequences

Theorem 2.28 THE BOLZANO-WEIERSTRASS THEOREM *Every bounded sequence has a converging subsequence.*

Outline of proof: Name the sequence s and let M be a bound for the sequence. That is, for all n, $|s_n| < M$. We will construct, using a bisection method, a sequence of non-empty, closed nested intervals whose lengths converge to 0 and a subsequence of s, such that k^{th} element of the subsequence is contained in the k^{th} interval. Necessarily, this subsequence converges to the common point of the intervals. More precisely, we define inductively a sequence of non-empty, closed, nested intervals, $[u_k, v_k]$, such that each interval contains an infinite number of sequence values and such that each interval is half the length of the previous interval. Along the way, we define the required subsequence, $s_{n_k} \in [u_k, v_k]$.

The base case is $u_0 = -M$ and $v_0 = M]$. Since all sequence points are in this interval, pick one, s_{n_0}. Assume $[u_k, v_k]$ has been defined. Let m be the midpoint of the interval $[u_k, v_k]$. There are two cases. They are not mutually exclusive, so pick the first if both are true.

1. If there is an infinite number of $s_n \in [m, v_k]$ let $u_{k+1} = m$ and $v_{k+1} = v_k$.

2. If there is an infinite number of $s_n \in [u_k, m]$ let $u_{k+1} = u_k$ and $v_{k+1} = m$.

In either case, then $[u_{k+1}, v_{k+1}]$ contains an infinite number of s_n and hence will contain one whose index value, n_{k+1} is greater than n_k. (There are only a finite number with index value less than n_k.) So, inductively, we have the required sequence of intervals. We also have constructed a subsequence, s_{n_k}. We know there is a unique common point, s, of all the intervals, $[u_k, v_k]$ and that $u_k \to s$ and $v_k \to s$. Because $u_k \le s_{n_k} \le v_k$, we conclude by the squeeze theorem that $s_{n_k} \to s$. We have constructed a converging subsequence for the sequence. □

Note on Bolzano-Weirstrass: The limit point, s, is by no means unique as we have seen sequences may have even an infinite number of limit points. The proof is also not deterministic in the sense that it does not really help us to construct a limit point. This is because it offers no procedure for determining which interval contains an infinite number of points. Still, it helps to know that a convergent subsequence exists. The theorem is also useful for understanding the following condition that guarantees that a sequence converges without explicitly finding the limit.

Definition We say that the sequence s_n *is Cauchy* whenever the following hold:

For all $\epsilon > 0$, *there exists a real number,* N, *such that*

$$n,\, m > N \implies |s_n - s_m| < \epsilon.$$

Example 2.19 The sequence $a_n = n$ is not Cauchy because $|a_n - a_{n+1}| = 1$.

Example 2.20 The sequence, $b_n = \frac{1}{n}$ is Cauchy because for all positive integers, $m > n$, we have

$$|\frac{1}{n} - \frac{1}{m}| = \frac{|m-n|}{mn} < \frac{m}{mn} = \frac{1}{n}$$

The following two lemmas provide all the tools needed to prove our main theorem, 2.31.

Lemma 2.29 *A Cauchy sequence is bounded.*

Proof. HINT: Mimic the proof of Theorem 2.2. □

Lemma 2.30 *If any subsequence of a Cauchy sequence converges, then the sequence itself converges.*

Proof. HINT: Consider: $|s_n - s| \le |s_n - s_{n_k}| + |s_{n_k} - s|$. The first term can be made small by the Cauchy criteria and second because the subsequence converges to s. □

Theorem 2.31 *A sequence converges if and only if it is a Cauchy sequence*

Outline of proof: $\Longrightarrow$ We will show that if there is a sequence, $s_n \to s$, then the sequence is Cauchy. So, let $\epsilon > 0$ be given. By convergence of s_n find N so that $n > N \implies |s_n - s| < \frac{\epsilon}{2}$. Then we have,

$$|s_n - s_m| \le |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So the sequence is Cauchy.

$\Longleftarrow$ By Lemma 2.29 the sequence is bounded. By THE BOLZANO-WEIERSTRASS THEOREM the sequence has a converging subsequence. By Lemma 2.30 the sequence also converges. □

The following theorem generalizes the nested interval theorem to closed sets.

Theorem 2.32 *The countable intersection of nested, non-empty bounded closed sets is not empty.*

Proof. Let C_n be the closed sets. Since the sets are nested there is a common bounded for all of them. since each is non-empty pick $s_n \in C_n$. This sequence is bounded so there is a subsequence that converges to some real number s. This point is a limit point for the intersection. Since the intersection is closed s is contained in it. □

2.4 Series and Power series

Series are special kinds of sequences where one keeps a running sum of a sequence, a_n to create a new sequence, s_n:

Definition The number

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n$$

is called the n^{th} partial sum of the *generating sequence*, a_n. If the sequence, s_n, converges to a point, s, we say the the series converges to s and we write

$$s = \sum_{k=0}^{\infty} a_k$$

Definition A *power series* is a special kind of series where the generating sequence is of the form $a_n = c_n \cdot r^n$. If the c_n's are constant, $c_n = c$ we call it a *geometric series*.

Theorem 2.33 ALGEBRAIC PROPERTIES OF SERIES *If* $s = \sum_{k=0}^{\infty} a_k$ *and* $t = \sum_{k=0}^{\infty} b_k$, *then*

1. $s + t = \sum_{k=0}^{\infty}(a_k + b_k)$.

2. *If* $c \in \mathbb{R}$*, then* $c \cdot s = \sum_{k=0}^{\infty} c \cdot a_k$.

Exercise 2.32 Give examples of each of the following:

a. A series that diverges to $+\infty$.

b. A series whose partial sums oscillate between positive and negative numbers.

Exercise 2.33 If $a_n \geq 0$ for all n, then $s_n = \sum_{k=0}^{n} a_k$ is an increasing sequence.

Exercise 2.34 If $s = \sum_{k=0}^{\infty} a_k$ and $t = \sum_{k=0}^{\infty} b_k$, write a possible formula for the terms of a series that might be $s \cdot t$. Prove that your series converges and is equal to $s \cdot t$.

Exercise 2.35 Show that $\sum_{k=0}^{\infty} \frac{1}{k}$ diverges.

2.4.1 Convergence of Geometric series

Lemma 2.34 *For all* $r \neq 1$,

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Proof.

$$\begin{aligned}
\text{Let } s_n &= \sum_{k=0}^{n} r^k && \text{then we see that} \\
r \cdot s_n &= \sum_{k=0}^{n} r^{k+1} && \text{distributive law across sums} \\
&= \sum_{k=1}^{n+1} r^k && \text{adjusting indices} \\
s_n - r \cdot s_n &= 1 - r^{n+1} && \text{the two sums have the same terms except} \\
& && \text{for the first in } s_n(r^0 = 1) \text{ and last in } r \cdot s_n(r^{n+1}).
\end{aligned}$$

Solving for s_n gives the result. □

Theorem 2.35 *If* $0 \leq |r| < 1$, *then*

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Proof. EFS □

2.4.2 Decimals

Definition A *decimal representation* looks like $a_0.a_1a_2a_3\cdots a_n\cdots$, where a_0 is an integer and a_n are integers between 0 and 9, inclusively. It represents the number which is given by the power series where $r = \frac{1}{10}$. We write

$$a_0.a_1a_2a_3\cdots a_n\cdots = a_0 + a_1\cdot(\frac{1}{10})^1 + a_2\cdot(\frac{1}{10})^2 + a_3\cdot(\frac{1}{10})^3 + \cdots = \sum_{n=0}^{\infty} a_n\cdot(\frac{1}{10})^n$$

Exercise 2.36 Prove: 0.999999... = 1.

Theorem 2.36 *Every decimal representation is a convergent power series and hence every decimal representation is a real number.*

Proof. The partial sums are bounded by $a_0 + \sum_{k=1}^{n} 9\cdot(\frac{1}{10})^k < a_0 + 1$. The partial sums are increasing. Every bounded increasing sequence converges by MONOTONE CONVERGENCE THEOREM 2.13. □

As well, every real number has a decimal representation that converges to it, more precisely

Theorem 2.37 *Given any* $r \in \mathbb{R}$, *there exists a sequence of integers* a_n, *where* $0 \leq a_k \leq 10$ *for all* $k \geq 1$ *such that*

$$\sum_{n=0}^{\infty} a_n\cdot(\frac{1}{10})^n = r$$

Proof. Use Theorem 1.59. and THE BEST NESTED INTERVAL THEOREM. □

Notice that nothing is said about the representation of a real number being unique. In fact any rational number that has a representation that ends with an infinite string of 0's has another representation that ends in a string of $9's$. And vice versa.

Exercise 2.37 What changes must be made to the procedure used to find the decimal representation of a real number, as described in the proof of Theorem 2.37, to produce a ending string of 0's instead of 9's?

Exercise 2.38 Make a flowchart of theorems and connective arguments that take us from the Completeness Axiom to the representation of real numbers as decimals.

2.4.3 B-ary representation of numbers in $[0, 1]$

There is really nothing special about the choice of 10 for representing real numbers as power series. If instead, we picked 2 we would get binary representation which is of course of great value in computer science. If you follow the procedure for this choice of base, you'll notice the procedure for finding the representation of a real number uses the bisection method. Octal representation, base 8, and hexidecimal representation, base 16, are also extensively used in computer science. In the next chapter, we will have occasion to use tertiary, or base 3, expansions.

Right now, we restate the theorems about decimal representation using a generic base B, where B is any positive integer greater than 1. We call these *B-ary representations*.

Exercise 2.39 Pick a single digit number for B to use to complete all the exercises and proofs in this section.

Definition A *B-ary representation* looks like $a_0.a_1a_2a_3\cdots a_n\cdots$, where a_0 is an integer and a_n are integers between 0 and $B-1$, inclusively. It represents the number which is given by the power series where $r = \frac{1}{B}$. We write

$$a_0.a_1a_2a_3\cdots a_n\cdots = a_0 + a_1\cdot(\frac{1}{B})^1 + a_2\cdot(\frac{1}{B})^2 + a_3\cdot(\frac{1}{B})^3 + \cdots = \sum_{n=0}^{\infty} a_n\cdot(\frac{1}{B})^n$$

Exercise 2.40 If a is the symbol that represents the $B-1$, prove: $0.aaaaaaa... = 1$.

Theorem 2.38 *Every B-ary representation is a convergent power series and hence every B-ary representation is a real number.*

Proof. The partial sums are bounded by $a_0 + \sum_{k=1}^{n}(B-1)\cdot(\frac{1}{B})^k < a_0 + 1$. The partial sums are increasing. Every bounded increasing sequence converges. □

As well, every real number has a B-ary representation that converges to it, more precisely

Theorem 2.39 *Given any $r \in \mathbb{R}$, there exists a sequence of integers a_n, where $0 \le a_k \le B$ for all $k \ge 1$ such that*

$$\sum_{n=0}^{\infty} a_n\cdot(\frac{1}{B})^n = r$$

Proof. Use Theorem 1.59. and THE BEST NESTED INTERVAL THEOREM. □

Notice that nothing is said about the representation of a real number being unique. In fact any rational number that has a representation that ends with an infinite string of 0's has another representation that ends in a string of a's. And vice versa. (a is still the symbol for $B-1$.)

Exercise 2.41 What changes must be made to the procedure used to find the B-ary representation of a real number, as described in the proof of Theorem 2.39, to produce a ending string of 0's instead of a's?

Chapter 3

Counting

3.1 Finite vs Infinite

We will be using the notion of a $1-1$ correspondence of two sets. This is just a bijection between the sets.

Definition Recall that we say that a set, S, is *finite* whenever there exists a integer, $N \geq 0$, and a $1-1$ correspondence between S and $\{n \in \mathbb{Z}^+ : n \leq N\}$. N is then the number of elements in the set. We say that a set is *infinite* if it is not finite.

editor: define cardinality

Definition We say that a set, S, is *countable* whenever there exists $1-1$ correspondence between S and a subset of $\mathbb{Z}^+$.

Exercise 3.1 Prove the following theorem.

Theorem 3.1 *Given two countable sets, U and V*

1. *Any subset of U is countable.*
2. *$U \cup V$ is countable*
3. *$U \cap V$ is countable*

Proof. □

True or False 13

Which of the following statements are true? Prove or give a counterexample.

a) If $A_0 \supset A_1 \supset A_2 \supset A_3 \supset A_4 \cdots$ are all infinite subsets of $\mathbb{R}$, then $\cap_{n=1}^{\infty} A_n$ is infinite.

b) If $A_0 \supset A_1 \supset A_2 \supset A_3 \supset A_4 \cdots$ are all infinite subsets of $\mathbb{R}$, then $\cap_{n=1}^{\infty} A_n$ is empty.

c) If $A_0 \supset A_1 \supset A_2 \supset A_3 \supset A_4 \cdots$ are all infinite subsets of $\mathbb{R}$, then $\cap_{n=1}^{\infty} A_n$ is finite but not empty.

d) $\cup_{n=1}^{\infty} A_n$ is infinite.

3.1.1 The rational numbers

Theorem 3.2 *The rational numbers are countable.*

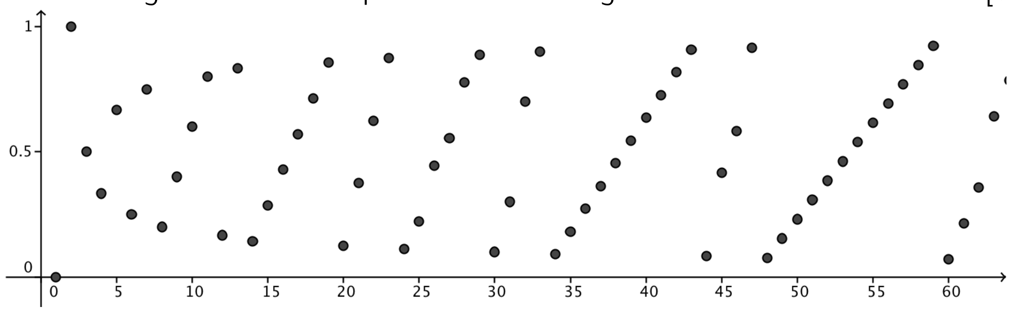
Proof. Consider this array that contains all the positive rational numbers. The first row contains all the rationals, in order, that have a denominator of 1. The second row contains all the rationals that have a denominator of 2. And so on. The fractions are not reduced so a particular rational number will appear more than once - in fact infinitely often. The rational number in the n^{th} row and m^{th} column, s_{nm}, is $\frac{m}{n}$.

By counting along the diagonals, as indicated by the overwriting, we can assign each rational number a positive integer to a rational number. As we move along these diagonals we do not count those rational numbers that we have counted before.

1 $\frac{1}{1}$	2 $\frac{2}{1}$	6 $\frac{3}{1}$	7 $\frac{4}{1}$	15 $\frac{5}{1}$	16 $\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	$\frac{9}{1}$	$\frac{10}{1}$	$\frac{11}{1}$	.	.	.
3 $\frac{1}{2}$	5 $\frac{2}{2}$	8 $\frac{3}{2}$	14 $\frac{4}{2}$	17 $\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	$\frac{9}{2}$	$\frac{10}{2}$	$\frac{11}{2}$	.	.	.
4 $\frac{1}{3}$	9 $\frac{2}{3}$	13 $\frac{3}{3}$	18 $\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{9}{3}$	$\frac{10}{3}$	$\frac{11}{3}$	.	.	.
10 $\frac{1}{4}$	12 $\frac{2}{4}$	19 $\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\frac{9}{4}$	$\frac{10}{4}$	$\frac{11}{4}$	.	.	.
11 $\frac{1}{5}$	20 $\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$\frac{10}{5}$	$\frac{11}{5}$	.	.	.
22 $\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{9}{6}$	$\frac{10}{6}$	$\frac{11}{6}$	.	.	.
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	$\frac{9}{7}$	$\frac{10}{7}$	$\frac{11}{7}$	.	.	.
$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$	$\frac{9}{8}$	$\frac{10}{8}$	$\frac{11}{8}$	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.	.

This procedure counts any rational number more than once but that doesn't matter. We still have a $1-1$ correspondence between that rationals and a subset of $\mathbb{Z}^+$. Each rational number corresponds to the integer on our list that is associated with the the rational number in lowest terms. So $\mathbb{Q}$ is countable. $\square$

Figure 3.1: A sequence whose image is all rational numbers in [0,1]



Example 3.1 This procedure that gives a list of all positive rational numbers is an example of a sequence whose image is all positive rational numbers. This sequence clearly diverges as we can find a subsequence that is unbounded by picking out the sequence elements that are integers. We can also find subsequences that converge to any given positive number.

Exercise 3.2 Using the sequence of rational numbers given above, describe a procedure to find a subsequence that converges to a particular real number, r.

Example 3.2 There is a sequence whose image is the rational numbers between 0 and 1. List these numbers by first listing in order all fractions that have denominator 2, followed by those with denominator 3, and so on. In Figure 3.1 we show this sequence. In this case we eliminated duplicates as we listed the fractions.

Exercise 3.3 Using a similar argument show that $\mathbb{N}x\mathbb{N}$ is countable.

Exercise 3.4 Using a similar argument show that the countable union of countable sets is countable.

3.1.2 How many decimals are there?

Theorem 3.3 *The real numbers are not countable.*

Proof. We will in fact show that the interval $(0, 1)$ is not countable. The method of proof is called 'Cantor's diagonal argument' after the mathematician who first used it. Suppose we had had a correspondence between $\mathbb{Z}^+$ and the real numbers, using the decimal representation of the real numbers we could list them like this:

1	$0.a_{11}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\, a_{16}\, a_{17}\, a_{18}\, a_{19} \cdots$
2	$0.a_{21}\, a_{22}\, a_{23}\, a_{24}\, a_{25}\, a_{26}\, a_{27}\, a_{28}\, a_{29} \cdots$
3	$0.a_{31}\, a_{32}\, a_{33}\, a_{34}\, a_{35}\, a_{36}\, a_{37}\, a_{38}\, a_{39} \cdots$
4	$0.a_{41}\, a_{42}\, a_{43}\, a_{44}\, a_{45}\, a_{46}\, a_{47}\, a_{48}\, a_{49} \cdots$
5	$0.a_{51}\, a_{52}\, a_{53}\, a_{54}\, a_{55}\, a_{56}\, a_{57}\, a_{58}\, a_{59} \cdots$
6	$0.a_{61}\, a_{62}\, a_{63}\, a_{64}\, a_{65}\, a_{66}\, a_{67}\, a_{68}\, a_{69} \cdots$
7	$0.a_{71}\, a_{72}\, a_{73}\, a_{74}\, a_{75}\, a_{76}\, a_{77}\, a_{78}\, a_{79} \cdots$
8	$0.a_{81}\, a_{82}\, a_{83}\, a_{84}\, a_{85}\, a_{86}\, a_{87}\, a_{88}\, a_{89} \cdots$
⋯	⋯

To see that this list cannot possibly contain all of the real numbers, we will construct a decimal representation that is not there. Consider the real number $\in [0, 1]$ with the representation: $0.a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8 \cdots$ where we define

$$a_k = a_{kk} + 1 \mod 10.$$

This number can not be on our list because it differs from the k^{th} element on the list in the k^{th} decimal place. □

3.2 Cantor Sets

Draw a picture of the recursive procedure that defines the Cantor Set.

View the Cantor set as the intersection of a countable number of closed sets. Conclude that the Cantor Set is closed.

The Cantor Set consists of all real numbers in $[0, 1]$ whose tertiary expansion can be written with no 2's Show that the Cantor Set is not countable by writing the numbers using their tertiary expansion and mimicking Cantor's argument.

The complement of the Cantor Set has accumulated length of 1.

The dimension of the Cantor Set is log2 /log3

Exercises: construct other Cantor Sets.

Chapter 4

Functions

4.0 Limits

Definition Let $f : D \to \mathbb{R}$.

We write $\lim_{x \to p} f(x) = L$ and say the *limit as x approaches p of $f(x)$ is L*, whenever,

$$\begin{gathered}\text{For all } \epsilon > 0, \text{ there exists } \delta > 0, \text{ so that for } x \in D, \\ 0 < |x - p| < \delta \implies |f(x) - L| < \epsilon.\end{gathered}$$

To emphasize the domain, we may say: the limit as x approaches p *in D*.

Note: The $0 <$ is included because it is not necessary for the limit to be equal to $f(p)$. It is not even necessary for f to be defined at p. This is important later on when we define derivatives as limits. The concern disappears when we use limits to define continuity and derivatives.

We don't go through the step of $L = 0$ as we did for limits of sequences because we are so much better at deconstructing limit statements now. However, it is true that

Theorem 4.1 $\lim_{x \to p} f(x) = L \iff \lim_{x \to p} (f(x) - L) = 0$

Definition We say that $f(x)$ *converges to L sequentially as $x \to p$* whenever,

$$\begin{gathered}\text{for any sequence, } x_n \in D - \{p\} \\ x_n \to p \implies f(x_n) \to L\end{gathered}$$

We do not consider values of p for the sequence because $f(p)$ may not be equal to L. Again, this won't be a concern with continuity.

Theorem 4.2 $\lim_{x \to p} f(x) = L$ *if and only if $f(x)$ converges to L sequentially as $x \to p$*

Proof. EFS □

Now, either definition of limit can be used in the proof of the following theorem.

Theorem 4.3 ALGEBRAIC PROPERTIES OF LIMITS 3 *Given two functions f and g defined on a domain, D, with* $\lim_{x\to p} f(x) = A$ *and* $\lim_{x\to p} g(x) = B$*, the following are true:*

1. $\lim_{x\to p}(f+g)(x) = A + B$
2. $\lim_{x\to p}(f\cdot g)(x) = A\cdot B$
3. *If* $f(x) \neq 0$ *for any* $x \in D$ *and if* $A \neq 0$*, then* $\lim_{x\to p}(\frac{1}{f})(x) = \frac{1}{A}$

Proof. A good way to prove these theorems is using sequential convergence and the corresponding theorems about sequences. □

Theorem 4.4 SQUEEZE THEOREM *Given two functions f and g defined on a domain, D, with* $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = C$*, and another function, h, defined on D, with* $f(x) \leq h(x) \leq g(x)$ *for all* $x \in D$*, then* $\lim_{x\to p} h(x) = C$

Proof. Given $\epsilon > 0$, find $\delta > 0$ such that

$$|x-p| < \delta \implies |f(x) - C| < \epsilon \text{ and } |f(x) - C| < \epsilon$$

The following inequalities demonstrate that $|h(x) - C| < \epsilon$, so $\lim_{x\to p} h(x) = C$

$$-\epsilon \leq -|f(x) - C| \leq f(x) - C \leq h(x) - C \leq g(x) - C \leq |g(x) - C| < \epsilon$$

□

Exercise 4.1 Because we are taking advantage of sequential limits, we did not need to prove the following theorem to prove Theorem 4.3 2. Prove this theorem:

Theorem 4.5 *If* $\lim_{x\to p} f(x) = A$*, then f is bounded in some open interval about p.*

4.1 Continuity

For the following, consider a function, $f : D \to \mathbb{R}$.

Definition CONTINUITY AT A POINT We say that the f is continuous at $p \in D$ if

$$\lim_{x\to p} f(x) = f(p).$$

Example 4.1 The function x^2 is continuous at the point $p = 3$.

Proof. directly from the definition: We need to show that $\lim_{x\to 3} x^2 = 9$.
Given $\epsilon > 0$, find a $\delta < \min\{1, \frac{\epsilon}{7}\}$. Then we can see that if $|x - 3| < \delta$ we have

$$|x^2 - 9| = |x - 3| \cdot |x + 3| < \delta \cdot 7 < \frac{\epsilon}{7} \cdot 7 = \epsilon.$$

[Justifications: Because $\delta < \min\{1, \frac{\epsilon}{7}\}$, we know that δ is less than both 1 and $\frac{\epsilon}{7}$. if $|x - 3| < 1$, then $|x + 3| = |x - 3 + 6| \leq |x - 3| + |6| < 1 + 6 = 7$. This justifies the first '$<$'. The second '$<$' is justified because $\delta < \frac{\epsilon}{7}$]
So $|x - 3| < \delta \implies |x^2 - 9| < \epsilon$, as we needed to show. □

Exercise 4.2 Directly from the definition of limit, show that the function $\sqrt{x}$ is continuous at the point $p = 4$.

Definition CONTINUITY ON A SUBSET OF THE DOMAIN We say that a function, f, is *continuous on* $E \subset D$ if it is continuous at all points of E.

Exercise 4.3 The function x^2 is continuous on $\mathbb{R}$.

Exercise 4.4 The function $\sqrt{x}$ is continuous on $\mathbb{R}^{\geq}$.

Exercise 4.5 Give an example of a function is continuous on all but 3 points in $\mathbb{R}$.

4.1.1 Sequential continuity

We could have used a sequentially definition for continuous at a point. Here it is said in a theorem.

Theorem 4.6 *A function f is* continuous at a point $p \in D$ *if and only if*

$$\textit{for any sequence, } x_n \in D$$
$$x_n \to p \implies f(x_n) \to f(p)$$

Proof. □

Another way to state the conclusion of this theorem is: *If f continuous on E and $s_n \in E$ converges to a point in E, then*

$$f(\lim_{n\to\infty} s_n) = \lim_{n\to\infty} f(s_n)$$

Exercise 4.6 Use the sequential definition of continuity to show that the function x^2 is continuous on $\mathbb{R}$.

Exercise 4.7 Give an example of a function, f, and a sequence of points, s_n, is the domain of f, such that $f(s_n)$ converges, but s_n does not.

Exercise 4.8 Prove the basic facts about continuous functions stated in the following theorem:

Theorem 4.7 ALGEBRAIC PROPERTIES OF CONTINUITY *Given two functions f and g defined and continuous on a domain, D, the following are true:*

1. *$f + g$ is continuous on D.*
2. *$f \cdot g$ is continuous on D.*
3. *If $f(x) \neq 0$ for any $x \in D$, $\frac{1}{f}$ is continuous on D.*

Proof. A good way to prove these theorems is to use the corresponding theorems about limits of functions. □

Exercise 4.9 Prove the following theorem.

Theorem 4.8 *The addition functions, $s_b : x \to x + b$, and multiplication functions, $t_m : x \to m \cdot x$, are continuous on $\mathbb{R}$.*

Theorem 4.9 *Polynomials are continuous.*

Proof. EFS First make a flowchart of the proof. What simpler facts should you prove first? What is the overall plan of attack. □

Theorem 4.10 *Rational functions are continuous wherever they are defined.*

Proof. EFS Use theorem 4.9 □

4.1.2 More Examples and Theorems

An aside to discussion inverse functions

The composition of two functions and the inverse of a function can be defined in the most abstract settings. We review those concepts here before considering the continuity of compositions and inverse functions.

Definition Given two functions, $f : D \to E$ and $g : E \to F$, the function, $g \circ f : D \to F$ is defined for each $x \in D$ by

$$g \circ f(x) = g(f(x)).$$

We write $g \circ f$ and say *composition of g with f*.

Theorem 4.11 *If $f : D \to E$ and $g : E \to F$ are continuous on their respective domains, then $g \circ f : D \to F$ is continuous on D.*

Proof. □

Definition We say that a function, $f : D \to E$ has an *inverse function*, if there exists a function, $f^{-1} : E \to D$ such that

$$\text{for all } d \in D, f^{-1} \circ f(d) = d.$$

Exercise 4.10 The proof of the following theorem is fundamental to understanding inverse functions.

Theorem 4.12 *If $f : D \to E$ has an inverse $f^{-1} : E \to D$, then f^{-1} has an inverse and it is f, that is*

$$\textit{for all } e \in E, f \circ f^{-1}(e) = e.$$

Proof. □

Exercise 4.11 There are not many examples available to us now without Υ. However, we can note that the addition function, $s_b : x \to x + b$ has an inverse. Show that

$$s_b^{-1} = s_{-b}.$$

Also, the multiplication function, $t_m : x \to m \cdot x$, has an inverse as long as $m \neq 0$. Show that

$$t_m^{-1} = t_{m^{-1}}.$$

Exercise 4.12 Write a formula for the composition function $t_m \circ s_b$ and another for $s_b \circ t_m$ What are the inverses of these composition functions?

Theorem 4.13 *Given two functions, $f : D \to E$ and $g : E \to F$, If both have inverses, so does $g \circ f : D \to F$. In fact,*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof. EFS □

A library of functions

It is good to have a well stocked library of functions to help think about the various properties and theorems.

1. polynomials and rational functions

2. piecewise continuous: step functions

3. piecewise differentiable: absolute value
4. characteristic function of the rationals
5. cantor function
6. inverse functions, like $\sqrt{x}$
7. Υ exponential, logarithms and trigonometric
8. power series

4.1.3 Uniform Continuity

Later we will need a stronger type of continuity:

Definition We say that a function f is *uniformly continuous on* $E \subset D$ whenever

$$\text{For all } \epsilon > 0, \text{ there exists a real number, } \delta > 0, \text{ such that}$$
$$\text{for all } x \text{ and } p \in E, \ |x - p| < \delta \implies |f(x) - f(p)| < \epsilon.$$

This defines a stronger condition than the just continuity. Given $\epsilon > 0$ one must find a δ that works for *all* $p \in E$. If one does not want to show *uniform* continuity, one can find δ that is dependent on the particular value of p.

Exercise 4.13 $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$.

Exercise 4.14 $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$.

Exercise 4.15 $f(x) = x^2$ is not uniformly continuous on $[1, \infty)$.

Exercise 4.16 $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Uniform continuity will be needed when we talk about integrals. The following theorem is very helpful.

Theorem 4.14 *A continuous function on a closed and bounded set is uniformly continuous on that set.*

Proof. Let $f : K \to \mathbb{R}$, where K is a closed and bounded subset of $\mathbb{R}$, be continuous at all points in K. Assume f is not uniformly continuous on K. Then there exists an $\epsilon > 0$ and sequences $x_n \in K$, $p_n \in K$ and $\delta_n \to 0$ with $|x_n - p_n| < \delta_n$ but $|f(x_n) - f(p_n)| \geq \epsilon$. x_n is bounded and so has a converging subsequence, x_{n_k}. Now p_{n_k} is bounded and so has a converging subsequence, $p_{n_{k_m}}$. For all m, we have

$$|x_{n_{k_m}} - p_{n_{k_m}}| < \delta_{n_{k_m}}.$$

So the limits are same:

$$\lim_{m\to\infty} x_{n_{km}} = \lim_{m\to\infty} p_{n_{km}}.$$

By sequential continuity, $\lim_{m\to\infty} |f(x_{n_{km}}) - f(y_{n_{km}})| \to 0$. So the difference cannot be bounded below by ϵ. ☐

Exercise 4.17 Is a linear combination of two uniformly continuous functions uniformly continuous? Explain.

Exercise 4.18 What are the other analogous facts for continuous functions and uniformly continuous functions from Theorem 4.3? Are they true or not?

4.2 Intermediate Value Theorem

Theorem 4.15 THE INTERMEDIATE VALUE THEOREM *A function that is continuous on a closed, bounded interval* $[a, b]$ *attains every value on the interval* $[f(a), f(b)]$.

Proof. If $f(a) = f(b)$, we are done. So assume, without lose of generality, that $f(a) < f(b)$ and let c be any number in $[f(a), f(b)]$. We will show that there exists two nested sequences of closed intervals, $[a_n, b_n]$ and $[f(a_n), f(b_n)]$ with $c \in [f(a_n), f(b_n)]$ such that the lengths of both go to zero.

We will then use the extensions of the nested interval theorem to show that the unique number in all the $[f(a_n), f(b_n)]$ is $f(c)$. We define the intervals inductively:

Base case: Let $a_0 = a$ and $b_0 = b$. c was chosen to be in $[f(a_n), f(b_n)]$. Assume that $[a_n, b_n]$ and $[f(a_n), f(b_n)]$ are defined as required, proceed inductively:

Let m be the midpoint of the interval $[a_n, b_n]$. There are three cases:

1. If $f(m) = c$, we are done.
2. If $f(m) < c < f(b_n)$, let $a_{n+1} = m$ and $b_{n+1} = b_n$.
3. If $f(a_n) < c < f(m)$, let $a_{n+1} = a_n$ and $b_{n+1} = m$.

In either case of the last two cases, $f(a_{n+1}) < c < f(b_{n+1})$. This new interval, $[a_{n+1}, b_{n+1}]$ is half the length of the previous because m in the midpoint. Because the intervals are cut in half at each step, the lengths converge to 0 and the extension of the Nested Interval Theorem: there is a point, p, in all of the intervals and $a_n \to p$ and $b_n \to p$. By sequential continuity $f(a_n) \to f(p)$ and $f(b_n) \to f(p)$. By the squeeze theorem, $f(p) = c$. ☐

Exercise 4.19 Show that there exists a real number c such that $c^4 - c^2 = 3$. Is c unique?

Theorem 4.16 *Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree. There exists a real number, c, such that $p(c) = 0$*

Proof. EFS ☐

Exercise 4.20 Use the THE INTERMEDIATE VALUE THEOREM to show that there was a time in your life when your height in inches was equal to your weight in pounds.

Exercise 4.21 Prove: If $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$, $f(a) < g(a)$ and $g(b) < f(b)$, there exists a real number, $c \in (a, b)$, such that $f(c) = g(c)$.

Theorem 4.17 *If a function,$f : [a, b] \to \mathbb{R}$ is strictly increasing on the interval, $[a, b]$, then there exists an* inverse *function, $f^{-1} : [f(a), f(b)] \to [a, b]$. f^{-1} is increasing on the interval $[f(a), f(b)]$.*

Proof. Since f is increasing on the interval, $f(a)$ is the minimum value of f and $f(b)$ is the maximum value of f. We will define $f^{-1}(y)$ for each $y \in [f(a), f(b)]$, and then show that f^{-1} is indeed the inverse of f.

For $y \in f(a), f(b)]$, there is an unique $x \in [a, b]$ such that $f(x) = y$. The THE INTERMEDIATE VALUE THEOREM (Theorem 4.15) guarantees there is at least one such x. There cannot be two because the function is strictly increasing. So define $f^{-1}(y) = x$. Clearly, $f^{-1} \circ f(y) = y$. ☐

True or False 14

Which of the following statements are true? Explain.

a) A decreasing function defined on $[a, b]$ has an inverse defined on $[f(b), f(a)]$?

b) An increasing function defined on (a, b) has an inverse defined on $(f(a), f(b))$?

c) If a function has an inverse on an interval, it must be either increasing or decreasing on that interval.

d) The function $f : x \to x^2$ has an inverse.

4.3 Continuous images of sets

Subsets of $\mathbb{R}$ that are both closed and bounded are also called *compact*. There is a more general definition for compact in other topological spaces but, for this book, we stick to closed and bounded. Compact sets are important in topology for reasons given in the next few theorems.

Theorem 4.18 *The continuous image of a closed bounded set is closed and bounded.*

Proof. Setup: Let C be a closed bounded set and let $f : C \to \mathbb{R}$ be continuous. We will show that $f(C)$ is also closed and bounded.

First, $f(C)$ is bounded: Assume the image of C is not bounded. Then there exists a sequence of points in $f(C)$ that converge to $+\infty$ (or $-\infty$). Now, the x_n live back in C and so, by THE BOLZANO-WEIERSTRASS THEOREM(Theorem 2.28), have a convergent subsequence, say $x_{n_k} \to x$. Because C is closed $x \in C$. Now, f is continuous on C so $f(x_{n_k}) \to f(x)$. But a convergent sequence is bounded so $f(x_{n_k})$ cannot be a subsequence of a sequence that converges to $+\infty$ (or $-\infty$). This is a contradiction, so $f(C)$ is bounded.

$f(C)$ is closed: Let y be a limit point of C. So $y_n = f(x_n) \to y$. We will show that $y \in f(C)$ which will prove that $f(C)$ is closed. Because $x_n \in C$ and C is bounded, there is a subsequence $x_{n_k} \to x$ (BW again). Since C is closed, $x \in C$. Since f is continuous on C, $f(x_{n_k}) \to f(x)$. So $y = f(x)$ and $y \in f(C)$ □

Theorem 4.19 *The inverse image, under a continuous function, of an open set is open.*

True or False 15

Which of the following statements are true? Explain.

a) The continuous image of an open set is open.

b) The continuous image of an unbounded set is unbounded.

c) The inverse image, under a continuous function, of an closed set is closed.

d) The inverse image, under a continuous function, of a bounded set is bounded.

4.4 Optional: Connected Sets

Theorem 4.20 *The continuous image of a connect set is connected.*

An alternative proof of the intermediate value theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. The image of $[a, b]$ is connected so it is an interval. It is bounded, so it is a finite interval. Finally it is closed, so it is a closed interval. So this interval must contain all of the interval $[f(a), f(b)]$. Hence every number in $[f(a), f(b)]$ is in the image of f. □

4.5 Existence of extrema

Theorem 4.21 *A continuous function defined on a closed, bounded set obtains a maximum (and a minimum value) on that set.*

Proof. □

Exercise 4.22 Find the maximum and minimum value of the function, $f(x) = x^2 - bx + c$ on the closed interval, $[0, 1]$. You will have to handle cases for different values of b and c.

4.6 Derivatives

4.6.1 Definitions

Definition DERIVATIVE AT A POINT We say that a function, $f : D \to \mathbb{R}$ is *differentiable* at a point $p \in D$, whenever the following limit exists.

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}. \tag{4.1}$$

Notation We call the limit 4.1 the *derivative* of f at p, and write $f'(p)$.

$$f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}. \tag{4.2}$$

Definition THE DERIVATIVE FUNCTION If the limit 4.1 exists for all p in a subset, E, of the domain of f, it defines a function which we write as f' and we say that f is *differentiable* on E

Theorem 4.22 *If $f : \mathbb{R} \to \mathbb{R}$ is constant on any open interval, then f' exists on the interval and $f'(x) = 0$ for all x in the interval.*

Exercise 4.23 Directly from the definition, find a formula for the derivative of the following functions at a point, p, in the domain of the function.

a) x^2

b) $\sqrt{x}$

c) x

4.6.2 Applications

editor: Ask students to bring in examples from their own discipline

practical significance The difference quotient represents an average rate – especially useful when the domain is a time variable. The derivative at a point p, $f'(p)$ represents an *instantaneous speed*.

geometric significance The derivative at a point p, $f'(p)$, represents the *slope* of the tangent line to the graph of f at the point $(p, f(p))$. [[insert picture]]

4.6.3 Basic Theorems

Theorem 4.23 *If $f : D \to \mathbb{R}$ is differentiable on D, then f is continuous on D.*

Theorem 4.24 *Given two real-valued functions f and g defined on a domain, $D \subset \mathbb{R}$ and $c \in \mathbb{R}$.*

1. *$c \cdot f + g$ is differentiable on D, and $(c \cdot f + g)'(x) = c \cdot f'(x) + g'(x)$*
2. *$f \cdot g$ is differentiable on D, and $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$*

Exercise 4.24 Restate the following theorem to include a formula for the derivative of a polynomial. Prove the theorem.

Theorem 4.25 *Polynomials are differentiable.*

Exercise 4.25 If $f(x) = \frac{1}{x}$, then f is differentiable on $(0, +\infty)$ and $f'(x) = -\frac{1}{x^2}$

Exercise 4.26 Carefully state a theorem that describes the derivative of a composition of two functions. Prove your theorem.

True or False 16

If $f'(x) > 0$ on an interval $(a, b) \subset D$, then $f(x)$ is increasing on (a, b).

4.6.4 Zero Derivative Theorem

The following series of theorems aims to show the converse of Theorem 4.22: If a function has a derivative that is zero on an open interval then the function is constant on that interval.

Theorem 4.26 *If a function, f, differentiable on (a, b) has a maximum value at $x_0 \in (a, b)$, then $f'(x_0) = 0$.*

Proof. □

Exercise 4.27 Give an example of a function that has a maximum value but the derivative at that point is not zero. Give one where the function is not differentiable and one where the interval is closed instead of open.

Exercise 4.28 Give an example of a differentiable function, f, and a point, p, such that $f'(p) = 0$, but f does not have a maximum or minimum value at p.

Theorem 4.27 ROLLE'S THEOREM *f:$[a, b] \to \mathbb{R}$, f continuous on $[a, b]$, and f differentiable on (a, b).*

$$f(a) = f(b) \implies \text{There exists } x^* \in (a, b) \text{ with } f'(x^*) = 0$$

Definition The *mean value of a function, f, over the interval* $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

Note that this is the slope of the line connecting $(a, f(a))$ and $(b, f(b))$.

This is an example of an existence theorem. The proof relies ultimately on BW which was constructive but unwinding that construction may be hard. In calculus, we find typically find those points where a function has maximum by solving $f'(x) = 0$.

Exercise 4.29 Write an equation for the straight line determined by the two points, $(a, f(a))$ and $(b, f(b))$. Use the point-slope formula for the equation of a line.

Example 4.2 average speed

Theorem 4.28 THE MEAN VALUE THEOREM *f:$[a, b] \to \mathbb{R}$, f continuous on $[a, b]$, and f differentiable on (a, b).*

$$\text{There exists } x^* \in (a, b) \text{ with } f'(x^*) = \frac{f(b) - f(a)}{b - a}$$

Exercise 4.30 Where on the interval $[-1, 1]$ does the derivative of $f(x) = x^3$ assume mean value of the function? Sketch a graph.

Theorem 4.29 UNIQUENESS THEOREM *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on the interval.*

Proof. Let z_1 and z_2 be any two points in the interval, (a, b). By THE MEAN VALUE THEOREM there is a point, $x^* \in (z_1, z_2)$, such that

$$f'(x^*) = \frac{f(z_2) - f(z_1)}{z_2 - z_1}.$$

But $f'(x^*) = 0$ by hypothesis. This then means that $f(z_1) = f(z_2)$. Since they were any two points in (a, b), f must be constant. □

Chapter 5

Integration

5.0 Definition of Riemann Integral

5.0.1 tagged partitions of an interval

Definition A *partition of the interval* $[a, b]$ is a finite, increasing sequence of numbers, x_j, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The points divide $[a, b]$ into n disjoint open subintervals, (x_i, x_{i+1}), such that the closed intervals cover $[a, b]$, i.e. $[a, b] = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}]$

The *mesh* of a partition is the length of the largest sub-interval. If P is a partition of $[a, b]$ then we write $|P|$ for the mesh of P. The mesh is used as a way of ordering partitions. This kind of ordering is called a partial ordering because there is not trichotomy, i.e. two different partitions can have the same mesh. This ordering is useful because we want to talk about the concept of the limit as the mesh of the partitions goes to zero.

A *tagged partition of the interval* $[a, b]$ is a partition with the additional n points, $t_k^* \in [x_k, x_{k-1}]$. So we have

$$a = x_0 \le t_1^* \le x_1 \le t_2^* \le x_2 \le t_3^* \cdots t_{n-1}^* \le x_{n-1} \le t_n^* \le x_n = b.$$

There are lots of tagged partitions of an interval. The x's may be placed with any spacing between them and the t^*'s can be any point with the interval. They are used to help us define the integral of a function. We need to consider, at least at the beginning, all possible tagged partitions because we want to know that as long as we choose a reasonable way to compute the integral we get the same answer. Our definition of integral must make it so.

editor: stick in some pictures

5.0.2 Definition of Riemann Integral

Definition Given any function, $f\colon [a,b] \to \mathbb{R}$, and a tagged partition of $[a,b]$, we define

$$P(f) = \sum_{k=1}^{n} f(t_k^*) \cdot (x_k - x_{k-1})$$

we call any such $P(f)$ a *Riemann sum of f over* $[a,b]$.

editor: insert picture

Definition We say the a function, $f\colon [a,b] \to \mathbb{R}$ is *Riemann-integrable* whenever there exists $I \in \mathbb{R}$ such that

For all $\epsilon > 0$, there exists $\delta > 0$ so that, for any tagged partition, P

$$|P| < \delta \implies |P(f) - I| < \epsilon$$

We notate I by

$$\int_a^b f(x)dx$$

Named after Riemann xxxx was the first to give a precise definition of integral

Example 5.1 This is a pretty unmanageable definition but we can show that

$$\int_a^b 1 \cdot dx = b - a,$$

for let P be any tagged partition. Since the function value at each t_i^* is 1, the Riemann sum is just the sum of the lengths of all the intervals or $b - a$. And so all R-sums are the same.

With a definition that involves the existence of something over a seemingly unmanageable sets as the set of all tagged partitions of an interval, it is easier to find examples of what is not than to prove something is. Here are two examples of functions that are not integrable:

Example 5.2 The function

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$$

is not integrable on $[0,1]$.

Proof that f is not integrable: Consider a sequence of partition, P_n, with n equal-length intervals, given by points, $x_i = \frac{i}{n}$. Let $t_0^* = \frac{1}{n^2}$. For any choice of the rest of the t_k^*'s, the Riemann sum,

$$P_n(f) = \sum_{k=1}^{n} f(t_k^*) \cdot (x_k - x_{k-1}) > \frac{1}{\frac{1}{n^2}} \cdot \frac{1}{n} = n.$$

Although $mesh(P_n) \to 0$, $P_n \to \infty$. So f is not Riemann integrable. □

Similar considerations allow us to conclude the following theorem:

Theorem 5.1 *If f is integrable on $[a, b]$, then f is bounded on $[a, b]$.*

Proof. Hint: Assume f is not bounded and, for every $n > 0$, find a tagged partition, P_n such that $|P_n(f)| < \frac{1}{n}$, but $P_n(f) > n$. □

Example 5.3 The function

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not integrable on $[0, 1]$.

Proof that g is not integrable: Let P_n be any tagged partition with mesh less than $\frac{1}{n}$ and t_i^* all rational numbers. Let Q_n be the same except the tagged numbers are all irrational. Then $P_n(g) = 1$ and $Q_n(g) = 0$. We have given to sequences of Riemann sums that converge to different numbers, so g cannot be Riemann integrable. □

In each example, we only needed to find a sequence of partitions that converged to zero but that the corresponding Riemann sums did not converge. It would be so nice if we could simplify the existence of the integral by looking at a single sequence of partitions. That's what the next theorem allows:

Theorem 5.2 SEQUENTIAL DEFINITION OF RIEMANN INTEGRAL *A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every sequence of tagged partitions, P_n, with $|P_n| \to 0 \implies P_n(f)$ converges.*

Proof. NOTE: If all tagged partitions converge they necessarily converge to a common limit because we can intertwine the sequences. □

Example 5.4 Consider The function

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \text{if } x \in (0, 1] \\ 0, & \text{if } x = 0. \end{cases}$$

Find two sequences of tagged partitions of $[0, 1]$, one diverging, one converging.

Now if we only knew that a function were integrable, we could pick out a convenient sequence of partitions and life would be easier. The next theorem gives us what we need for a large class of functions.

Theorem 5.3 *Continuous functions are Riemann integrable*

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous on [a,b]. We know by Theorem XXX, that f is uniformly continuous on $[a, b]$.
Let P_n be a sequence of tagged partitions such that mesh(P_n)$\to 0$. We will show that the corresponding Riemann sums is Cauchy and hence converges.
Given $\epsilon > 0$, use the uniform continuity of f to find δ such that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$
Now, find a positive integer N such that

$$n > N \implies \text{mesh}(P_n) < \frac{\delta}{2}.$$

Consider any two partitions, P_n and P_m, with $n, m > N$. Label the tags from P_n with t^* and the tags from P_m with s^*. Consider a refinement of the two partitions, Q_{n+m} with $n + m$ elements, x_k, the combined points in the two given partitions. The difference between the Riemann sums will look like:

$$P_n(f) - P_m(f) = \sum_{k=1}^{n+m}(f(t_k^*) - f(s_k^*) \cdot (x_k - x_{k-1})$$

where the t_k^* is the t^* assigned to whatever P_n interval contains $[x_k, x_{k-1}]$ and similarly for s_k^*. We claim that $|t_k^* - s_k^*| <$ mesh(P_n)+mesh(P_m) $< \delta$. Proof of claim: Each interval in the refinement is contained in a unique interval from each of P_n and P_m. If both endpoints are from P_n, then t^* comes from that interval so $|t_k^* - s_k^*| <$ mesh(P_m). Similarly, $|t_k^* - s_k^*| <$ mesh(P_n) if both x_k and x_{k+1} are in P_m. If each endpoints is from a different partition, t^* and s^* come from two overlapping intervals – they may be as far apart as the length of one plus the length of the other. Since the lengths are less than respective mesh, we have $|t_k^* - s_k^*| <$ mesh(P_n)+mesh(P_m). Which means we also have:

$$|t_k^* - s_k^*| < \delta \implies |f(t_k^*) - f(s_k^*)| < \frac{\epsilon}{b-a}$$

Putting it all together:

$$\begin{aligned}
|P_n(f) - P_m(f)| &= |\sum_{k=1}^{n+m}(f(t_k^*) - f(s_k^*) \cdot (x_k - x_{k-1})| \\
&\leq \sum_{k=1}^{n} |f(t_k^*) - f(s_k^*)| \cdot (x_k - x_{k-1}) \\
&\leq \epsilon \cdot \sum_{k=1}^{n+m}(x_k - x_{k-1}) \\
&\leq \epsilon(b - a)
\end{aligned}$$

□

Now we are free to use any of the techniques that are normally seen in a calculus courses: Consider partitions that are evenly spaced, tagged values that are endpoints, or maximum values or whatever.

Example 5.5 Compute $\int_a^b x^2 dx$

Exercise 5.1 Compute $\int_a^b x dx$ (Please do note that the answer is the area bounded by the curve, the x-axis, and the vertical lines $x = a$ and $x = b$.)

5.0.3 Theorems

Theorem 5.4 ALGEBRAIC PROPERTIES OF RIEMANN INTEGRALS *Given two functions, f and g, both integrable on the closed interval $[a, b]$. The following are true:*

1. $\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$
2. $\int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$
3. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

Theorem 5.5 ORDER PROPERTIES OF RIEMANN INTEGRALS *Given two functions, f and g, both integrable on the closed interval $[a, b]$. The following are true:*

1. *If $f(x) \leq g(x)$for all $x \in [a, b]$, then*
$$\int_a^b f(x) dx \leq \int_a^b g(x) dx)$$
2. *If M and m are respectively upper and lower bounds for f on $[a, b]$, then*
$$m \cdot (b - a) \leq \int_a^b f(x) dx \leq M \cdot (b - a)$$
3. $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$

We may know what integrable functions are but it can be cumbersome if not impossible to show the exact (or even approximate) value for a particular function, or class of functions. That is where the theorems of the next section make life easier.

There is nothing in the definition of the Riemann Integral that prohibits $b \leq a$.

Theorem 5.6 *For any function, f, integrable on $[a, b]$,*

1. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\displaystyle\int_a^a f(x)dx = 0.$

Proof. □

5.1 Fundamental Theorem of Calculus

5.1.1 Integrals as Functions

Given a function, f, integrable on an interval $[A, B]$ and $a \in [A, B]$, we define a new function $F : [A, B] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(x)dx$$

Theorem 5.7 *For any function, f, integrable on $[A, B]$ and F,*

1. *On any interval where f is positive, F is increasing.*

2. $F(a) = 0$

Proof. □

5.1.2 Statement of the Theorem

Theorem 5.8 FIRST FUNDAMENTAL THEOREM OF CALCULUS, AN EXISTENCE THEOREM *Let f be integrable on $[a, b]$ and continuous on (a, b). For $x \in [a, b]$ define*

$$F(x) = \int_a^x f(t)dt.$$

Then the derivative of F exists for all $x \in (a, b)$ and

$$F'(x) = f(x).$$

editor: The conclusion holds for any point, x, where f is continuous. f need not be continuous on all of the open interval.

Proof. □

Theorem 5.9 SECOND FUNDAMENTAL THEOREM OF CALCULUS, A UNIQUENESS THEOREM *Let f be continuous on an open interval, I, let F be an antiderivative for f on I, that is $F'(x) = f(x)$ for all $x \in I$. Then, for $a \in I$ and each $x \in I$,*

$$F(x) = F(a) + \int_a^x f(t)dt.$$

editor: Well, unique up to a constant.

5.2 Computing integrals

5.3 Application: Logarithm and Exponential Functions

Definition We define a function, $\log : \mathbb{R}^+ \to \mathbb{R}$, using Riemann integral of $\frac{1}{x}$. That is,

$$\log(x) = \int_1^x \frac{1}{t} dt$$

NOTE: Because $\frac{1}{t}$ is not integrable on any interval, $(0, a)$, the log is not defined for negative values.

Common properties Υ of logarithms can be derived from properties of integrals.

Exercise 5.2 Prove the follow theorem:

Theorem 5.10 *1.* $\log(1) = 0$

2. For all $x, y \in \mathbb{R}^+$, $\log(xy) = \log(x) + \log(y)$

5.4 Flowchart