Activity 1 **A Grab-Bag Graph**

The chart below shows the 24 participants at a grab-bag event, and whose gift each of them picked. Draw a graph that represents this situation, determine whether or not the graph is a cycle, and provide an explanation for your conclusion.

Name	.. picked the gift brought by ..		Name	.. picked the gift brought by ..
A	Q		M	P
B	U		N	T
C	I		O	L
D	C		P	N
E	K		Q	S
F	D		R	X
G	O		S	R
H	W		T	M
I	A		U	G
J	B		V	F
K	H		W	J
L	E		X	V

Is your graph a cycle? (Circle one) YES NO

Explain your conclusion. ____________________

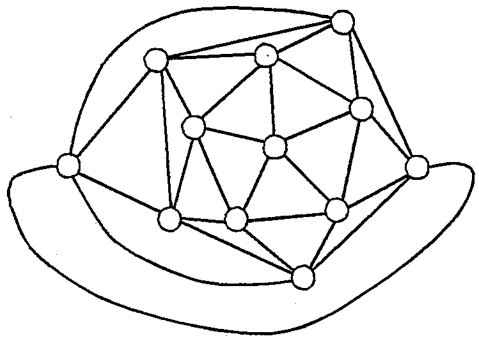
____

____

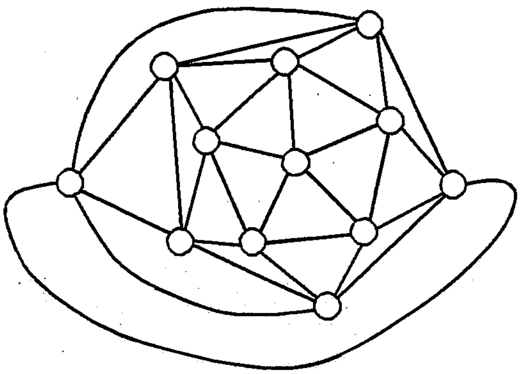
____

Activity 2 **Counting Vertices and Edges**

1. Verify that the graph is regular.



2. Count the number of edges using two different methods:
   a) count every edge individually, numbering the edges as you count them; and



   b) count the number of edges by multiplying the number of vertices by the degree of the vertices.

Are these two answers the same? Explain.

Activity 3 **3-Regular Graphs**

1. Construct a 3-regular graph with 4 vertices.

2. Construct two different 3-regular graphs with 6 vertices.

3. a. A graph has 12 vertices each of degree 5. How many edges does it have?

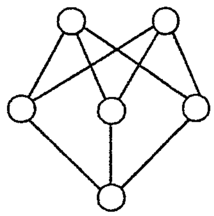
   b. A graph has 10 vertices each of which has degree 3 or 4. Explain why the graph has to have at least 15 edges and at most 20 edges.

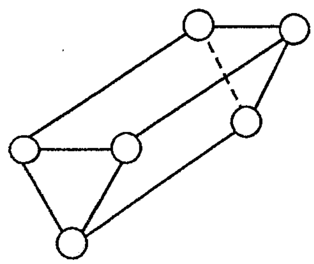
Activity 4 **Constructing Graphs**

Construct two different graphs each of which has six vertices – one vertex of degree 5, two vertices of degree 4 and three vertices of degree 3.

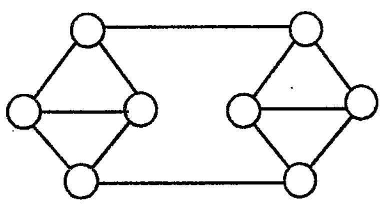
Activity 5 **Non-Isomorphic Graphs**

1. Show that the two graphs below are not isomorphic.

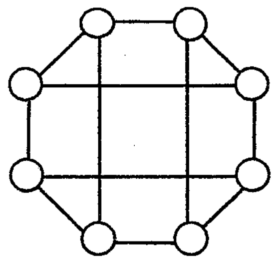




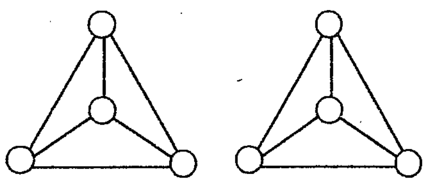
2. Below are four 3-regular graphs with 8 vertices.



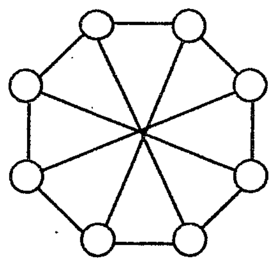
Graph A



Graph B



Graph C



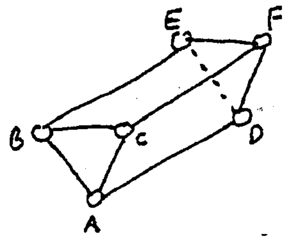
Graph D

Explain why no two of them are isomorphic. (To do this, you have to look at each pair of graphs and explain why those two graphs are not isomorphic. Use the back of the page for more space.)

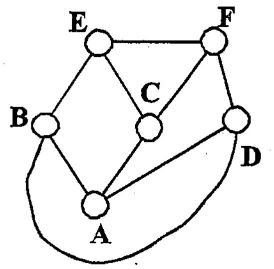
How many pair-wise comparisons will you have to make? ______

Activity 6 **Isomorphic Graphs**

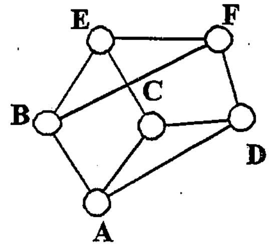
1.



Graph A



Graph B

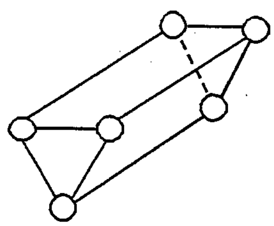


Graph C

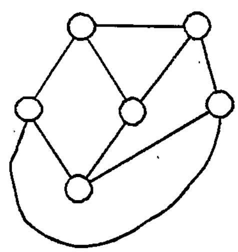
a. Explain why the labeling of Graph C does not establish an isomorphism with Graph A

b. Explain why the labeling of Graph B does not establish an isomorphism with Graph A.

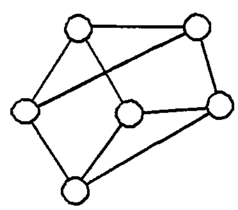
c. Find a labeling for Graphs B and C that will establish an isomorphism with Graph A.



Graph A



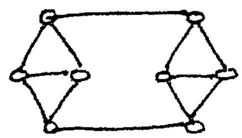
Graph B



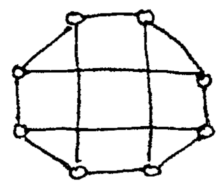
Graph C

Activity 6 **Isomorphic Graphs (cont.)**

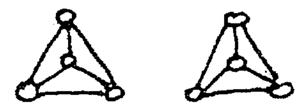
2. Which of the four 3-regular graphs with 8 vertices (Graphs A through D below) are isomorphic to the cube graph?



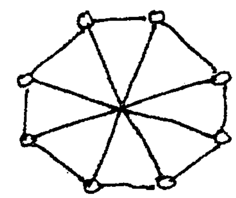
Graph A



Graph B

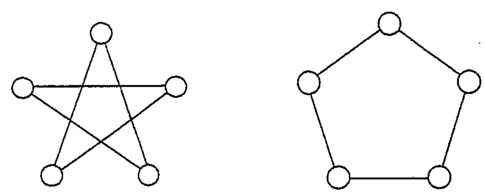


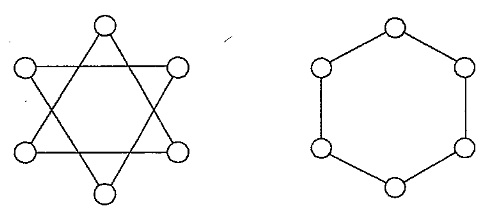
Graph C

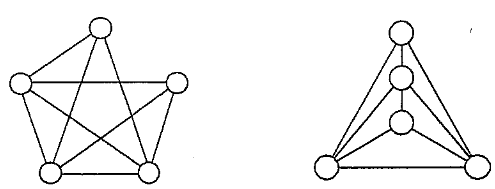


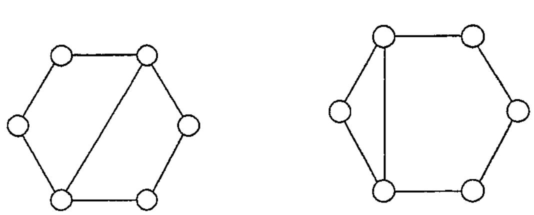
Graph D

3. For each pair of graphs, either show that the two graphs are isomorphic (using one of the methods above), or show that they are not isomorphic (by finding a characteristic of the vertices and edges of one graph that is not shared by the vertices and edges of the other graph).









Activity 7 **Constructing Graphs**

Construct all the different graphs that have six vertices – two vertices with degree 3 and four vertices with degree 2.

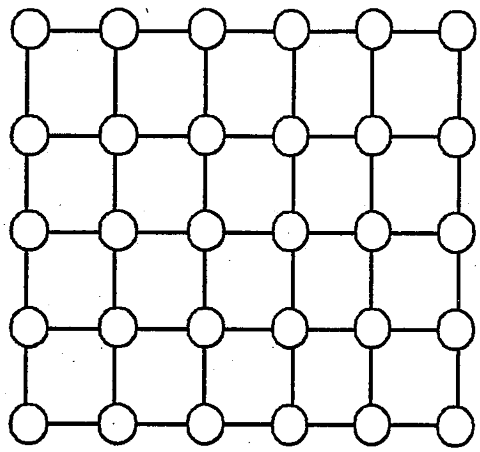
Activity 8 **Soccer Ball Graph**

1. How many edges are there in the soccer ball graph?

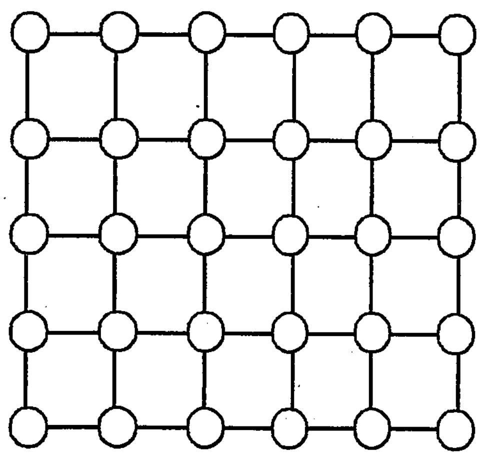
2. How many hexagons are there on the soccer ball?

Activity 9 **A Grid Graph**

1. The grid graph below has thirty vertices in a 5x6 array. Find the degrees of all the vertices in the grid graph and add them together to find the total number of edges in the graph.



2. Count directly the number of edges in your graph.



Does your answer agree with your conclusion in question 1? Explain.

____

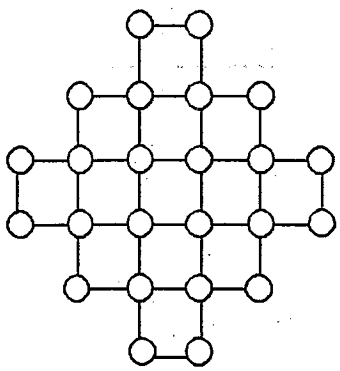
____

____

Activity 10 **Vertices and Edges**

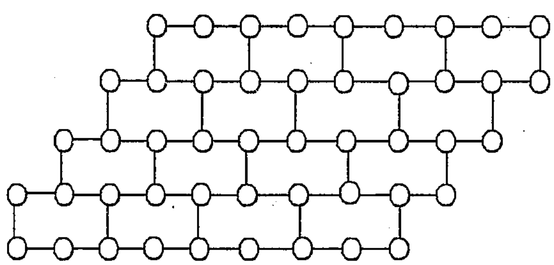
1. Find the number of vertices and the number of edges in each of the graphs below, first by degree counting the edges, and then, as a check, by counting directly the vertices and edges.

Degree Counting

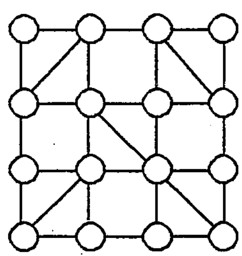


Graph A

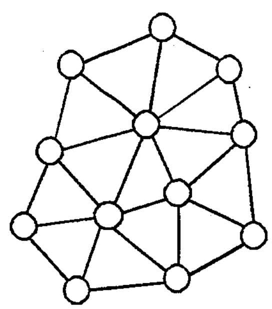
	Number of Vertices	**Number of Edges**
Graph A		
Graph B		
Graph C		
Graph D		



Graph B



Graph C



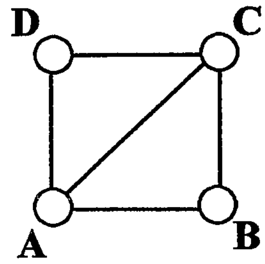
Graph D

2. Try to draw a graph that has three vertices of degree 2, three vertices of degree 3, and three vertices of degree 4. Do you think it can be done?

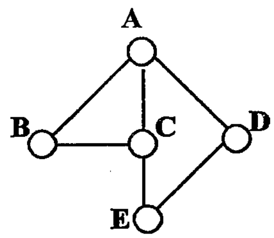
Activity 11 **Odd and Even Plates**

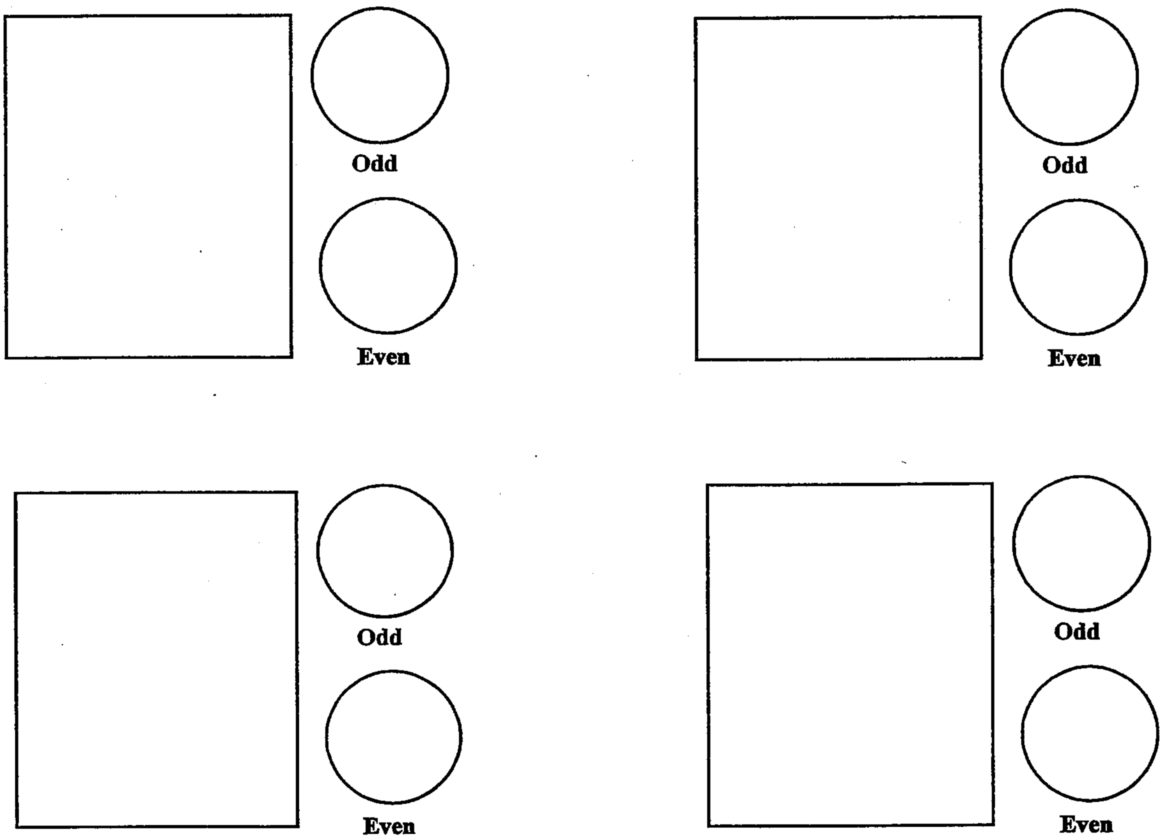
When we analyzed the construction of the graph to the right using odd plates and even plates, we learned that the number of vertices on the odd plate was always even.

However, since there were four vertices altogether, the number of vertices on the even plate was also always even.

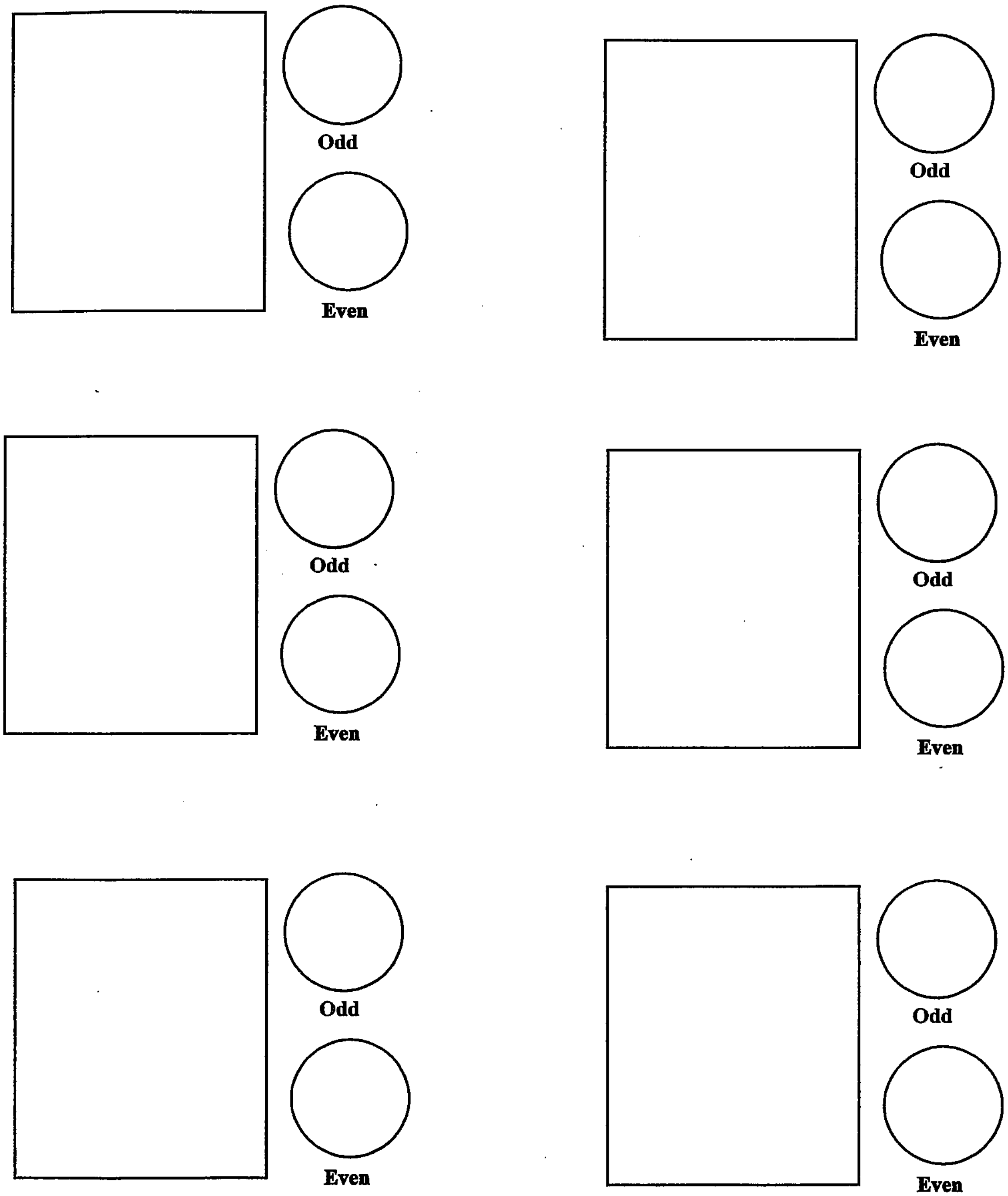


Carry out the same step-by-step construction of the graph to the right to verify that at each step of the construction the number of vertices on the odd plate is even. (In this example, with five vertices, the number of vertices on the even plate will now be always odd.)





Activity 11 **Odd and Even Plates (Cont.)**



Activity 12 **Constructing Graphs**

1. How many different graphs can you find with four vertices? (Use the problem solving strategy of "Break the Problem into Sub-problems", one for each possible number of edges in the graph.)

2. Construct a graph with seven vertices – two vertices of degree 2, two of degree 3, and three of degree 4.

3. Can there be a graph with five vertices – two vertices of degree 2 and three of degree 3? If yes, construct such a graph; if no, explain why.

Activity 12 **Constructing Graphs (Cont.)**

4. Construct a graph with six vertices – one vertex of degree 5, one of degree 4, two of degree 3, one of degree 2, and one of degree 1. How many different graphs like this are there?

5. How many different graphs can you find with six vertices – ~~four~~ two vertices of degree 2 and ~~two~~ four vertices of degree 3.

6. How many different 3-regular graphs with eight vertices can you find?

Activity 13 **Sprouts**

1. a. Suppose that faced with the situation in Figure 88 Player I creates a new vertex E as in Figure 91 instead of as in Figure 89. Find a winning move for Player II in this new position.

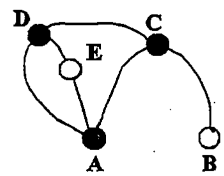


Figure 89

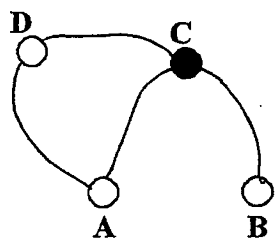


Figure 88

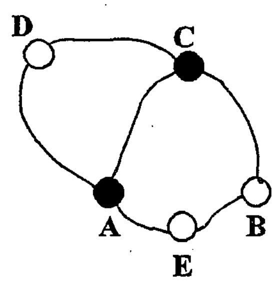


Figure 91

Player II creates new vertex D   Player I creates new vertex E   Player II's winning move

b. Thus the move for Player I described in Figure 91 above is not good for Player I, since it allows Player II to follow with a winning move. Suppose instead that Player I creates a new vertex E as in Figure 92. Show that no matter what Player II does next, Player I's subsequent move will win the game.

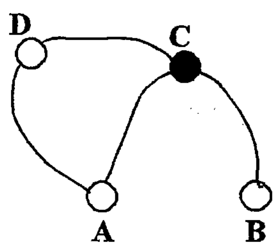


Figure 88

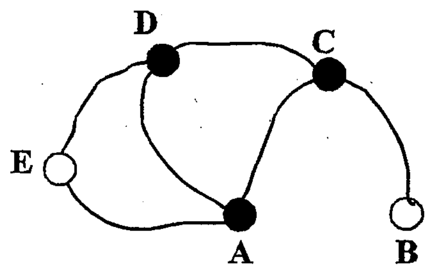


Figure 92

Player II creates new vertex D   Player I creates new vertex E   Player II's possible next moves

Player I's subsequent move which wins the game

Activity 13 **Sprouts (Cont.)**

c. Player II's move in Figure 88 – connecting D to A and C – was therefore a losing move for Player II. Find a winning move for Player II in Figure 87 – that is, a move that will guarantee a win for Player II no matter what Player I does.

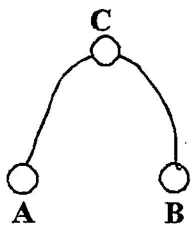


Figure 87

Player I creates new vertex C

Player II creates ...

Player I creates ...

Player II creates ...

Player I creates ...

Player II creates ...

Player I creates ...

Player II creates ...

Player I creates ...

Activity 13 **Sprouts (Cont.)**

2. a. Calculate the number of vertices and edges after each move in a game of Sprouts using the table below. Determine the pattern in this table.

After Move #	the number of vertices is	and the number of edges is
0	2	0
1	3	2
2	4	
3		
4		
5		
6		

Describe the pattern you found: ______________________________

____

____

b. Play several games of Sprouts (either with a real partner or a pretend partner), trying to find the largest number of vertices that you can have before the game is over. If needed, use the back of this page for more space.

Activity 13 **Sprouts (Cont.)**

c. Use Rule 1 and the table on the previous page to find the maximum number of moves that can take place in a game of Sprouts.

Rule 1: No vertex can have degree greater than 3.

3. Suppose that Sprouts started with three vertices instead of two vertices. What's the maximum number of moves in the game? (Play some games first, then make a chart, and then try to use Rule 1 and the chart to determine the maximum number of moves in this version of Sprouts.)

After Move #	the number of vertices is	and the number of edges is
0		
1		
2		
3		
4		
5		
6		

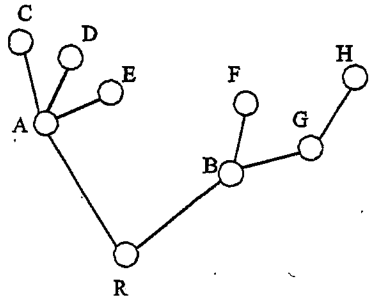
Describe the pattern you found: ______________________________

____

____

Activity 14 **Levels in Trees**

The trees in the Figures below are the same as the tree in the Figure to the right, except that whereas in the tree in Figure 1 is exactly the same as the tree to the right, in the other Figures, the vertex H has been relocated; instead of being adjacent to G (as in Figure 1) it is adjacent to A (in Figure 2), B (in Figure 3), C (in Figure 4), or F (in Figure 5).



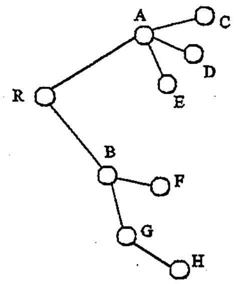
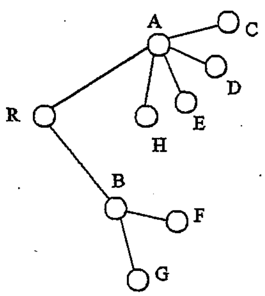
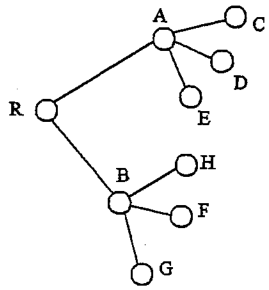
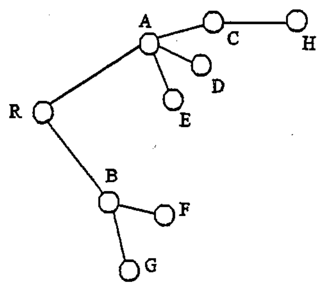
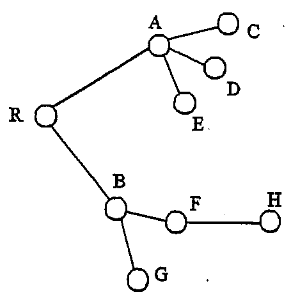


Figure 1

1. Determine whether each of these trees is isomorphic to the tree in Figure 1 and explain your conclusions.

Tree	Isomorphic to Figure 1? YES or NO	Explain why or why not
 **Figure 2**		

Activity 14 **Levels in Trees (cont.)**

Tree	**Isomorphic to Figure 1?** **YES or NO**	**Explain why or why not**
 **Figure 3**		
 **Figure 4**		
 **Figure 5**		

Activity 14 **Levels in Trees (cont.)**

2. Draw two different (i.e., non-isomorphic) trees with twelve vertices, one at level 0, two at level 1, five at level 2, and four at level 3. (Use the back of this page for more space.)

3. Draw another tree that meets the conditions in problem 2 and with a different choice of root has one vertex at level 0, four at level 1, two at level 2, two at level 3, and three at level 4.

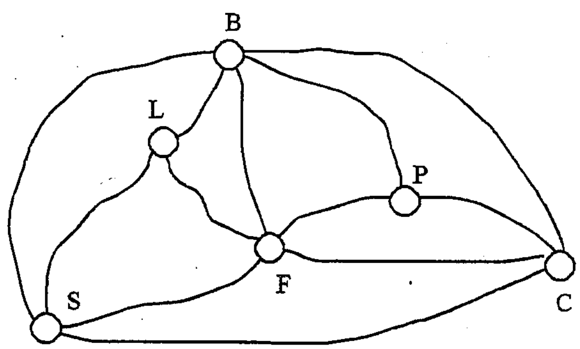
4. Draw another tree with twelve vertices, one at level 0, two at level 1, five at level 2, and four at level 3 (just as in part 2) but with a different choice of root has one vertex at level 0, five at level 1, five at level 2, and one at level 3.

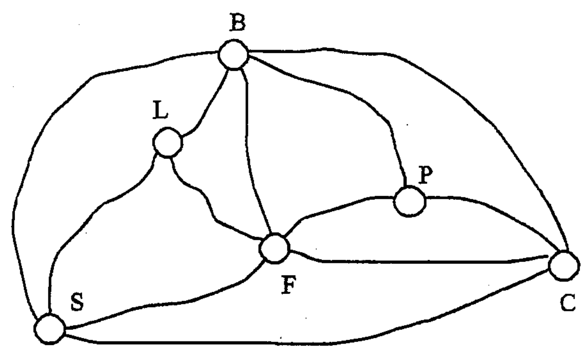
5. If the height of a tree is 4, and a different vertex is chosen as the root, what is the maximum height of the tree that results?

Activity 15 **A Muddy City Problem**

Suppose for example that the graph below represents a map of Muddy City that shows its six critical sites and all the unpaved roads that connect them. The six sites are the city hall (C), police station (P), school (S), bank (B), firehouse (F) and library (L).

a. Find the fewest number of roads that the governing body needs to pave in order to make it possible for people to travel from any site to any other site on paved roads. Two copies of the graph are provided.





b. The governing body wants you to justify that your solution involves the fewest possible number of roads, that is, that your solution uses a minimum number of roads. Prove to them that your solution is indeed minimal. (Hint: Imagine that you had to supply a proof not just for the configuration above, but for all possible Muddy Cities.)

Activity 16 **Tree Diagrams**

1. a. Make a tree diagram consisting of all the *folders* in your computer's C drive. (Since many of the folders will contain a large number of files, you should not try to include files in your tree diagram.)

   b. How many levels are there in this tree? ______________

(Remember that the root is at level 0 and that all vertices adjacent to the root are at level 1.)

   c. What is the length of the longest path in this tree?____________________________

2. You have four shirts, three pairs of pants, and two jackets. Make a tree diagram that shows all of the possible outfits you can wear if you first choose a shirt, then a pair of pants, and then a jacket. (You will need to give each item of clothing its own name so that you can use the names as labels in the tree diagram.)

   a. How many levels are there in this tree? _____ How many vertices at each level?_____

   b. How many branches are there in this tree? ______________

Activity 16 **Tree Diagrams (cont.)**

3. A spinner is divided into four sections – red, yellow, blue, and green.

a. Make a tree diagram that shows all the possible outcomes of spinning three times.

b. How many levels are there in this tree?__________

How many vertices are at each level? __________

c. How many vertices would there be in each of the next two levels if the spinner was used two more times?

d. How many branches are there in the tree you made in part a? _____

How many branches contain vertices that are all labeled with the same color?_____

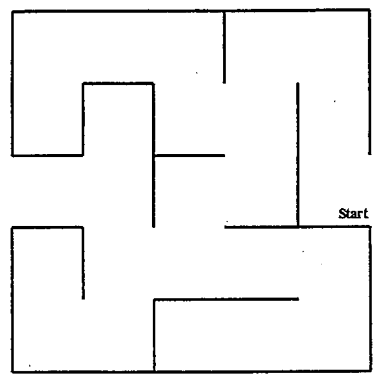
How many branches contain vertices that are all labeled with different colors? ______

What would be the answers to these questions if the spinner was used one more time?

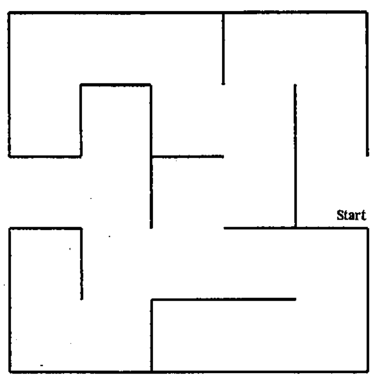
Two more times?

Activity 17 **Mazes and Trees**

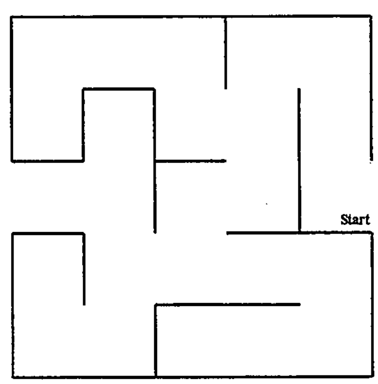
1. Partition the maze into a 5 x 5 grid graph and label each of the resulting twenty-five squares in the following way: From left to right, use the letters A to E in the top row, letters F to J in the second row, K to O in the third row, P to T in the fourth row, and U to Y in the bottom row.



2. What path would you take through the maze below if you applied the hand-on-the-wall technique using your left hand? Draw this path in the maze and also generate a list of letters that show how the path moves through the squares.

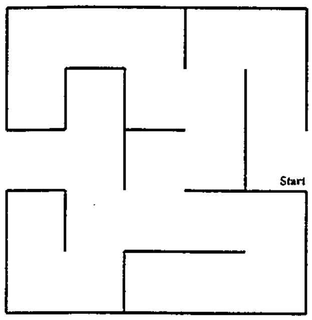


3. What path would you take through the maze below if you applied the hand-on-the-wall technique using your right hand? Draw this path in the maze and also generate a list of letters that show how the path moves through the squares.

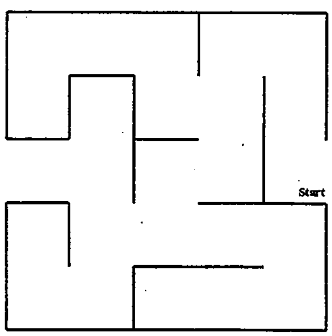


Activity 17 **Mazes and Trees**

4. Find a tree that represents the maze below which has the property that a depth-first search through that tree would give the same path as provided by the hand-on-the-wall technique in problem 2.

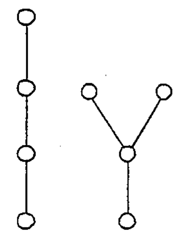


5. What path would you take through the maze if you applied a depth-first search through this tree for the exist, but if instead of always favoring "left" you always favored "right"? Compare this path with your answer to problem 3.



Activity 18 **Growing Trees**

1. Construct all trees with five vertices by adding one vertex and one edge in all possible ways to the two trees with four vertices. Determine which of the trees you constructed are isomorphic to each other, and then indicate how many different (i.e., non-isomorphic) trees there are with five vertices.



2. Construct all trees with six vertices. Determine which of the trees you constructed are isomorphic to each other, and then indicate how many different (i.e., non-isomorphic) trees there are with six vertices.

Activity 18 **Growing Trees (cont.)**

3. Construct all forests with six vertices. Determine which of the forests you constructed are isomorphic to each other, and then indicate how many different (i.e., non-isomorphic) forests there are with six vertices. Explain why no two forests on your list are isomorphic.

Activity 19 **Proof by Contradiction**

1. A **star** is a tree that has a vertex that is adjacent to all the other vertices; this vertex is called the **core** of the star. Use proof by contradiction to show that in a star every vertex except for the core has degree 1.

Step #1:

Step #2:

Step #3:

Step #4:

Step #5:

Step #6:

Activity 19 **Proof by Contradiction (cont.)**

2. Can a star have more than one core? _________
   Use proof by contradiction to show that, with one exception, a star has only one core.

Step #1:

Step #2:

Step #3:

Step #4:

Step #5:

Step #6:

Activity 19 **Proof by Contradiction (cont.)**

3. Show that if A and B are two vertices in a star, then there is a path of length 1 or length 2 from A to B.

4. Construct all stars with six vertices. What general conclusion can you make about the number of stars with 25, 100, or *n* vertices?

5. What would you call a graph if every one of its components was a star?______________

Activity 20 **Is the Graph Bipartite?**

1. For each of the following graphs (Figures 155-162), determine whether or not it is bipartite. If it is bipartite, assign the vertices to the two sets X and Y so that every edge of the graph joins a vertex in the set X with a vertex in the set Y. One way of doing this is simply to shade the vertices that are in X; you have done this correctly if each edge in the graph has a shaded vertex at one end and an unshaded vertex at the other end.

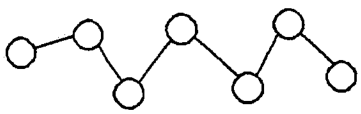


Figure 2.155

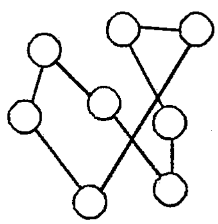


Figure 2.156

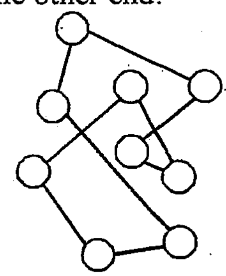


Figure 2.157

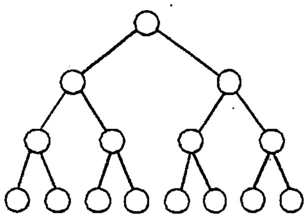


Figure 2.158

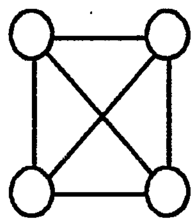


Figure 2.159

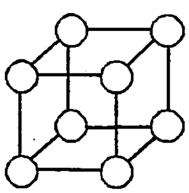


Figure 2.162

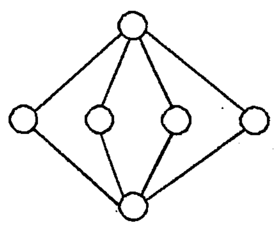


Figure 2.160

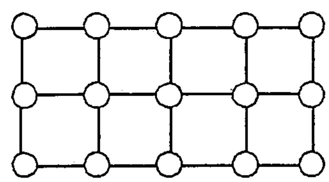


Figure 2.161

2. Draw the complete bipartite graphs $K_{2,4}$ and $K_{4,2}$ and explain why they are isomorphic. What is the appropriate general conclusion?

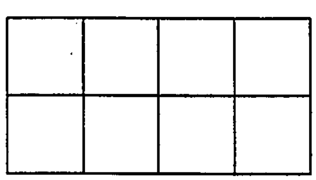
Activity 21 **Grid Graphs**

1. How many vertices and edges are there in $G_{3,5}$, $G_{7,8}$, and $G_{m,n}$?

2. How many small squares are there in these graphs?

3. What is the relationship between $G_{3,5}$ and $G_{5,3}$? What is true in general?

4. Although the figure below looks a bit like a grid graph, it is not a grid graph – vertices are not drawn at the intersections. It is instead a map consisting of eight small square countries whose boundaries form a grid. Draw and describe the graph associated with this map.



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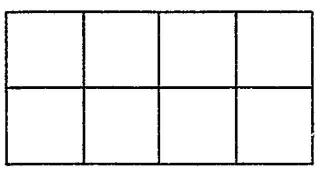


Activity 22 **Checkerboard Problems (cont.)**

3. A cube of cheese has dimensions 3" by 3" by 3". It consists of 27 smaller cubes (each with dimensions 1" by 1" by 1") of 27 different delectable cheeses. A mouse starts in one of the bottom corner cubes and eats all of the cheese in that cube. Then it proceeds to an adjacent cube and eats of the cheese there. If it always has to go to an adjacent cube, can it eat all 27 cubes of cheese, finishing with the cube in the center?

Activity 23 **Factor Trees**

1. Construct a factor tree for 90 and a factor tree for 900.

2. Construct a factor tree for 72 where the prime factors, reading left to right across the leaves of the tree, are 3, 2, 2, 2, 3. How many different factor trees can you construct with this property?

Activity 23 **Factor Trees (cont.)**

3. Let us insist that when the number on a vertex is factored, the smaller number always goes on the left. In that case, how many different factor trees are there for 70? How many different factor trees are there for 72?

4. Find a factor tree for 900 whose height is as large as possible. Find a factor tree for 900 whose height is as small as possible. Show that these factor trees do indeed have maximum and minimum heights.

Activity 24 **Connected Graphs and Induction**

1. Show by induction (on the number of vertices in a tree) that any tree with more than one vertex has at least two leaves.

2. Show by induction (on the number of vertices in a tree) that if a tree has a vertex of degree k then the tree has at least k leaves.

Activity 2.4 **Connected Graphs and Induction (cont.)**

3. Show by induction that a connected graph on n vertices has at least $n - 1$ edges.

Activity 24 **Connected Graphs and Induction (cont.)**

3. Show by induction that a connected graph on n vertices has at least $n - 1$ edges.

Activity 25 **Factor Graphs**

1. a. Draw the factor graphs F_{12}, F_{18}, F_{36}, and F_{100}.
b. Each of the graphs in part a. is isomorphic to a grid graph. Find each grid graph and label the vertices to show that it is isomorphic to the factor graph you drew for part a.

	Factor Graph	Isomorphic Grid Graph
F_{12}		
F_{18}		
F_{36}		
F_{100}		

Activity 26 **Matchings**

The chart below indicates which of the seven employees of the Ruckmaking Corporation is qualified to do each of the seven jobs that need to be done. Find all possible job assignments that will result in each person being assigned to a job for which he or she is indeed qualified.

Job Number	People Qualified for the Job
1	Gerald, Laurel, Roberta
2	Karen, Gerald
3	Laurel, Chris
4	Karen, Roberta, Sam
5	Joe, Chris
6	Joe, Karen
7	Sam, Gerald, Laurel

(You should draw the graph, but only to verify that it turns out to be a mess. It's the sort of mess that computers can deal with easily, which is why there are many programs that will solve such assignment and scheduling problems. But it's not the sort of mess that we humans like to deal with.)

Activity 27 **Maximum Matchings**

1. Find a maximum matching in each of the graphs in Figures 200–204. In each case, explain why your matching is a maximum matching and whether it is a perfect matching.

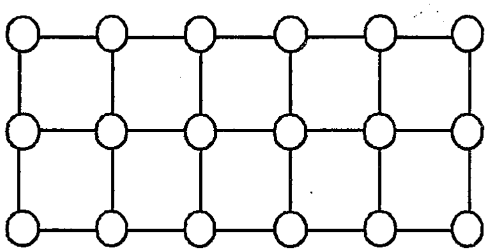


Figure 2.200

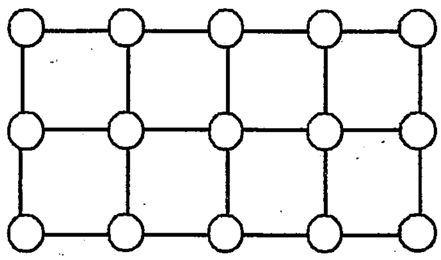


Figure 2.201

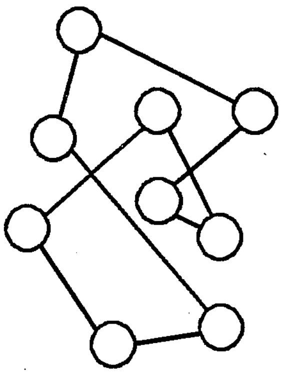


Figure 2.202

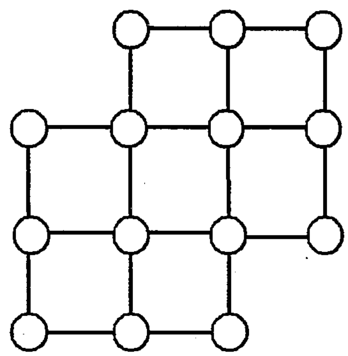


Figure 2.203

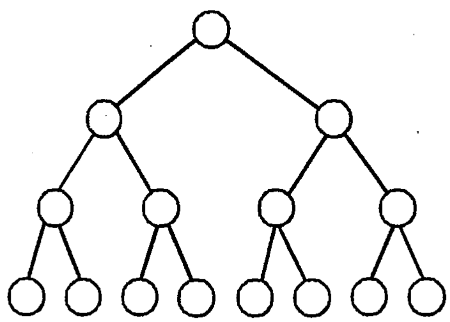


Figure 2.204

Activity 27 **Maximum Matchings (cont.)**

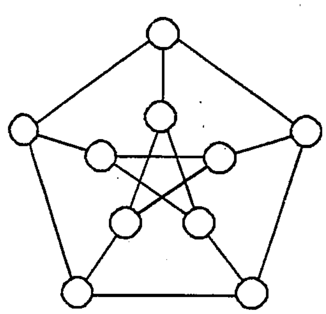
2. How many different maximum matchings are there in each of the following graphs?

   a. The path P_n

   b. The cycle C_n

   c. The wheel W_n

   d. The Petersen graph



Activity 28 **Joe's Game**

Joe's Game:

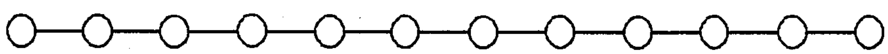
Player 1 and Player 2 alternate picking vertices in a graph, with Player 1 going first, subject to the following two rules:

Rule 1. Neither player can pick a vertex that has been picked earlier in the game.
Rule 2. Every vertex must be adjacent to the vertex picked at the previous turn.

The winner is the last player who can pick a vertex.

Explain the rules to this game to a friend and try out the game on each of the graphs in this Activity.

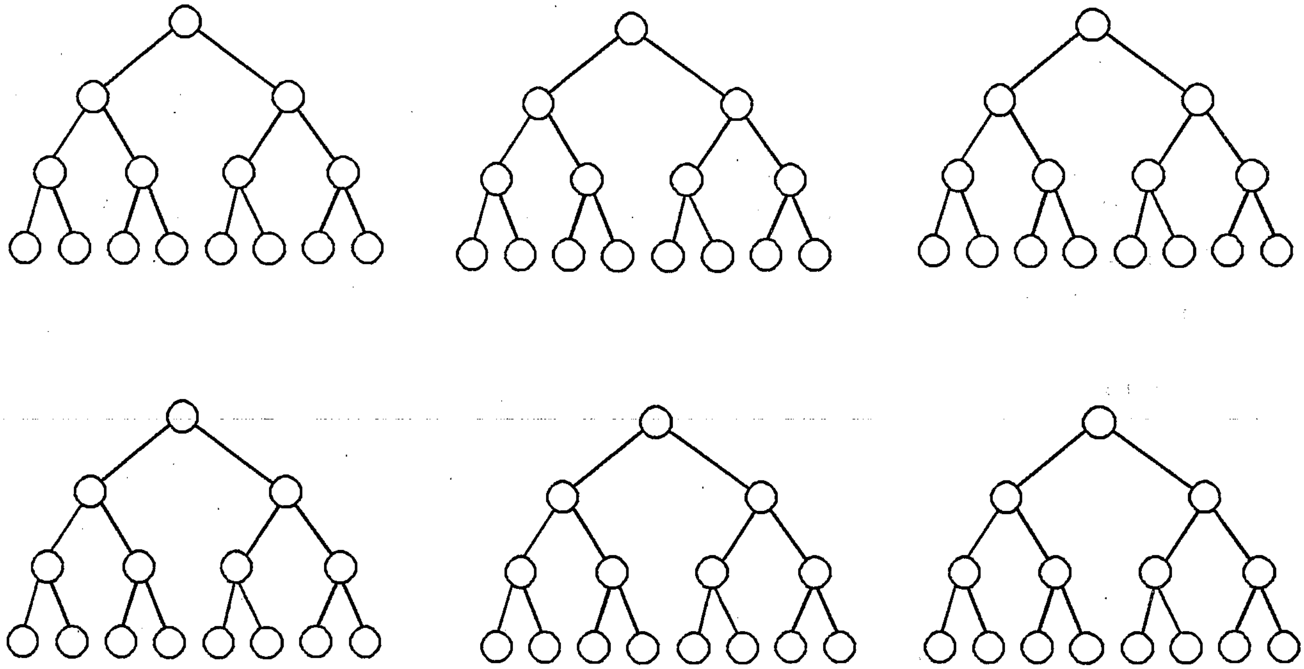
1. In the game on the graph P_{12}, how should Player 2 respond if Player 1 picks one of the vertices in the middle?



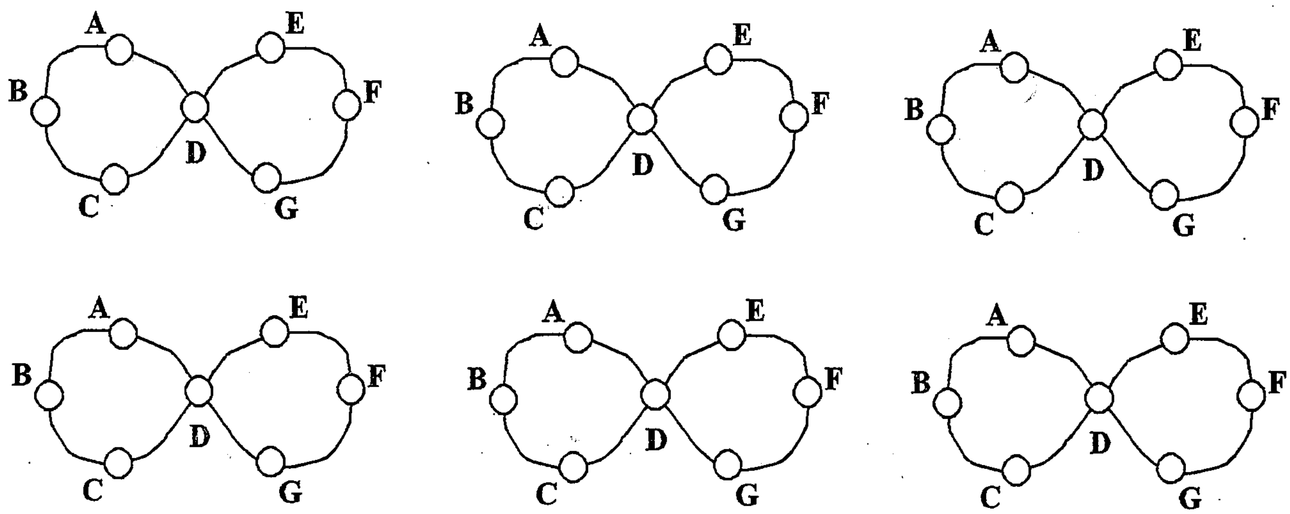
2. Which player has the winning strategy on the graph P_n, for each n?

Activity 28 **Joe's Game (cont.)**

3. Which player has the winning strategy on this graph? Six copies of the graph are provided.



4. Which player has the winning strategy on this graph? Six copies are provided.



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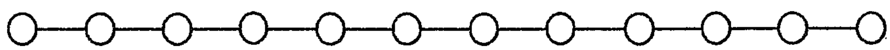
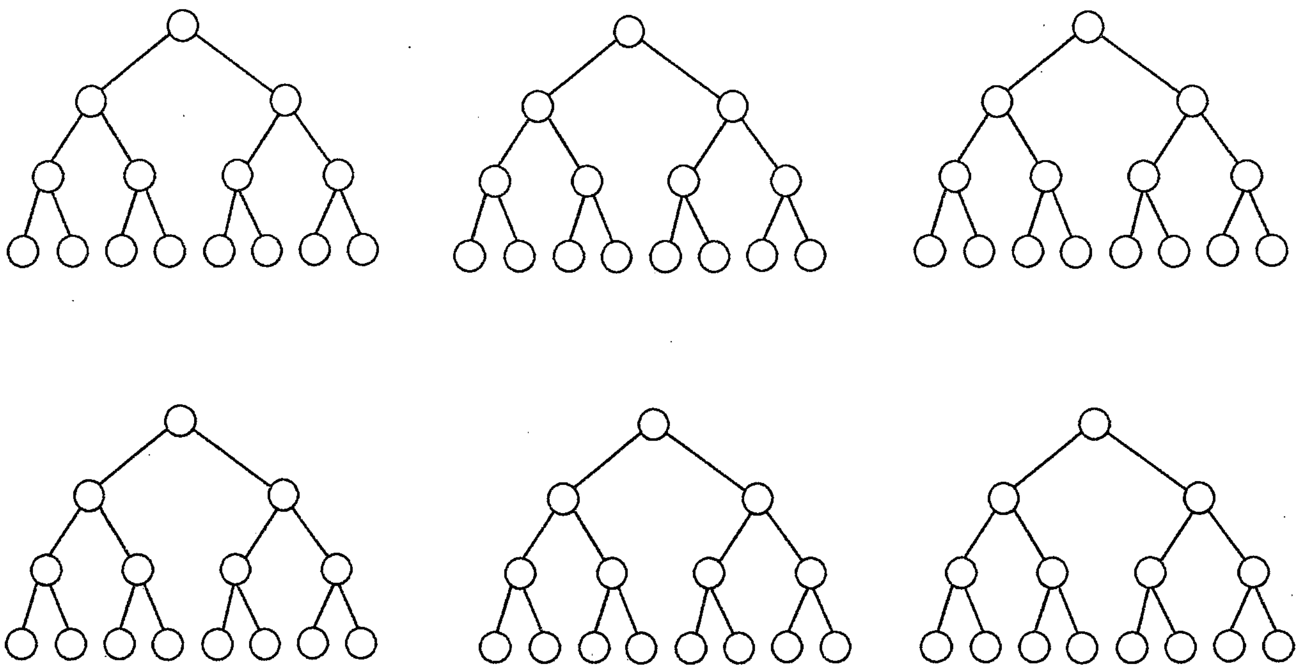


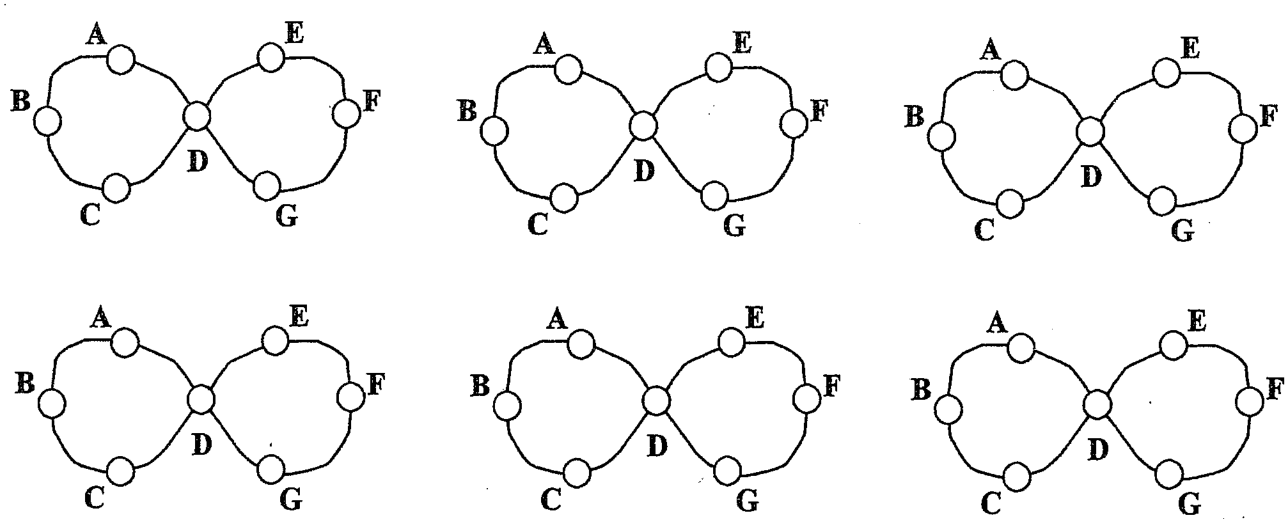
2. Which player has the winning strategy on the graph P_n, for each n?

Activity 28 **Joe's Game (cont.)**

3. Which player has the winning strategy on this graph? Six copies of the graph are provided.

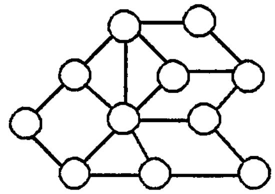


4. Which player has the winning strategy on this graph? Six copies are provided.

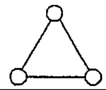
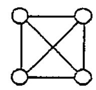
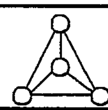
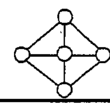
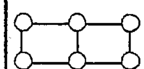
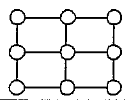
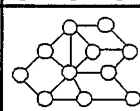
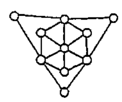


Activity 30 **Regions, Vertices, and Edges**

1. Determine the number of regions in this graph and the number of regions whose borders are isomorphic to C_n for each n.



2. Calculate the number of regions, vertices, and edges for each of the indicated graphs and enter the results in the table.

Graph Name	Graph Picture	R	V	E
K_3				
K_4				
W_3				
W_4				
$G_{2,3}$				
$G_{3,3}$				
				
				

3. Can you find a relationship involving R, V, and E that is true for all these graphs?

(To answer this question, you need to look closely at the data in the table, and see if you can find a pattern involving the entries in the R, V, and E columns. There are a few extra rows in the table so that you can draw your own planar graphs, and enter the additional data generated by those graphs.)

Activity 29 **Planarity of Complete Graphs**

3. What is the minimum number of crossings with which you can draw the complete bipartite graph $K_{3,3}$.

4. What is the minimum number of crossings with which you can draw the complete graph K_6?

Activity 32 **Crossing Number**

1. Use software like http://illuminations.nctm.org/mathlets/graphcreator/index.html to find a drawing of K_5 that has only one crossing and in which all the edges are straight line segments. After you find this graph, draw it in the space below.

2. Use the software and your answer to Problem 1 to find drawings of K_6, K_7, and K_8 that have the minimal number of crossings indicated in the chart below and in which all the edges are straight line segments.

	Rectilinear Crossing Number	**Crossing Number**	**Drawing**
K_6	3	3	
K_7	9	9	
K_8	19	18	

3. Can you find a drawing of K_8 in which there are only 18 crossings?

Activity 31 **The Utilities Problem**

Show that $K_{3,3}$ is not planar, and that therefore there is no way of connecting the supply lines without any crossings. (Your argument should be similar to the argument used for K_5 found on pages 87-88. You will need to use the fact that a bipartite graph like $K_{3,3}$ contains no *odd* cycles.)