**Homework problems ##4 and 5**
**Math 536**
**Due Friday, Oct. 2, 2009**

**4.** Let $Y$ be a Riemann surface and let $\Gamma$ be a group which acts biholomorphically, freely and properly discontinuously on $Y$. In class we showed, for the special case where $Y$ is an open subset of $\mathbf{C}$, that there is a natural atlas on $Y/\Gamma$.

**(a)** Generalize the construction given in class to define a natural atlas on $Y/\Gamma$.

**(b)** Show that $Y/\Gamma$ is Hausdorff in the topology defined by the natural atlas, and is therefore a Riemann surface in a natural way.

**(c)** Show that the projection $p : Y \to Y/\Gamma$, defined by taking $p(y)$ to be the orbit of $y$ for every $y \in Y$, is an analytic map of Riemann surfaces.

**5.** This problem concerns the Riemann sphere $\widehat{\mathbf{C}} := \mathbf{C} \cup \infty$.

**(a)** Show that every rational function $f$ on $\mathbf{C}$ extends uniquely to a meromorphic function $\widehat{f}$ on $\widehat{\mathbf{C}}$.

**(b)** Let $P_1, \dots, P_m, Q_1, \dots, Q_n$ be distinct points of $\widehat{\mathbf{C}}$, let $d_1, \dots, d_m, e_1, \dots, e_n$ be positive integers, and suppose that $d_1 + \cdots + d_m = e_1 + \cdots + e_n$. Show that there exists a meromorphic function on $\widehat{\mathbf{C}}$ of the form $\widehat{f}$, where $f$ is rational on $\mathbf{C}$, such that $\widehat{f}$ has a zero of order $d_i$ at each $P_i$, a pole of order $e_j$ at each $Q_j$, and no other zeros or poles. (This will go much better if you think about multiplication and division of meromorphic functions rather than addition.)

**(c)** Show that every meromorphic function $g$ on $\widehat{\mathbf{C}}$ has the form $g = \widehat{f}$ for some rational function $f$ on $\mathbf{C}$. (First use results about analtic maps between Riemann surfaces, proved in class on Friday 9/25 or Monday 9/28, to show that $g$ has only finitely many zeros $P_1, \dots, P_m$ and finitely many poles $Q_1, \dots, Q_n$; and that if $d_i$ denotes the order of the zero of $g$ at $P_i$, and $e_j$ denotes the order of the pole of $g$ at each $Q_j$, then $d_1 + \cdots + d_m = e_1 + \cdots + e_n$. Then consider $g/\widehat{h}$, where $h$ is a suitably chosen rational function.)