Chapter 1
On the energy equality for weak solutions of the 3D Navier-Stokes equations

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Abstract We prove that the energy equality holds for weak solutions of the 3D Navier-Stokes equations in the functional class $L^3([0,T);\mathscr{D}(A^{5/12}))$, where $\mathscr{D}(A^{5/12})$ is the domain of the fractional power of the Stokes operator.

1.1 Introduction

We consider weak solutions to the three dimensional 3D incompressible Navier-Stokes equations (NSE)

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = 0 \end{cases} \tag{1.1}$$

on a smooth bounded domain $\Omega \subset \mathrm{R}^3$ subject to the Dirichlet boundary condition. We assume $f \in L^1([0,T);L^2(\Omega))$. The classical Leray-Hopf solutions to (1.9) that belong to $u \in L^\infty L^2_x \cap L^2_t H^1_x$ are known to only satisfy the energy inequality

$$|u(t)|_2^2 + 2\nu \int_{t_0}^{t} |\nabla u(s)|_2^2 ds \le |u(t_0)|_2^2 + 2\int_{t_0}^{t} (f(s) \cdot u(s)) ds, \tag{1.2}$$

for all $t \in [0,T)$ and almost all $t_0 \in [0,t)$ including $t_0 = 0$ (see [2]). The problem of proving exact energy equality for such solutions is an open problem. In a sequence of papers by Serrin [8], Lions [7], Shinbrot in [9], and more recently, Kukavica [6]

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it was settled in a stricter regularity class for velocity or pressure. Namely, $u \in L^r_t L^s_x$, with $2/s + 2/r \leq 1$ and $s \geq 4$, or $p \in L^2_{t,x}$. The main technical obstacle in proving energy equality is to show vanishing of the energy flux due to the nonlinear term. In context of the related question of energy conservation for weak solutions of the Euler equation ($\nu = 0$, $f = 0$) the optimal regularity of u has been found in terms of Besov-type spaces (see [1, 3, 4, 5]):

$$\lim_{y \to 0} \frac{1}{|y|} \int_{\Omega \times [0,T]} |u(x-y,t) - u(x,t)|^3 dx dt = 0. \tag{1.3}$$

where $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. The differential dimension of this space is equivalent to that of $L^3_t L^{9/2}_x$, which breaks the previously known scaling. On the Sobolev scale the corresponding space is $H^{5/6}$. For Navier-Stokes equation with Dirichlet boundary conditions more practical spaces to use are fractional domains of the Stokes operator A (see [2]). In the scale of those spaces $H^{5/6}$ corresponds to $D(A^{5/12})$. We note that the methods used previously for proving the energy equality in the case of $\mathbb{R}^3$ or $\mathbb{T}^3$ are not applicable to Dirichlet boundray conditions. For such boundary conditions the following theorem is proved in this present paper.

Theorem 1. *Every weakly continuous weak solution* $u : [0,T) \to L^2(\Omega)$ *to* (1.9) *which belongs to the regularity class* $L^3_t \mathscr{D}(A^{5/12})_x \cap L^2_t H^1_x$ *satisfies the energy equality.*

We remark that the energy inequality (1.2) is not needed in the proof of Theorem 1.

1.2 Preliminaries

In this section we briefly recall some standard facts (see [2] for details). Let us denote

$$H = \{u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial\Omega} = 0\}, \tag{1.4}$$

and let $\mathbb{P} : L^2(\Omega) \to H$ be the L^2-orthogonal projection. Let A be the Stokes operator defined by

$$Au = -\mathbb{P}\Delta u. \tag{1.5}$$

The Stokes operator is a self-adjoint positive sectorial operator with a compact inverse. Hence, there exists an orthonormal basis of eigenvectors $\{w_n\}$ in H, and a sequence of positive eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty,$$

such that

$$Aw_n = \lambda_n w_n, \qquad w_n \in \mathscr{D}(A), \tag{1.6}$$

Henceforth we will use the notation $u_n = (u, w_n)$. For $s > 0$ we define the operator A^s by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n w_n, \tag{1.7}$$

and the space

$$V^s = \{u \in H : u = \sum_{n=1}^{\infty} u_n w_n,\ \|u\|_s^2 = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2 < \infty\}, \tag{1.8}$$

which is exactly the domain $\mathscr{D}(A^{s/2})$. We denote $V = V^1$ and V' its dual. Let us put $B(u,v) := \mathbb{P}(u \cdot \nabla v) \in V'$ for $u,v \in V$. We can rewrite (1.9) as the following differential equation in V':

$$\partial_t u + \nu A u + B(u,u) = g, \tag{1.9}$$

where u is a V-valued function of time and $g = \mathbb{P} f$. Finally, we denote $b(u,v,w) = \langle B(u,v), w\rangle$. This trilinear form is anti-symmetric:

$$b(u,v,w) = -b(u,w,v), \qquad u,v,w \in V,$$

in particular, $b(u,v,v) = 0$ for all $u,v \in V$.

1.3 The proof of Theorem 1

Define

$$P_\kappa u = \sum_{n:\lambda_n \le \kappa^2} u_n w_n, \qquad u \in H. \tag{1.10}$$

Let $u \in V^\beta$ and denote $u_\kappa^{\mathrm{l}} = P_\kappa u$, $u_\kappa^{\mathrm{h}} = u - u_\kappa^{\mathrm{l}}$. Observe the following inequalities:

$$\begin{aligned} \|u_\kappa^{\mathrm{l}}\|_\beta &\le \kappa^{\beta-\alpha} \|u_\kappa^{\mathrm{l}}\|_\alpha \\ \|u_\kappa^{\mathrm{h}}\|_\alpha &\le \kappa^{\alpha-\beta} \|u_\kappa^{\mathrm{h}}\|_\beta, \end{aligned} \tag{1.11}$$

whenever $\beta > \alpha$.

Lemma 1. *Let* $u : [0,T) \to H$ *be a weakly continuous weak solution of* (1.9) *on* $[0,T)$. *Then*

$$\begin{aligned} &|u(t)|^2 + 2\nu \int_{t_0}^t \|u\|^2 \, ds \\ &\qquad = |u(t_0)|^2 + 2\int_{t_0}^t (g,u)\, ds + 2 \lim_{\kappa \to \infty} \int_{t_0}^t b(u, u_\kappa^{\mathrm{l}}, u)\, ds, \end{aligned} \tag{1.12}$$

for all $0 \le t_0 \le t < T$.

Proof. One can see from our assumption that $u_\kappa^{\mathrm{l}} \in C([0,T);V)$ and $\partial_t u_\kappa^{\mathrm{l}} \in L^2([0,T);V)$. Thus, using u_κ^{l} as a test function (allowed by the standard approximation argument) we obtain

$$|u^{\mathrm{l}}_\kappa(t)|^2 - |u^{\mathrm{l}}_\kappa(t_0)|^2 + 2\nu\int_{t_0}^t \|u^{\mathrm{l}}_\kappa\|^2\,ds - 2\int_{t_0}^t (g, u^{\mathrm{l}}_\kappa)\,ds = 2\int_{t_0}^t b(u, u^{\mathrm{l}}_\kappa, u)\,ds. \quad (1.13)$$

From this we see that the limit of the right hand side exists as $\kappa \to \infty$, which completes the proof of the lemma. $\square$

Let u^{l}_κ and u^{h}_κ be defined as before. In view of Lemma 1, it suffices to show that

$$\lim_{\kappa\to\infty}\int_0^T |b(u, u^{\mathrm{l}}_\kappa, u)|\,ds = 0. \quad (1.14)$$

To this end let us write

$$b(u, u^{\mathrm{l}}_\kappa, u) = b(u^{\mathrm{h}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa) + b(u^{\mathrm{l}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa) + b(u^{\mathrm{h}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{l}}_\kappa) + b(u^{\mathrm{l}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{l}}_\kappa).$$

The last two terms vanish, so it suffices to estimate only the first two. We use the standard estimate found, for example, in [2]:

$$|b(u,v,w)| \leq \|u\|_{s_1}\|v\|_{s_2+1}\|w\|_{s_3} \quad (1.15)$$

where $s_1 + s_2 + s_3 \geq 3/2$. To estimate the first term let us set $s_1 = s_2 = s_3 = 1/2$, then

$$|b(u^{\mathrm{h}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa)| \leq \|u^{\mathrm{h}}_\kappa\|^2_{1/2}\|u^{\mathrm{l}}_\kappa\|_{3/2},$$

and by (1.11) we have

$$\|u^{\mathrm{h}}_\kappa\|_{1/2} \leq \kappa^{-1/3}\|u^{\mathrm{h}}_\kappa\|_{5/6} \quad (1.16)$$

$$\|u^{\mathrm{l}}_\kappa\|_{3/2} \leq \kappa^{2/3}\|u^{\mathrm{l}}_\kappa\|_{5/6}. \quad (1.17)$$

So,

$$|b(u^{\mathrm{h}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa)| \leq \|u^{\mathrm{h}}_\kappa\|^2_{5/6}\|u^{\mathrm{l}}_\kappa\|_{5/6},$$

which tends to zero a.e. in t as $\kappa \to \infty$. Since in addition,

$$|b(u^{\mathrm{h}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa)| \leq \|u\|^3_{5/6}$$

for all t, by the Dominated Convergence Theorem,

$$|b(u^{\mathrm{h}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa)| \to 0, \qquad \text{as} \qquad \kappa \to \infty,$$

in $L^1([0,T))$. As to the second term, similar estimates with $s_1 = 5/6$, $s_2 = 0$, $s_3 = 2/3$, yield

$$|b(u^{\mathrm{l}}_\kappa, u^{\mathrm{l}}_\kappa, u^{\mathrm{h}}_\kappa)| \leq \|u^{\mathrm{l}}_\kappa\|^2_{5/6}\|u^{\mathrm{h}}_\kappa\|_{5/6},$$

which also tends to zero in $L^1([0,T))$ as $\kappa \to \infty$. This finishes the proof of (1.14) and the theorem.

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