Isolated Blocks in Finite Classical Groups, II

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To the memory of Walter Feit

1. INTRODUCTION

Let $\mathbb{G}$ be a connected, reductive algebraic group defined over $\mathbb{F}_q$ and G the finite reductive group of $\mathbb{F}_q$-rational points of $\mathbb{G}$. Let ℓ be a prime not dividing q. Each ℓ-block of G determines a conjugacy class (s) in a dual group G^* of G, where $s \in G^*$ is an ℓ'-semi simple element. The block is said to be isolated if $C_{\mathbb{G}^*}(s)$ has the same semisimple rank as $\mathbb{G}^*$. If a block is not isolated, the characters in the block can be obtained by Lusztig induction from a Levi subgroup of G which is a dual of $C_{G^*}(s)$. Thus it is important to classify the isolated blocks of G.

If $\mathbb{G}$ is a classical group with a connected center, a description of all the ℓ-blocks of G (ℓ odd, q odd) was given in [6] in combinatorial terms using the language of symbols. On the other hand, Cabanes and Enguehard [4] have given descriptions in terms of Lusztig induction of the blocks of arbitrary finite reductive groups, with some restrictions on ℓ. In an earlier paper of the author [10] the isolated blocks (ℓ odd, q odd) of $Sp(2n, q)$ or $SO^{\pm}(2n, q)$ were described via Lusztig induction. A combinatorial description of the blocks was, however, not given in [4] or [10] for these groups since such a description of Lusztig induction was not available. However, a combinatorial description of Lusztig induction in the cases of $Sp(2n, q)$ and $O^{\pm}(2n, q)$ was recently given by Waldspurger [11], provided q in large, during his proof of the Lusztig conjecture for these groups. Using this description we give a parameterization of the isolated blocks of $Sp(2n, q)$ and $O^{\pm}(2n, q)$ (ℓ odd, q odd) by means of pairs of symbols which are e-cores, where e is the order of q^2 mod ℓ. We also give a description of the characters in a block which lie in the Lusztig rational series $\mathcal{E}(G, (s))$, where (s) is the ℓ'-semisimple class corresponding to the block, again by pairs of symbols. Thus this paper represents a completion of the project started in [6].

We note that the case of $SO(2n+1,q)$ was treated in [6]. Finally we remark that a combinatorial description of blocks of this kind fits in with a philosophy of Broué, by which representation-theoretic data on a finite reductive group should be determined by data independent of q.

Notation: If G is a finite group, $Irr(G)$ is the set of (complex) irreducible characters of G, and $\mathcal{C}(G)$ is the space of complex-valued class functions on G with the usual inner product $\langle\ ,\ \rangle$.

2. ISOLATED SEMISIMPLE CLASSES

Let $\mathbb{G}$ denote the algebraic group $Sp(2n)$ or $SO(2n)$, and let $G = Sp(2n,q)$ or $SO^\eta(2n,q)$ q odd, $\eta = \pm 1$. We will usually omit q and write $G = Sp(2n)$ or $SO^\eta(2n)$. The set $Irr(G)$ is partitioned into geometric series by Deligne-Lusztig theory, and further into rational series $\mathcal{E}(G,(s))$ where $s \in G^*$ is an ℓ'-semisimple element (see [4, 8.23]). Then the set $\mathcal{E}_\ell(G,(s))$ which is the union of the rational series $\mathcal{E}(G,(t))$ such that $t_{\ell'} = s$ is a union of ℓ-blocks of G (see [4, 9.12]). We note that in our cases if $G = Sp(2n)$ (resp. $SO^\eta(2n)$) then $G^* = SO(2n+1)$ (resp. $SO^\eta(2n)$). Each ℓ-block of G (block, for short) then determines an ℓ'-semisimple class $(s) \subset G^*$. If $C_{\mathbb{G}^*}(s)$ has the same semisimple rank as G^* we say (s) is an isolated class and the corresponding block is an isolated block.

In these cases, suppose G^* acts on a vector space V^*. Then an isolated class consists of elements whose eigenvalues are ± 1; in other words, an element s of such a class satisfies

$s^2 = 1$.

We now consider the centralizers of isolated semisimple elements (see [6, p. 126], [11, p. 57]). Let H_0 be the group of $\mathbb{F}_q$-rational points of $C^0_{\mathbb{G}^*}(s)$, and $H = C_{G^*}(s)$. We have the following cases:

Case 1. $G = Sp(2n)$. Then $H_0 \cong SO(2m+1) \times SO^\eta(2k)$ for some m, k with $m + k = n$ and $H \cong (O(2m+1) \times O^\eta(2k)) \cap SO(2n+1)$.

Case 2. $G = O^\eta(2n)$. We note that $s \in SO^\eta(2n)$. Then $H_0 \cong SO^\epsilon(2m) \times SO^{\eta\epsilon}(2k)$ for some m, k with $m + k = n$, $\epsilon = \pm 1$, $H \cong O^\epsilon(2m) \times O^{\eta\epsilon}(2k)$.

Case 3. $G = SO^\eta(2n)$. Then H_0 is as in Case 2 and $H = \widetilde{H} \cap G$, where $\widetilde{H} \cong O^\epsilon(2m) \times O^{\eta\epsilon}(2k)$.

3. SQUARE-UNIPOTENT CHARACTERS

By a theorem of Lusztig [8], if $G = Sp(2n)$ or $SO^\eta(2n)$ the characters in $\mathcal{E}(G, (s))$ are in bijection with $\mathcal{E}(H, (1))$. If $G = O^\eta(2n)$ we define the characters in $\mathcal{E}(G, (s))$ to be those which cover the characters in $\mathcal{E}(SO^\eta(2n), (s))$. Following Waldspurger, we call the characters in $\mathcal{E}(G, (s))$ where $G = Sp(2n)$, $SO^\eta(2n)$ or $O^\eta(2n)$, where $s^2 = 1$, square-unipotent characters (quadratiques-unipotent in [11]).

We give the classification of square-unipotent characters as in [11]. We first consider the case $s = 1$, i.e. the unipotent characters. By the work of Lusztig [7] the unipotent characters of classical groups are parameterized by equivalence classes of symbols. We refer to [3, p. 375, pp. 466-476; 2, p. 48] for a description of the symbols associated with unipotent characters of $Sp(2n)$ and $SO^{\eta}(2n)$, including definitions of the equivalence relations on symbols and the rank and defect of a symbol.

We denote a symbol by $\Lambda = (S, T)$ where $S, T \subseteq \mathbb{N} \cup \{0\}$. If Λ is unordered, it is regarded as the same as (T, S) and also the same as the symbol obtained by a shift operation from itself (see [3], p. 375). The defect of Λ is $|S| - |T|$. The unipotent characters of $Sp(2n)$, $SO^{+}(2n)$, $SO^{-}(2n)$ are in bijection with unordered symbols of rank n and odd defect, defect $\equiv 0 (\text{mod } 4)$ and defect $\equiv 2 (\text{mod } 4)$ respectively, except that in the case of $SO^{+}(2n)$ there are two characters corresponding to each $\Lambda = (S, S)$. We say $\Lambda = (S, T)$ is non-degenerate if $S \neq T$ and degenerate if $S = T$.

The unipotent characters of $O^{\eta}(2n)$ are then parameterized by (equivalence classes of) ordered symbols. In particular, a degenerate symbol corresponds to just one character. This parameterization was given by Asai [1, p. 552] for $O^{+}(2n)$.

Following [11] we define a map σ on ordered symbols by $\sigma(S, T) = (T, S)$. We now describe the parameterization of square-unipotent characters of $Sp(2n)$ and $O^{\eta}(2n)$ given in [11, 4.4, 4.11]. Let $\widetilde{S}_{n,d}$ be the set of ordered symbols of rank n and defect d, and let $S_{n,d} = \widetilde{S}_{n,d} \cup \widetilde{S}_{n,-d}$, module the relation $\Lambda \sim \sigma(\Lambda)$.

Let

$$
\begin{aligned}
S_{n,odd} &= \bigcup_{\substack{d \in \mathbb{N} \\ d \text{ odd}}} S_{n,d}, \widetilde{S}_{n,even} = \bigcup_{\substack{d \in \mathbb{Z} \\ d \text{ even}}} \widetilde{S}_{n,d}, \\
\widetilde{S}\widetilde{S}_{n,\text{ even}} &= \bigcup_{n_1+n_2=n} \widetilde{S}_{n_1,\text{ even}} \times \widetilde{S}_{n_2,\text{ even}}\ , \\
S\widetilde{S}_{n,\text{ mix}} &= \bigcup_{n_1+n_2=n} S_{n_1,\text{ odd}} \times \widetilde{S}_{n,\text{ even}}.
\end{aligned}
$$

We then have:

(3.1). There is a bijection π of $\widetilde{S}\widetilde{S}_{n,\text{ even}}$ onto the set of square-unipotent characters of $O^+(2n) \cup O^-(2n)$. If $(\Lambda_1, \Lambda_2) \in \widetilde{S}\widetilde{S}_{n,\text{ even}}$, $\pi(\Lambda_1, \Lambda_2) \in \mathcal{E}(O^\eta(2n), (s))$ where $\eta = (-1)^{(d_1-d_2)/2}$, d_1, d_2 being the defects of Λ_1, Λ_2. Then $s \in SO^\eta(2n)$ has 2 rank (Λ_1) (resp. 2rank (Λ_2)) eigenvalues equal to 1 (resp. -1).

(3.2). There is a bijection π of $S\widetilde{S}_{n,mix}$ onto the set of square-unipotent characters of $Sp(2n)$. If $(\Lambda_1, \Lambda_2) \in S\widetilde{S}_{n,mix}$, $\pi(\Lambda_1, \Lambda_2) \in \mathcal{E}(Sp(2n), (s))$ where s has 2 rank (Λ_1) (resp. 2 rank (Λ_2)) eigenvalues equal to 1 (resp. -1).

These parameterizations are obtained by Harish-Chandra induction from cuspidal square-unipotent characters of suitable Levi subgroups, and the labelling of the cuspidal character is not unique.

Remark. We have taken the liberty of replacing "pair" by "even" and "imp" by "odd" in [11].

(3.3). We now describe the restriction of $\pi(\Lambda_1, \Lambda_2) \in \mathcal{E}(O^\eta(2n), (s))$ to $SO^\eta(2n)$ [11, 9.8]. The restriction is irreducible unless both Λ_1, Λ_2 are degenerate, in which case it is the sum of two irreducible characters. The characters in the restriction are in bijection with the unipotent characters of H lying above the character of H_0 parameterized by (Λ_1, Λ_2).

4. LUSZTIG INDUCTION

For any finite reductive group G and Levi subgroup L we have the Lusztig twisted induction map (see e.g. 4, p. 125) $R_L^G : \mathbb{Z}Irr(L) \to \mathbb{Z}Irr(G)$. The adjoint map is denoted by ${}^*R_L^G$: $\mathbb{Z}Irr(G) \to \mathbb{Z}Irr(L)$. The definition of the R_L^G map involves the algebraic group $\mathbb{G}$ and a parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$ with a Levi subgroup $\mathbb{L}$ whose subgroup of $\mathbb{F}_q$-rational points is L. It is not known in general if the definition of R_L^G is independent of the choice of $\mathbb{P}$, except for large q. If we have Levi subgroups L, M of G arising from Levi subgroups $\mathbb{L}, \mathbb{M}$ of $\mathbb{G}$ with $\mathbb{L} \leq \mathbb{M}$, with suitable choices of parabolic subgroups containing $\mathbb{L}$ and $\mathbb{M}$ we have transitivity $R_M^G \cdot R_L^M = R_L^G$ (see [11], p. 10). A more general definition of the R_L^G map applicable to disconnected groups such as $O(2n)$ can be found in ([11], p. 9) or ([5], p. 364).

The R_L^G map was known explicitly for classical groups in the following cases:

(i) unipotent characters [1]

(ii) all characters, classical groups with a connected center [9].

These results were used in [6] to give a combinatorial description of the blocks.

We now describe the R_L^G map for square-unipotent characters of $Sp(2n)$ and $O^\eta(2n)$, due to Waldspurger.

Let $a \neq 0$ be an integer. Asai [1] has introduced operations I_a, I_a^-, J_a on symbols. The operations I_a, J_a can be regarded as "adding an a-hook" and "adding an a-cohook" respectively to a symbol Λ. They can be described as follows (see [6, p. 159]). Let $\Lambda = (S, T)$. We say a symbol Λ' is obtained from Λ by adding an a-hook if it is obtained by deleting a member x of S (or T) and inserting $x + a$ in S (or T). We say Λ' is obtained from Λ by adding an a-cohook if it is obtained from Λ by deleting a member x of s (or T) and inserting $x + a$ in T (or S).

Then $I_a(\Lambda)$, $J_a(\Lambda)$ are $\mathbb{Z}$-linear combinations of symbols obtained from Λ by adding an a-hook, a-cohook, respectively. These are given in [2, p. 49, p. 52] and [11, 2.3].

If Λ' is obtained from Λ by adding an a-hook or a-cohook, we say Λ is obtained from Λ' by deleting an a-hook or a-cohook. If it is not possible to delete an a-hook (resp. a-cohook) from Λ we say Λ is an a-core (resp. a-cocore). From now on we will use the notation a-core for either an a-core or an a-cocore, and a-hook for either an a-hook or a-cohook, depending on the context, unless it is necessary to specify one of the two. We have a Fourier transform $\mathcal{F}$ on symbols introduced by Lusztig (see [3, p. 384]).

(4.1). We then have $\mathcal{F}J_a\mathcal{F} = I_a, \mathcal{F}I_a^-\mathcal{F} = \sigma J_a$ where, as before $\sigma(S,T) = (T,S)$ [W, p. 18].

We now take $G = O^\eta(2n)$ or $Sp(2n)$, $L = T_1 \times T_2 \times L_0$ where $L_0 \cong O^{\eta'}(2n)$ or $Sp(2m)$, and T_1 and T_2 are both tori of orders $q^a - 1$ or $q^a + 1$. We remark that if we can describe the R_L^G map in this case, we can then describe the map where T_1 and T_2 are replaced by products of tori, by transitivity.

Consider a square-unipotent character λ of L as follows. Let $\lambda = 1 \times \zeta \times \psi$, where 1 is the trivial character of T_1, ζ is the non-trivial character of order 2 of T_2 and $\psi = \pi(\mu_1, \mu_2)$ is a square-unipotent character of L_0.

(4.2). Theorem [11]. *Let $G = Sp(2n)$ or $O^\eta(2n)$. Let $q > 2n$. With the above notation, the constituents of $R_L^G(\lambda)$ are the $\pi(\Lambda_1, \Lambda_2)$ where Λ_1, Λ_2 are obtained from μ_1, μ_2 respectively by adding an a-hook ($q^a - 1$ case) or an a-cohook ($q^a + 1$ case).*

The theorem follows from the commutative diagram for character sheaves in [11, 8.2 and 13.8]. Since Waldspurger proves the Lusztig conjecture for G, the map k_n in the diagram can be replaced by $\pi_n \cdot \mathcal{F}$ (here π_n is a linear map on the space spanned by symbols, induced by π; see [11, 5.1]). We then use (4.1).

Thus we have an analogue of Asai's result for unipotent characters in this case, but for large q. This condition is assumed in order to use Lusztig induction for character sheaves.

Remark. Waldspurger proves, after proving Lusztig's conjecture, that there is a uniform labelling of the square-unipotent characters of the Levi subgroups of $O^\eta(2n)$ such that Theorem 4.2 holds. This is not apparent at first in the case of $O^\eta(2n)$, or indeed even in the case of $Sp(2n)$ since the centralizers of isolated elements can be disconnected groups.

5. ISOLATED BLOCKS OF $Sp(2n)$, $SO(2n)$

We now discuss the isolated ℓ-blocks of $Sp(2n)$ and $SO^\eta(2n)$, $\eta = \pm 1$, where as before q and ℓ are odd. Let $G = Sp(2n)$ or $SO^\eta(2n)$, B an isolated block of G associated with a conjugacy class $(s) \subset G^*$ such that $s^2 = 1$. This then implies that $B \cap \mathcal{E}(G,(s)) \neq \emptyset$ [4, Th. 9.12]. We note that $B \cap \mathcal{E}(G,(s))$ is the set of square-unipotent characters in B.

Certain subgroups L, Q of G were considered in [10] in this context. These subgroups were already introduced in [6] and we describe them now. Let R be the defect group of B. Subgroups C, C', Q were defined in [6, p. 178] as follows: $C = C_G(R)$, C' is the centralizer in G of the "base group" of R and $Q = C_G(z)$ where z is a suitable element of order ℓ in the center of R. We have factorizations

$$Q = Q_+ \times Q_0, C = C_+ \times C_0, C' = C'_+ \times C_0 \ \text{ with}$$

$$C_+ \leq C'_+ \leq Q_+, Q_0 = C_0, \text{ and thus } \ C \leq C' \leq Q.$$

The subgroup L defined in [10] can be identified with C'. We set $L_+ = C'_+, L_0 = C_0$, so that we have a factorization $L = L_+ \times L_0$. Let e be the order of q^2 mod ℓ. Then L_+ is a product of k tori of order $q^e - 1$ or of order $q^e + 1$, i.e. $L_+ \cong GL(1, q^e)^k$ or $U(1, q^e)^k$, and L_0 is of the same type as G, i.e. $L_0 \cong Sp(2m)$ or $SO^{\eta'}(2m)$ for some m. Here $\eta = \eta'$ or $\eta = (-1)^k \eta'$ according as we are in the linear or unitary case in L_+. We also have $Q_+ \cong GL(k, q^e)$ or $U(k, q^e)$.

We have a character λ of L which factorizes as $\lambda = \lambda_+ \lambda_0$, where λ_+ is a linear character of order dividing 2 of L_+ and λ_0 is a character of ℓ-defect 0 of L_0. We note that $\lambda \in \mathcal{E}(L, (s))$.

The following was proved in [10].

(5.1). Theorem. *The square-unipotent characters in B are the constituents of $R_L^G(\lambda)$. The pair (L, λ) is determined by B up to G-conjugacy.*

6. ISOLATED BLOCKS OF $O(2n)$

We now consider the blocks of $O^\eta(2n)$ covering the block B of $SO^\eta(2n)$. Let $G = O^\eta(2n)$, $G_0 = SO^\eta(2n)$, and B as before. We embed L in a subgroup M of G such that $M = M_+ M_0$, $M_+ = L_+$, $L_0 \leq M_0$ and $M_0 \cong O^{\eta'}(2m)$, so that $(M_0 : L_0) = (M : L) = 2$.

We recall that $\lambda = \lambda_+ \lambda_0$ where λ_0 is a character of L_0. Then λ_0 is covered by one or two characters of M_0. Let $\widetilde{\lambda}_0$ be one such character covering λ_0, and set $\widetilde{\lambda} = \lambda_+ \widetilde{\lambda}_0$, a character of M covering λ. Since λ_+ is a linear character of a product of tori, we can parameterize $\widetilde{\lambda}$ and $\widetilde{\lambda}_0$ by the same pair of symbols.

By the parameterization given in (3.1) we can write $\widetilde{\lambda} = \widetilde{\lambda}_0 = \pi(\kappa_1, \kappa_2)$ where κ_1, κ_2 are symbols. The next lemma describes these symbols in terms of e.

Lemma. *κ_1, κ_2 are e-cores, i.e. no e-hooks can be removed from them.*

Proof: It was proved in [10] that λ is e-cuspidal, i.e. that ${}^*R_K^L(\lambda) = 0$ for any e-split Levi subgroup K of G with $K \leq L$. In our case this means, in particular, that for any subgroup $K \leq L$ of the form $K = K_+K_0$ where K_+ is a product of a torus of order $q^e - 1$ or $q^e + 1$ and K_0 is a special orthogonal group, we have ${}^*R_K^L = 0$.

We now embed K in a subgroup N of M with $N = N_+ \times N_0$, $N_+ = K_+$, $K_0 \leq N_0$, N_0 being an orthogonal group with $(N_0 : K_0) = 2$.

We have the following formula which connects the maps ${}^*R_N^M$ and ${}^*R_K^L$ [5, Corollaire 2.4 (iii)]:

$$ {}^*R_K^L Res_L^M = Res_K^N {}^*R_N^M. \tag{6.1} $$

We apply this to $\widetilde{\lambda}$. Then $Res_L^M(\widetilde{\lambda})$ is either equal to λ or the sum of λ and another e-cuspidal character, and so the left hand side is 0. We consider the right hand side.

Now ${}^*R_N^M(\widetilde{\lambda}) = \sum\limits_i a_i\widetilde{\mu}_i$, where the $\widetilde{\mu}_i$ are square-unipotent characters of N. Each $\widetilde{\mu}_i$ covers one or two characters of K. We wish to show $Res_K^N(\sum\limits_i a_i\widetilde{\mu}_i) = 0$ implies each $a_i = 0$. If $\widetilde{\mu}_i$ covers 2 characters of K, then no other $\widetilde{\mu}_j$ has the same restrictions to K and then $a_i = 0$. Suppose we have a pair $\widetilde{\mu}_i$, $\widetilde{\mu}_j$ which restrict to the same character μ of K. We analyze this possibility, using the description of the R_N^M map and the description of restrictions from N to K (see (3.3)). Here $\widetilde{\mu}_i$, $\widetilde{\mu}_j$ are parameterized by pairs of symbols, at least one of which is non-degenerate. Suppose $\widetilde{\mu}_i = \pi(\Lambda_1, \Lambda_2)$, $\widetilde{\mu}_j = \pi(\Lambda_1', \Lambda_2')$ where Λ_1, Λ_1' are non-degenerate, and where $\Lambda_1 = (S, T)$, $\Lambda_1' = (T, S)$ with $S \neq T$. We have $\widetilde{\lambda} = \pi(\kappa_1, \kappa_2)$, then κ_1 is obtained either from Λ_1 or Λ_1' by adding an e-hook. We show this is not possible, if Λ_1 and Λ_1' are of the above form, and similarly for κ_2.

Case 1. Suppose an e-hook is added to (S, T) and (T, S) to get the same symbol $\kappa_1 = \kappa$, for simplicity. Let $S = \{x_1, x_2, \dots, x_r\}$, $T = \{y_1, y_2, \dots, y_t\}$. Then we obtain from (S,T) either (X_1, Y_1) or (X_2, Y_2), where

$$X_1 = \{x_1, x_2, \dots, \widehat{x}_i, \dots x_r, x_i + e\}, Y_1 = T$$

$$X_2 = S, Y_2 = \{y_1, y_2, \dots, \widehat{y}_j, \dots, y_t, y_t + e\}.$$

From (T,S) we obtain either (X_3, Y_3) or (X_4, X_4), where $X_3 = \{y_1, y_2, \dots, \widehat{y}_e, \dots, y_t, y_t + e\}$, $Y_3 = S$ $X_4 = T$, $Y_4 = \{x_1, x_2, \dots, \widehat{x}_\kappa, \dots, x_r, x_k + a\}$. Suppose $(X_1, Y_1) = (X_3, Y_3)$ or (X_4, Y_4). We consider this case, the case of (X_2, Y_2) being similar. If $(X_1, Y_1) = (X_3, Y_3)$ then $S = T$, which is not the case. If $(X_1, Y_1) = (X_4, Y_4)$ then $r = t$ and $X_1 = Y_4$. This gives $\{x_k, x_i + e\} = \{x_i, x_k + e\}$, which is not possible unless $k = i$. If $k = i$, $X_1 = Y_4$, $Y_1 = X_4$, so κ is degenerate. If κ is degenerate and (S,T) and (T,S) are obtained from it by deleting an e-hook, they occur with the same sign (see [2, (3.5)]). Then no cancellation occurs from $\widetilde{\mu}_i$ and $\widetilde{\mu}_j$ when restricted to K.

Case 2. Suppose an a-cohook is added to (S,T) and (T,S) to get the symbol κ. In this case, from (S,T) we obtain (X_1, Y_1) or (X_2, X_2) where

$$\begin{aligned} X_1 &= \{x_1, x_2, \dots \widehat{x}_i, \dots, x_r\},\ Y_1 = \{y_1, y_2, \dots, y_t, x_i + e\}, \\ X_2 &= \{x_1, x_2, \dots, x_r, y_j + e\},\ Y_2 = \{y_1, y_2, \dots, \widehat{y}_j, \dots, y_t\}. \end{aligned}$$

From (T,S) we obtain (X_3, Y_3) or (X_4, Y_4) where

$$\begin{aligned} X_3 &= \{y_1, y_2, \dots, \widehat{y}_\ell, \dots, y_t\},\ Y_3 = \{x_1, x_2, \dots, x_r, y_\ell + e\} \\ X_4 &= \{y_1, y_2, \dots, y_t, x_k + e\},\ Y_4 = \{x_1, x_2, \dots, \widehat{x}_k, \dots, x_r\}. \end{aligned}$$

Suppose $(X_1, Y_1) = (Y_3, Y_3)$ or (X_4, Y_4). Again, we consider only the case of (X_1, Y_1) the case of (X_2, Y_2) being similar.

If $(X_1, Y_1) = (X_3, Y_3)$ then $r = t$, and $\{x_i, y_\ell + e\} = \{y_\ell, x_i + e\}$. So $x_i = y_\ell$ and $S = T$. If $(X_1, Y_1) = (X_4, Y_4)$, $r - 1 = t + 1$ or $t = r - 2$.

Suppose $i \neq k$. Since $x_k \neq x_k + e$, from $X_1 = X_4$ we get $x_k \in T$. Similarly from $Y_1 = Y_4$ we get $x_i \in T$. But then again from $X_1 = X_4$ and $Y_1 = Y_4$ we get $r - 1$ elements from S to be in T. Since $|T| = r - 2$ we get a contradiction.

Suppose $i = k$. Then $X_1 = Y_4$ and $Y_1 = Y_4$, so we obtain a degenerate symbol. Then we argue as in Case 1.

Finally we note that by the transitivity of Lusztig induction, it is sufficient to choose K as above. Then ${}^*R_N^M(\widetilde{\lambda}) = 0$ for any e-split Levi subgroup K of L and $K \leq N$ as above. This shows that if $\widetilde{\lambda} = \pi(\kappa_1, \kappa_2)$ then κ_1, κ_2 are e-cores and proves the lemma. $\square$

We return to the blocks of $G = O^\eta(2n)$ covering the block B of $G_0 = SO^\eta(2n)$ and consider two cases.

Case 1. Suppose λ_0 extends to two characters $\widetilde{\lambda}_0, \widetilde{\lambda}_0'$ of M_0, and so λ extends to $\widetilde{\lambda}$ and $\widetilde{\lambda}'$ in $\mathcal{E}(M, (s))$. If $\widetilde{\lambda} = \pi(\kappa_1, \kappa_2)$, $\widetilde{\lambda}' = \pi(\kappa_1', \kappa_2')$, at least one of κ_1, κ_2 and one of κ_1', κ_2' is non-degenerate. We have $C_G(R) = C_+ \times M_0$, where R is the defect group of B. Then there are two blocks of $C_G(R)$ covering a block of $C_{G_0}(R) = C_+ \times L_0$ which induces to B, and each of these two blocks is fixed under $N_G(R)$. Hence there are two blocks $\widetilde{B}, \widetilde{B}'$ of G covering B, by Brauer's First Main Theorem. By (5.1) $B \cap \mathcal{E}(G_0, (s))$ consists of the constituents of

$R_L^{G_0}(\lambda)$. We use the adjoint of the formula (6.1), applied to G, G_0, L and M:

$$Ind_{G_0}^G R_L^{G_0} = R_M^G Ind_L^M, \tag{6.2}$$

and apply both sides to λ. By (4.2), on the right hand side we get characters in $\mathcal{E}(G,(s))$ of the form $\pi(\Lambda_1, \Lambda_2)$ where Λ_i has core κ_i or κ_i' $(i = 1, 2)$. Since degenerate symbols have degenerate cores, at least one of Λ_1, Λ_2 is non-degenerate. On the left hand side we get characters in the blocks $\widetilde{B}, \widetilde{B}'$. Thus we have

$$(\widetilde{B} \cup \widetilde{B}') \cap \mathcal{E}(G,(s)) = Irr(R_M^G(\widetilde{\lambda}) \cup R_M^G(\widetilde{\lambda}')).$$

In order to prove that the constituents of $R_M^G(\widetilde{\lambda})$ and $R_M^G(\widetilde{\lambda}')$ lie in different blocks, we recall the Brauer map d_G^z where $z \in G_\ell$ (see [4, 5.7]). We choose $z \in R$ as a suitable ℓ-element with $Q = C_{G_0}(z)$ as before. Then $d_{G_0}^z : \mathcal{C}(G_0) \to \mathcal{C}(Q)_{\ell'}$ is defined by $(d_G^z f)(y) = f(zy)$, $y \in Q_{\ell'}$.

We need a generalization of the commuting of the Brauer map and Lusztig induction (see [4, 21.4]) to the disconnected group case. Let $P = C_G(z)$, so that $Q \leq P$, $P = Q_+P_0$, $Q = Q_+Q_0$ with $Q_0 = L_0, P_0 = M_0$ and $Q_+ \cong GL(k, q^e)$ or $U(k, q^e)$. We have $M \leq P$, and we define $d_G^z : \mathcal{C}(G) \to \mathcal{C}(P)_{\ell'}$ as above.

Then

$$^*R_M^P d_G^z = d_M^z \, ^*R_M^G \tag{6.3}$$

The proof is similar to that of the connected group case, where the character formula for the $^*R_M^G$ map (or its analog in that case) is used (see [4, loc. cit]). The more general character formula has been proved in [5, 2.6].

Let $\chi \in \widetilde{B} \cup \widetilde{B}'$ be such that $\langle \chi, R_M^G(\widetilde{\lambda})\rangle \neq 0$. If $\chi = \pi(\Lambda_1, \Lambda_2)$ then Λ_1, Λ_2 are obtained from κ_1, κ_2 respectively by adding e-hooks and κ_1, κ_2 are the e-cores of Λ_1, Λ_2. We have $P = Q_+P_0$, $M = L_+M_0$, and L_+ is a product of tori. Then ${}^*R_M^G(\chi)$ is a linear combination of characters of the form $\mu \times \widetilde{\lambda}_0$ where μ is a character of L_+ and $\mu \times \widetilde{\lambda}_0$ is $N_G(M)$-conjugate to $\widetilde{\lambda}$. Thus ${}^*R_M^G(\chi)$ is a linear combination of characters in blocks of M which induce to a fixed block of G, which we can take to be $\widetilde{B}$. The same then holds for functions in $d_M^z\, {}^*R_M^G(\chi)$, i.e. the right hand side of (6.3).

We now consider the left hand side of (6.3). By Brauer's Second Main Theorem $d_G^z(\chi)$ consists of functions in $\mathcal{C}(P)$ which are in blocks that induce to the block $B(\chi)$ of G containing χ. If ψ is such a function, ${}^*R_M^P(\psi)$ is a linear combination of functions which are in blocks inducing to $B(\chi)$, since $P_0 = M_0$ and L_+ plays the same role for Q_+ as L does for G_0. This shows that $B(\chi) = \widetilde{B}$. Thus we have shown the constituents of $R_M^G(\widetilde{\lambda})$ all lie in a block $\widetilde{B}$. Similarly the constituents of $R_M^G(\widetilde{\lambda}')$ lie in $\widetilde{B}'$.

Case 2. Suppose λ_0 is covered by a single character $\widetilde{\lambda}_0$ of M_0, so that $Ind_{L_0}^{M_0}(\lambda_0) = \widetilde{\lambda}_0$. Then in fact we have two characters λ_0, λ_0' of L_0 which induce to $\widetilde{\lambda}_0$, and then we have two characters $\lambda = \lambda_+\lambda_0, \lambda' = \lambda_+\lambda_0'$ of L which are e-cuspidal. However, λ and λ' are conjugate by an element $x \in N_{G_0}(L)$ which acts on L_0 like an element of $M_0 \backslash L_0$. As in Case 1, we consider $C_G(R) = C_+M_0$, $C_{G_0}(R) = C_+L_0$. In this case there is one block of $C_G(R)$ covering two blocks of $C_{G_0}(R)$, and thus there is one block $\widetilde{B}$ of G covering B. We now apply (6.2) and argue as in Case 1. It follows that the characters in $\widetilde{B} \cap \mathcal{E}(G,(s))$ are the constituents of $R_M^G(\widetilde{\lambda}_0)$. If $\widetilde{\lambda}_0 = \pi(\kappa_1, \kappa_2)$, these constituents are of the form $\pi(\Lambda_1, \Lambda_2)$ where Λ_1, Λ_2 have

e-cores κ_1, κ_2 respectively. We also remark that in this case κ_1 and κ_2 are both degenerate.

7. THE MAIN THEOREM

The square-unipotent characters of $G = Sp(2n)$ in a block B were described in Theorem 5.1. They are the constituents of $R_L^G(\lambda)$ where λ is e-cuspidal. Thus if $\lambda = \pi(\kappa_1, \kappa_2)$ then κ_1, κ_2 are e-cores, as follows from the description of the R_L^G map in (4.2).

We now state our main theorem.

Theorem. *Let $G = O^\eta(2n)$, $\eta = \pm 1$ or $Sp(2n)$, where q is odd and $q > 2n$. Let B be an isolated ℓ-block of G, where ℓ is odd. Let e be the order of q^2 mod ℓ. There is a unique pair $(\kappa_1, \kappa_2) \in \widetilde{S}\widetilde{S}_{n,even}$ ($O(2n)$-case) or $S\widetilde{S}_{n,mix}$ ($Sp(2n)$ case) such that the square-unipotent characters in B are of the form $\pi(\Lambda_1, \Lambda_2)$, where κ_i is the e-core of Λ_i $(i = 1, 2)$.*

The proof of the theorem follows from the results in §6 for the $O(2n)$ case, and from the above remarks for the $Sp(2n)$ case. The theorem describes, for $q > 2n$, the square-unipotent characters in an isolated ℓ-block in combinatorial terms, and is a natural extension of the results in [6] for the connected center case.

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