Preface

This is part catalog and part encyclopedia. I have listed most of the papers written about the Hrushovski construction and some background material. I have also attempted to categorize the different type of constructions and highlight some of the main ideas. Please be tolerant of inconsistent notation. I drew this together from talks over close to 20 years and have made only modest efforts at unification.

Outline

Contents

1 Introduction: weak ranks and strong submodels **1**

2 Ab Initio Constructions **6**
- 2.1 Ab Initio: Irrational Coefficients 7
- 2.2 Ab Initio: rational coefficients 11

3 Expansions & Fusions **14**

4 Infinitary Case **16**

1 Introduction: weak ranks and strong submodels

Prehistory

1. Fraïssé limits– countable homogenous universal relational structures
2. Jónsson: no restriction on cardinality
3. Algebras later– need countably many structures; locally finite

Grzegorczyk's question

How many $\aleph_0$-categorical theories are there? [A68]

Answer: $2^{\aleph_0}$

Ehrenfeucht[Ehr72], Glassmire [Gla71], and Henson [Hen72]

Henson's proof was $2^{\aleph_0}$ applications of the Fraïssé construction:

Let $L = \{E, P_n\}_{n<\omega}$ with E binary and P_n n-ary. Consider graphs. Let $A \prec_{\boldsymbol{K}} B$ in $\boldsymbol{K}_X$ if (for exactly those n in X) P_n picks out a maximal complete n-graph in A which remains maximal in B. The E-reduct of the generic is model complete.

Limitations

These examples obviously have the independence property.

Much later, Hrushovski [Hru89] showed there are only countably many *ω-stable* $\aleph_0$-categorical structures.

Extension Axioms

If $A \subseteq B$, every instance of A extends to an instance of B.

Dense Linear Order

$$(\forall v_0)(\exists z)\, v_0 < z$$
$$(\forall v_0)(\exists z)\, z < v_0$$
$$(\forall v_0, v_1)(\exists z)\, v_0 < z < v_1$$

The random graph

Axioms ϕ_k :

$$(\forall v_0 \dots v_{k-1} w_0 \dots w_{k-1})(\exists z)\ \wedge_{i<k} (Rzv_i \bigwedge \neg Rzw_i)$$

Language restrictions

What is the role of
finite?
relational?

Four Questions

1. Lachlan: Is there a strictly stable $\aleph_0$-categorical theory?
2. Zilber: Is there a strongly minimal set that is neither discrete, nor vector space-like nor field-like?
3. Cherlin: Do any two strongly minimal sets have a common expansion?
4. Cherlin-Nesin: Is there a bad field?

Two Directions: 'false' dichotomy

Ab Initio

A 'nice' countable model is constructed from a class of finite models.

Expansions/Fusions

A 'nice' countable model is constructed by expanding or fusing models of strongly minimal theories.

We expand [Bal02]. Other surveys [BS96, Poi02, Wag94].

Kueker and Laskowski [KL92] allow the basic class to be closed under chains rather than submodels.

The starting models

Let T_{-1} be a theory such that any subset X of a model N of T_{-1} is contained in a minimal submodel of N.

$\langle X\rangle_N$ denotes the submodel generated by X.

Two examples:

1. T_{-1} is universally axiomatized
2. T_{-1} is strongly minimal

Notation

$\overline{\boldsymbol{K}}_{-1} = \mathrm{mod}(T_{-1})$;
$\boldsymbol{K}_{-1}$ is the *finitely generated* members of $\overline{\boldsymbol{K}}_{-1}$.

Examples

T_{-1} is a universal theory in a finite relational language; $\boldsymbol{K}_{-1}$ is the finite models of T_{-1};

T_{-1} is a universal theory in a countable relational language with only countably many non-isomorphic finite models.; $\boldsymbol{K}_{-1}$ is the finite models of T_{-1}.

T_{-1} is Acf_p; $\boldsymbol{K}_{-1}$ contains those algebraically closed fields of finite transcendence degree;

More generally, T_{-1} is a strongly minimal, inductive theory with elimination of quantifiers and imaginaries and the definable multiplicity property; $\boldsymbol{K}_{-1}$ contains the models generated by finitely many independent elements.

Semimodularity[1]

Let $\langle K(N), \wedge, \vee\rangle$ be a lattice of substructures of a model N.

Let δ be a function from $K(N)$ into $\mathbb{N}$

We write $\delta(A/B) = \delta(A \vee B) - \delta(B)$.

δ is *lower* semimodular (or submodular) if:

$$\delta(A \vee B) - \delta(B) \leq \delta(A) - \delta(A \wedge B).$$

δ is *upper* semimodular if:

$$\delta(A \vee B) - \delta(B) \geq \delta(A) - \delta(A \wedge B).$$

We say δ is modular if both hold.

Lower semimodularity can be rewritten as, δ is monotonic: if $B \subseteq A, C \subseteq N$ and $A \wedge C = B$,

$$\delta(A/B) \geq \delta(A/C).$$

[1]These note reflect corrections to silly justifications of true statements in [Bal02] pointed out Alice Medvedev.

Examples

Examples of δ include:

1. modular
   (a) cardinality,
   (b) vector space dimension
2. lower semi-modular
   (a) transcendence degree (in ω-stable theories).
3. upper semi-modular
   (a) relation size

The simplest example of 'relation size' is just the number of edges in a (symmetric) graph.

Weak ranks

A *weak rank* is a lower semimodular function δ from $K(N)$ into a discrete subgroup of the reals ($\mathcal{R}$), which is defined on each N in a class $\boldsymbol{K}$.

A *positive* linear combination of lower semimodular functions is a weak rank.

Subtracting an upper semimodular lower function from a lower semimodular functions yields a weak rank.

With this observation, most of the examples of this construction can be seen as built up from the examples given earlier.

Strong Submodels

Definition For $N \models T_{-1}$, $K(N)$ is the substructures of N which are in $\boldsymbol{K}_{-1}$. For $A, B \in \overline{\boldsymbol{K}}_{-1}$, we say A is a *strong substructure* of B and write $A \prec_{\boldsymbol{K}} B$ if:

for every $B' \in \boldsymbol{K}_{-1}$ with $B' \subseteq B$, $\delta(B'/B' \cap A) \geq 0$.

Definition We denote by $\overline{\boldsymbol{K}}_0$ the members of $\overline{\boldsymbol{K}}_{-1}$ which have hereditarily positive rank and by $\boldsymbol{K}_0$ those which are finitely generated and have hereditarily positive rank. T_0 denotes the theory of $\overline{\boldsymbol{K}}_0$,

Properties of Strong Submodel

Theorem 1. *The notion of strong substructure has the following properties.*

- **A1**. *If $M \in \overline{\boldsymbol{K}}_{-1}$ then $M \prec_{\boldsymbol{K}} M$.*
- **A2**. *If $M \prec_{\boldsymbol{K}} N$ then $M \subseteq N$.*
- **A3**. *If $A, B, C \in \overline{\boldsymbol{K}}_{-1}$, $A \prec_{\boldsymbol{K}} B$, and $B \prec_{\boldsymbol{K}} C$ then $A \prec_{\boldsymbol{K}} C$.*

- **A4**. *If* $A, B, C \in \overline{\boldsymbol{K}}_{-1}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ *and* $A \subseteq B$ *then* $A \prec_{\boldsymbol{K}} B$.

Since $\prec_{\boldsymbol{K}}$ is imposed by δ,

- **A5**. *If* $A, B, C \in K(N)$, $A \prec_{\boldsymbol{K}} C$, $B \subseteq C$, *then* $A \cap B \prec_{\boldsymbol{K}} B$.

  We restrict to $\overline{\boldsymbol{K}}_0$ precisely to obtain:

- **A6**. $\emptyset \in \overline{\boldsymbol{K}}_0$ *and* $\emptyset \prec_{\boldsymbol{K}} A$ *for all* $A \in \overline{\boldsymbol{K}}_0$.

These yield Abstract Elementary Classes

If we close such a class under unions of $\prec_{\boldsymbol{K}}$-chains we get an abstract elementary class.

AMALGAMATION PROPERTY

The class $\boldsymbol{K}$ satisfies the *amalgamation property* if for any situation with $A, M, N \in \boldsymbol{K}$:

$$\begin{tikzcd}
 & N \\
A \arrow[ur] \arrow[dr] & \\
 & M
\end{tikzcd}$$

there exists an N_1 such that

$$\begin{tikzcd}
 & N \arrow[dr] & \\
A \arrow[ur] \arrow[dr] & & N_1 \\
 & M \arrow[ur] &
\end{tikzcd}$$

Generic vrs rich

Rich

Definition. The model M is finitely $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-*homogeneous* or *rich* if $A \prec_{\boldsymbol{K}} M, A \prec_{\boldsymbol{K}} B \in \boldsymbol{K}_0$ implies there exists $B' \prec_{\boldsymbol{K}} M$ such that $B \cong_A B'$.

Could also be called $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-*saturated*; same as model homogeneous in the aec.

Generic

The model M is *generic* if M is rich and M is an increasing union of finite closed substructures.

The usage is confused in the literature.

Finite Closures

Definition.

The class $(\overline{\boldsymbol{K}}_0, \leq)$ of relational structures has *finite closures* if for every $A \in \overline{\boldsymbol{K}}_0$ and every finite $A_0 \subset A$ there is a finite $A_1 \in \overline{\boldsymbol{K}}_{-1}$ with $A_0 \subseteq A_1 \prec_{\boldsymbol{K}} A$

Locally closed is the analogous (more general) notion when there are function symbols in the language.

This is true if the generic is ω-saturated. [Wag94] posits 'saturated generic' as a fundamental axiom but it fails for the stable random graph.

The situation becomes more complicated if functions are allowed.

Uniqueness

There is at most one generic model.

If $\boldsymbol{K}$ is locally closed all rich models are $L_{\infty,\omega}$ equivalent. So the generic is the unique countable rich model.

Existence

Theorem. If a class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the amalgamation property and the joint embedding property then there is a $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-homogeneous structure M.

There is a countable $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ generic model M if there are only countably many pairs $N_0 \prec_{\boldsymbol{K}} N_1$ of countable models of $\boldsymbol{K}_0$.

(E.g. if every member of $\boldsymbol{K}_0$ is finite.)

Compare the construction in [Vau61].

2 Ab Initio Constructions

Ab Initio

$$\delta = \alpha\delta_1 - \beta\delta_2$$

δ_1 is cardinality of a finite structure.

δ_2 is 'relation size'.
If there are a finite number of relations symbols
$\delta_2(B) = \Sigma\alpha_i|R_i|$.

Parameters for ab initio classes

1. The language may be finite or countable.
2. The α_i may be rational, irrational, or mixed.
3. The class $\boldsymbol{K}$ may be *proper* in the class of models with non-negative rank.

Setting some $\alpha_i = 0$ encompasses the 'expansion' case.

Intrinsic Closure
Definition

1. Let $A \subseteq M \in \boldsymbol{K}$. The *intrinsic* (or self-sufficient) closure of A in M, denoted $\mathrm{icl_M(A)}$ is the unique minimal N such that: $A \subseteq N$, $N \in \boldsymbol{K}$, $N \prec_{\boldsymbol{K}} M$.

2. We say B is a *minimal intrinsic extension* of A if $\delta(B/A) < 0$ but $\delta(B'/A) \geq 0$ for every B' with $B \supset B' \supseteq A$.

The intrinsic closure can be built up iteratively from minimal intrinsic extensions.

Key issue: Is $\mathrm{icl_M(A)}$ finite if A is finite? uniformly?

2.1 Ab Initio: Irrational Coefficients

Ab Initio: $\alpha = 1$, β irrational. I
$\delta(A) = |A| - \beta R(A)$.

1. $\boldsymbol{K}_0^1$: Hrushovski [Hru88] constructed a strictly stable $\aleph_0$-categorical theory.

   This refuted Lachlan's conjecture that a stable $\aleph_0$-categorical theory is ω-stable.

Ab Initio: $\alpha = 1$, β irrational. II

2. $\boldsymbol{K}_0^2$. Baldwin and Shi [BS96] modified the second Hrushovski construction to construct a stable theory T_β.

3. The exact connections with forking in this class and its CM-triviality are proved in [VY03].

4. [BS97] show this is the almost sure theory of random graphs with edge probability $n^{-\beta}$ (originally [SS88]).

5. Baldwin [Bal03] (see also Shelah [She00]) has generalized this argument to show a 0-1-law for expansions of successor by graphs with edge probability $n^{-\beta}$.

6. For extensions to other edge probabilities see [Bal97].

Subclasses and Algebraicity
Subclasses of $\overline{\boldsymbol{K}}_0$ are studied for two reasons:

1. To guarantee specific properties

2. To enforce algebraicity

Role of Subclass

The distinction between the Hrushovski and the Baldwin-Shi examples is that Hrushovski restricts to a subclass to bound the growth of $\mathrm{icl}_{\mathrm{M}}(\mathrm{A})$ and guarantee $\aleph_0$-categoricity.

Almost sure theories

Fix a finite relational language L. Let $\boldsymbol{K}_n$ be a collection of L-structures with universe n. Let P_n be a probability measure on $\boldsymbol{K}_n$.

For any formula ϕ, let

$$P_n(\phi) = \sum \{P_n(B) : B \models \phi, |B| = n\}.$$

E.g. $\boldsymbol{K}_n$ is all graphs of size n; P_n is the uniform distribution (edge probability 1/2).

T is an *almost sure theory* if for some $(\boldsymbol{K}_n, P_n)$, $\phi \in T$ iff $\lim_{n\to\infty} P_n(\phi) = 1$.

0-1 law for finite graphs (Glebski et al, Fagin [GKLT69, Fag76]):

The (theory of the)random graph is almost sure with respect to the uniform distribution as each extension axiom has limit probability 1.

Random Graphs

Let B be a graph with $|B| = n$. Let

$$P_n(B) = n^{-\alpha|e(B)|} \cdot (1 - n^{-\alpha})^{\binom{n}{2} - e(B)}.$$

Let α be irrational $0 < \alpha < 1$.

Theorem. [Spencer-Shelah] For each first order sentence ϕ, $\lim_{n\to\infty} P_n(\phi)$ is 0 or 1.

Theorem.[Baldwin-Shelah] The almost sure theory is stable and nearly model complete. (It does not have the finite cover property.)

Quantifier Reduction

Definition. T is *model complete* if every formula is equivalent in T to an existential formula.

Definition. T is *nearly model complete* if every formula is equivalent in T to a *Boolean Combination* of existential formulas.

Random graph: $n^{-\beta}$

Baldwin, Shi, Spencer, Shelah gave a π_3 axiomatization of the random graph with edge probability $n^{-\beta}$. This meant that a 'second moment' argument was necessary to prove the axioms almost surely true.

T_β is nearly model complete.

T_β is not model complete. [BS97]

Ab Initio: $\alpha = 1$, β irrational.

Laskowski's improvements I

Return to the original extension axioms:
If $A \prec_{\boldsymbol{K}} B$, every instance of A extends to an instance of B.

Building on ideas of Ikeda [Ike05], Laskowski axiomatizes T_β with these extension axioms.

Ab Initio: $\alpha = 1$, β irrational.

Laskowski's improvements II

1. T_β is π_2-axiomatizable;
2. this means verification of the 0-1 law is easy.
3. T_β is nearly model complete in a very specific way.
4. better proofs that this theory has the dimensional order property but not the finite cover property (originally [BS98]).

Laskowski III Existential closure

Locally finite means the 'model theoretic algebraic closure of a finite set is finite'.

For T_β

1. There is no strong embedding of any nonempty finite structure into an existentially closed model. (No e.c. model is locally finite.)
2. There are locally finite models that are not generic.
3. The generic model is locally finite.

Ab Initio: $\alpha = 1$, many irrational β

Herwig [Her95] varied the construction by allowing an infinite language to find a stable theory with infinite p-weight. This paper also contains the best published exposition of Hrushovski's $\aleph_0$-categorical stable theory. See also [Wag94].

Simple Theories

To construct (Hrushovski) strictly simple theories, make the inequality in the definition of strong substructure strict.

For $A, B \in \overline{\boldsymbol{K}}_{-1}$, we say A is a $*$-*strong substructure* of B and write $A \prec^*_{\boldsymbol{K}} B$ if for every $B' \in \boldsymbol{K}_{-1}$ with $B' \subseteq B$, $\delta(B'/B' \cap A) > 0$.

There is an $\aleph_0$-categorical strictly simple theory where forking is not locally modular. [Hru88],[Hru] [Pou00] [PW06].

An annoying open problem

Conjecture: If an *ab initio* generic structure is superstable then it is ω-stable.

Suppose α is irrational. (I.e. the α_i are $\mathbb{Q}$-linearly independent.
If

1. $\boldsymbol{K}$ contains all acyclic finite graphs (Ikeda [Ike05]) or
2. $\boldsymbol{K}$ is all finite graphs with non-negative rank (Laskowski [Las07])

the generic is strictly stable.

Ikeda's proof was the spark for Laskowski's work.

Closure under Quasisubstructure

Work of Anbo and Ikeda [AI].

Definition

$\boldsymbol{K}$ is closed under quasisubstructure if $A \in \boldsymbol{K}$ and $B \subset A$ and for every relation symbol R, $R(B) \subset R(A) \cap B$ then $B \in \boldsymbol{K}$.

Theorem (Anbo-Ikeda

If $\boldsymbol{K}$ is an ab initio class such that the generic is saturated and $\boldsymbol{K}$ is closed under quasi-substructures then if the theory of the generic is superstable, it is ω-stable.

Some more open problems

Baldwin[Bal03] and Shelah (independently) extended the 0-1 law for n^α to random expansions of successor.

Integrate Laskowski's idea to:

1. give a simple proof of the 0-1 law over successor.
2. Prove the 0-1 law over vector spaces.
3. What happens in proper subclasses $\boldsymbol{K}$ of $\boldsymbol{K}_\beta$?

2.2 Ab Initio: rational coefficients

Dimension Functions

A weak rank δ is a *predimension* if δ maps into the integers.

Definition.

1. For $M \in \overline{\boldsymbol{K}}_0$, $A \subseteq M$, $A \in \boldsymbol{K}_0$, $d_M(A) = \inf\{\delta(B) : A \subset B \subseteq M, B \in \boldsymbol{K}_0\}$.
2. For A, b contained M, $b \in \mathrm{cl}(A)$ if $d_M(bA) = d_M(A)$.

Extend to infinite sets by imposing finite character.

Dimension Function Properties

Lemma.

1. cl is monotone and idempotent.
2. If, in addition δ is a predimension:
   (a) if for any finite X, $d_M(X) \leq |X|$ then the closure system satisfies exchange.
   (b) For finite A, icl(A) is finite.

Saturation of the Generic

We discuss several strengthenings from [BS96] of the notion of amalgamation which imply the generic saturated.

Uniform Amalgamation I

We say A is n-strong in B, written $A \leq_n B$, if for any B' with $A \subseteq B' \subseteq B$ and $|B' - A| \leq n$, $A \leq B'$.

$(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ has the *uniform amalgamation property* (u.a.p.) if the following condition holds for every $A \leq B \in \boldsymbol{K}_0$. For every $m \in \omega$ there is an $n = f_B(m)$ such that if $A \leq_n C$ then there is a D, a strong embedding of C into D and an m-strong embedding of B into D that complete a commutative diagram with the given embeddings of A into B and C.

Uniform Amalgamation and ω-saturation

As pointed out by Herwig, Poizat, Wagner if $\boldsymbol{K}$ has finite closures then $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ has uap iff M is ω-saturated.

Is there a finitely closed class that has a.p. but not u.a.p.?

Kueker and Laskowski [KL92] prove that if the generic structure M is weakly saturated then M is saturated.

Sharp Amalgamation and ω-saturation

Definition

$(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ has the *sharp amalgamation property* if for every A, B, C in $\boldsymbol{K}_0$ with $A \leq B$ and $A \leq_{|B|-|A|} C$, if B is a primitive extension of A, then either $B \otimes_A C \in \boldsymbol{K}_0$ or there is a strong embedding of B into C over A.

Note that any one-point extension must be primitive. It is now straightforward to prove by induction that

Proposition

If $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ has the sharp amalgamation property then $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ has the uniform amalgamation property with $f_B(m) = m + |B - A|$.

Note that classes defined by a successful Hrushovski construction (with μ function) have u.a.p.

Open Question

Prove or Disprove
In the *ab initio* case with finitely many rational coefficients for any subclass $\boldsymbol{K}$ of $\overline{\boldsymbol{K}}_0$,

the generic is always saturated.

For expansions, this fails in general [BH00].

Counting Extensions

Suppose $A, B \in \boldsymbol{K}_0$. For any $M \in \boldsymbol{K}_0$ and let $\chi_M(A/B)$ denote the number of copies of A over B in M. Note:

- $\delta(A/B) < 0$ implies $\chi_M(A/B)$ is finite.
- $\delta(A/B) > 0$ implies $\chi_M(A/B)$ is infinite.
- $\delta(A/B) = 0$ implies $\chi_M(A/B)$ is undetermined.

If α is irrational the third case cannot occur.
If α is rational we control case iii).

Primitives

Definition Let $A, B \in \boldsymbol{K}_0$. We say A is *primitive* over B if $\delta(A/B) = 0$ and for any A' with $B \subset A' \subset A$, $\delta(A/A') < 0$.

In the *ab initio* case one needs to also minimize the base B; in the bicolored field case this falls out from the general theory of canonical bases.

The following description (accurate in the *ab initio* case) oversimplifies the statement in e.g., the bicolored field case, but expresses the spirit of the argument.

$\boldsymbol{K}^\mu$

To guarantee $\aleph_1$-categoricity of the generic, one studies the subclass $\boldsymbol{K}^\mu$ of those $M \in \boldsymbol{K}_0$ where for each primitive A/B,

$$\chi_M(A/B) \leq \mu(A/B)$$

for a given function μ from primitive pairs into $\mathbb{N}$.
If the generic model for $\boldsymbol{K}^\mu$ is ω-saturated, categoricity follows easily.

If μ is *not* finite-to-one, T may not be ω-stable [BH00].
So finite-to-one is assumed below.

Ab Initio: $\alpha = \beta = 1$.

$\delta_1(B)$ is the cardinality of a finite relational structure B and
$\delta_2(B)$ is the number of tuples which satisfy a fixed list of symmetric relations on B.

$\delta_1(B) - \delta_2(B)$ is the dimension function for the first application of the method: Hrushovski's new strongly minimal set

Ab Initio: $\alpha = \beta = 1$.

1. The class $\boldsymbol{K}_0^\mu$ depends on a function μ into $\mathbb{N}$ with a finite-to-one μ yields [Hru93] a strongly minimal set.

2. If the μ-function is relaxed to allow even one infinite value, the rank is infinite [BI94]. There are continuum many different theories of this sort depending on the choice of μ.

Ab Initio: $\alpha = \beta = 1$.

3. Working with the class of all structures $\boldsymbol{K}_0$ with hereditarily non-negative rank yields a theory of rank ω [Goo89]. There are countably many classes which satisfy a certain 'δ-invariance' condition; they are classified in [Are95, ABM99].

4. It is straightforward that Hrushovski's example does not admit elimination of imaginaries but Verbovskiy [Ver06] provides a variant which does.

5. There are minimal but not strongly minimal structures with arbitrary finite dimension [Ike01]

Ab Initio: α **an integer** $\beta = 1$.

1. Baldwin [Bal94] varied the method to construct almost strongly minimal projective planes which have no infinite definable groups of automorphisms. In [Bal95] he showed these planes had the least possible structure in the sense of the Lenz-Barlotti classification.

2. $\alpha = n - 1$, $\beta = n - 2$. Debonis and Nesin (for odd n) [MJDB98] and Tent [Ten00] (uniformly for all n) constructed almost strongly minimal generalized n-gons. The automorphism groups of Tent's structures were highly transitive even though they were not Moufang. Thus she showed that the analog of the Feit-Higman theorem [FH64] did not hold for finite Morley rank n-gons.

3 Expansions & Fusions

Fusions:

δ_1, δ_2 are Morley rank on two finite rank structures which share the same universe. Let,

$$\delta(\mathbf{x}) = \alpha\delta_1(\mathbf{x}) + \beta\delta_2(\mathbf{x}) - \lg(\mathbf{x}).$$

$\alpha = \beta = 1$

1. Hrushovski [Hru92] showed any two reasonable sm sets have a common expansion.

2. Holland [Hol97, Hol95] clarifies this construction and in [Hol99] proves that these theories (as well as the Hrushovski strongly minimal set) are model complete.

Groups

1. δ_1, δ_2 are the vector space dimension of a vector space E and an associated subspace of $\bigwedge^2 E$. $\delta = \delta_1 - \delta_2$. Baudisch [Bau95] constructs a nilpotent $\aleph_1$-categorical group which does not interpret a field.

2. In [Bau00], Baudisch analyzes some obstructions to extending Hrushovski's construction of a strictly stable structure to find a strictly stable $\aleph_0$-categorical group.

Fields

1. Poizat [Poi99] constructs an ω-stable field of rank $\omega \times 2$ with a proper definable subset (additive subgroup, multiplicative subgroup) [Poi01]

2. Baldwin-Holland [BH00][BH01] construct a rank 2 field with a proper definable subset.

3. Baldwin-Holland construct a rank k [BH03] field with a proper definable subset.

Model Completeness

Lindstrom's little theorem
If a π_2 theory is categorical in some infinite power then it is model complete.

Baldwin-Holland [BH04]:

1. show Poizat's infinite rank bicolored field is *not* model complete;

2. provide a sufficient condition for the $\aleph_1$-categorical expansions of strongly minimal sets to be model complete;

3. show an expansion by constants of Baldwin's projective plane is model complete.

The Second/third Generation
Given a q.e. strongly minimal theory with the definable multiplicity property.

(Note automatically π_2-axiomatizable.)

Expansion II

There is a finite rank expansion of an algebraically closed field with

1. a proper definable additive subgroup [BMPZ07b] (Baudisch, Martin-Pizzaro and Ziegler)

2. a proper definable *multiplicative* subgroup [BHMPW07] (Baudisch, Hils, Martin-Pizzaro and Wagner). BAD FIELD

Fusions: II

T is *good* if it has finite Morley rank with definable rank and degree. Ziegler [Zie08]

1. Any two good T_1 and T_2 have a common conservative expansion with rank a common multiple of their ranks. This implies:

2. the existence of a bicolored field.

3. every good theory can be interpreted in a strongly minimal set. [Has07].

Ziegler makes two 'technical' assumptions; without them it isn't known if T^μ is even complete.

The Additive Collapse

Bausdish [Bau] provides a unified treatment of:

1. basic fusion [BMPZ07a] [HH06]

2. fusions over vector spaces [BMPZ06]

3. finite rank expansions of an acf with a predicate for an additive subgroup [BMPZ07b]

4. construction of the Baudisch group. [Bau95]

4 Infinitary Case

Zilber Constructions

openprobSkip to Open problems

1. Quasiminimal Excellent Classes [Zil05, Bal, Kir07]

2. Covers of Abelian varieties [BZ00, Zil06, Zil03]

3. Pseudoexponentiation [Zil04]

QUASIMINIMAL EXCELLENCE

A class $(\boldsymbol{K}, \mathrm{cl})$ is *quasiminimal excellent* if cl is a combinatorial geometry which satisfies on each $M \in \boldsymbol{K}$:

1. there is a unique type of a basis;
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models;
3. (ccp) the closure of a finite set is countable;
4. and 'excellence': unique amalgamation of n independent countable models for all n.

Consequences

Let $A \prec_{\boldsymbol{K}} B$ if A is closed in B.
Note $\prec_{\boldsymbol{K}}$ 'is' the $\leq^*$ for $L_{\omega_1,\omega}(Q)$ in [Bal].

If $(\boldsymbol{K}, \mathrm{cl})$ satisfies 1) and 2) then $\boldsymbol{K}$ is $\aleph_1$-categorical.

Any QME class closed under unions of chains (and with an infinite dimensional model) is [Kir07] :

1. Categorical in all uncountable powers
2. axiomatizable in $L_{\omega_1,\omega}(Q)$

Context for Quasiminimal excellence

QME codifies some consequences for combinatorial geometry of the Hrushovski construction. It then adds others (homogeneity over models and excellence) which are immediate consequences of the construction. Excellence is expounded in a larger context in [Bal] and [She83a, She83b].

In particular, there is *no* use of a dimension function in the next example (covers). But there is in the second infinitary example.

Covers of Algebraic Groups

Definition A cover of a commutative algebraic group $A(\mathcal{C})$ is a short exact sequence

$$0 \to Z^N \to V \stackrel{\exp}{\to} A(\mathcal{C}) \to 1. \tag{1}$$

where V is a $\mathbb{Q}$ vector space and A is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

Axiomatizing Covers: first order

Let A be a commutative algebraic group over an algebraically closed field F.

Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q\in\mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $A(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of A).
3. exp is a group homomorphism from $(V, +)$ to $(A(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1,\omega}$

Add to T_A

$\Lambda = \mathbb{Z}^N$ asserting the kernel of exp is standard.

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m}\in\mathbb{Z}^N} \Sigma_{i<N} m_i x_i = y]$$

Finitary AEC

For any A:

$T_A + \Lambda = \mathbb{Z}^N$

1. has arbitrarily large models
2. has the amalgamation property

Algebraic Input

$A = (\mathcal{C}, \cdot)$**,** $A = (\overline{F}_p, \cdot)$

Number theoretic argument shows homogeneity over models and excellence [Zil06, BZ00]. So for this choice of A the class of covers is categorical in all powers.

other A

Open problems; serious algebra and model theory. [Gav06, Gav08]

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski like dimension function. Prove some $L_{\omega_1,\omega}$-sentence Σ is categorical and has quantifier elimination.

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

THE AXIOMS

$L = \{+, \cdot, E, 0, 1\}$

$(K, +, \cdot, E) \models \Sigma$ if

K is an algebraically closed field of characteristic 0.

E is a pseudo-exponential

$L_{\omega_1,\omega}$-axioms

E is a homomorphism from $(K, +)$ onto $(K^x, \cdot)$ and there is $\nu \in K$ transcendental over $\mathbb{Q}$ with $\ker E = \nu Z$.

K is strongly exponentially algebraically closed.

$L(Q)$-axioms

ccp: The closure of a finite set is countable.

PSEUDO-EXPONENTIAL

E is a pseudo-exponential if for any n linearly independent elements over $\mathbb{Q}$, $\{z_1, \ldots z_n\}$

$$d_f(z_1, \ldots z_n, E(z_1), \ldots E(z_n)) \geq n.$$

Schanuel conjectured that true exponentiation satisfies this equation.

CONSISTENCY AND CATEGORICITY

For a finite subset X of an algebraically closed field k with a partial exponential function. Let

$$\delta(X) = d_f(X \cup E(X)) - ld(X).$$

Apply the Hrushovski construction to the collection of such k with $\delta(X) \geq 0$ for all finite X.

The δ yields a combinatorial geometry. Further algebraic arguments yield that the class is quasiminimal excellent achieving Objective A.

Open Questions

1. Is there a strictly stable $\aleph_0$-categorical group?
2. Is dmp needed for the fusion construction?
3. When does the Hrushovski construction yield a first order theory? When is it model complete?
4. Is there an $\aleph_0$-homogeneous (over models) quasiminimal class which is *not* excellent?
5. Is there a 'Hrushovski construction' that is not $\aleph_0$-homogeneous (over models)?

Where to start?

1. $\aleph_0$-categorical strictly stable [Her91]
2. ab initio
   (a) rational/irrational [BS96] supplemented by [VY03].
   (b) rational [Wag94] (His framework doesn't handle the random graph.)
   (c) irrational α, 0-1-laws: [Las07]
3. fusions and expansions [BMPZ07a]? 2nd generation in any case.
4. Infinitary [Bal, Kir07]

Bibliography

References

[A68] Grzegorczyk A. Logical uniformity by decomposition and categoricity in $\aleph_0$. *Bul. Acad. Sci. Polon. Sci. Ser. Sci. Math. Astron. Phys.*, 16:687–92, 1968.

[ABM99] Roman Aref'ev, J.T. Baldwin, and M. Mazzucco. δ-invariant amalgamation classes. *The Journal of Symbolic Logic*, 64:1743–1750, 1999.

[AI] Y. Anbo and K. Ikeda. A note on the stability spectrum of generic structures. preprint.

[Are95] R.D. Aref'ev. $K_{(H,|A|-e(A),\leq)}$-homogeneous-universal graphs. In B.S. Baizhanov M.B. Aidarkhanov, editor, *Proceedings of Informatics and Control Problems Institute, Almaty*, pages 27–40. 1995. In Russian.

[Bal] John T. Baldwin. Categoricity. www.math.uic.edu/˜ jbaldwin.

[Bal94] J.T. Baldwin. An almost strongly minimal non-Desarguesian projective plane. *Transactions of the American Mathematical Society*, 342:695–711, 1994.

[Bal95] J.T. Baldwin. Some projective planes of Lenz Barlotti class I. *Proceedings of the A.M.S.*, 123:251–256, 1995.

[Bal97] J. T. Baldwin. Near model completeness and 0-1-laws. In R. Boppana and James Lynch, editors, *Proceedings of the DIMACS Workshop on Logic and Random Structures: 1995*. American Mathematical Society, 1997.

[Bal02] John T. Baldwin. Rank and homogeneous structures. In Katrin Tent, editor, *Tits Buildings and the Theory of Groups Wurzburg Sept 14-17 2000*. Cambridge University Press, 2002.

[Bal03] J.T. Baldwin. Expansions of geometries. *Journal of Symbolic Logic*, 68:803–827, 2003.

[Bau] A. Baudisch. The additive collapse. on Baudisch's webpage.

[Bau95] A. Baudisch. A new uncountably categorical group. *Transactions of the American Mathematical Society*, 348:889–940, 1995.

[Bau00] A. Baudisch. Closures in $\aleph_0$-categorical bilinear maps. *J. Symbolic Logic*, pages 914–922, 2000.

[BH00] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Rank 2 fields. *The Journal of Symbolic Logic*, 65:371–391, 2000.

[BH01] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Computing rank. *Fundamenta Mathematica*, 170, 2001.

[BH03] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Rank k. *Notre Dame Journal of Formal logic*, 44:139–147, 2003.

[BH04] J.T. Baldwin and K. Holland. Constructing ω-stable structures: Model completeness. *Annals of Pure and Applied Logic*, 125:159–172, 2004.

[BHMPW07] A. Baudisch, M. Hils, A. Martin-Pizarro, and F.O. Wagner. Die b ose Farbe. 2007.

[BI94] J.T. Baldwin and M. Itai. K-generic projective planes have Morley rank two or infinity. *Mathematical Logic Quarterly*, 40:143–152, 1994.

[BMPZ06] A. Baudisch, A. Martin-Pizarro, and M. Ziegler. Fusion over a vector space. *J. Math. Logic*, pages 141–162, 2006.

[BMPZ07a] A. Baudisch, A. Martin-Pizarro, and M. Ziegler. Hrushovski's fusion. In Torsten Schatz Frieder Haug, Benedikt Lwe, editor, *Algebra, Logic, Set Theory, Festschrift fr Ulrich Felgner zum 65. Geburtstag;*, Studies in Logic, pages 15–31, London, 2007. College Publications.

[BMPZ07b] A. Baudisch, A. Martin-Pizarro, and M. Ziegler. Red fields. *J. Symbolic Logic*, pages 207–225, 2007.

[BS96] J.T. Baldwin and Niandong Shi. Stable generic structures. *Annals of Pure and Applied Logic*, 79:1–35, 1996.

[BS97] J.T. Baldwin and S. Shelah. Randomness and semigenericity. *Transactions of the American Mathematical Society*, 349:1359–1376, 1997.

[BS98] J.T. Baldwin and S. Shelah. DOP and FCP in generic structures. *Journal of Symbolic Logic*, 63:427–439, 1998.

[BZ00] M. Bays and B.I. Zilber. Covers of multiplicative groups of an algebraically closed field of arbitrary characteristic. preprint: arXive math.AC/0401301, 200?

[Ehr72] A. Ehrenfeucht. There are continuum ω_0-categorical theories. *Bulletin de l'Acadamie Polonaise des sciences math. , astr., et phys.*, XX:425–427, 1972.

[Fag76] Ronald Fagin. Probabilities on finite models. *Journal of Symbolic Logic*, 41:50–58, 1976.

[FH64] W. Feit and G. Higman. The nonexistence of certain generalized polygons. *Journal of Algebra*, 1:114–131, 1964.

[Gav06] Misha Gavrilovich. *Model Theory of universal covering spaces of complex analytic varieties.* PhD thesis, Balliol college Oxford, 2006.

[Gav08] Misha Gavrilovich. A remark on transitivity of galois action on the set of uniquely divisible abelian extensions of $E(\overline{Q})$ by z^2. *Journal of K-theory*, 38:135–152, 2008.

[GKLT69] Y. Glebski, V. Kogan, M.I. Liogon'kii, and V.A. Taimanov. The extent and degree of satisfiability of formulas of the restricted predicate calculus. *Kiberneticka*, 2:17–27, 1969.

[Gla71] W. Glassmire. There are $2^{\aleph_0}$ countably categorical theories. *Bulletin de l'Acadamie Polonaise des sciences math. , astr., et phys.*, XIX:185–190, 1971.

[Goo89] John B. Goode. Hrushovski's Geometries. In Helmut Wolter Bernd Dahn, editor, *Proceedings of 7th Easter Conference on Model Theory*, pages 106–118, 1989.

[Has07] A. Hasson. Interpreting structures of finite morley rank in a strongly minimal set. *Annals of Pure and Applied Logic*, pages 96–114, 2007.

[Hen72] W. Henson. Countable homogeneous relational structures and $\aleph_0$-categorical theories. *The Journal of Symbolic Logic*, 37:494–500, 1972.

[Her91] B. Herwig. ω-saturated generic structures. preprint, 1991.

[Her95] B. Herwig. Weight ω in stable theories with few types. *The Journal of Symbolic Logic*, 60:353–373, 1995.

[HH06] A. Hasson and M. Hils. Fusion over sublanguages. *The Journal of Symbolic Logic*, 71:361–398, 2006.

[Hol95] Kitty Holland. An introduction to the fusion of strongly minimal sets: The geometry of fusions. *Archive for Mathematical Logic*, 6:395–413, 1995.

[Hol97] Kitty Holland. Strongly minimal fusions of vector spaces. *Annals of Pure and Applied Logic*, 83:1–22, 1997.

[Hol99] Kitty Holland. Model completeness of the new strongly minimal sets. *The Journal of Symbolic Logic*, 64:946–962, 1999.

[Hru] E. Hrushovski. Simplicity and the Lascar group. preprint.

[Hru88] E. Hrushovski. A stable $\aleph_0$-categorical pseudoplane. preprint, 1988.

[Hru89] E. Hrushovski. Totally categorical structures. *Transactions of the American Mathematical Society*, 313:131–159, 1989.

[Hru92] E. Hrushovski. Strongly minimal expansions of algebraically closed fields. *Israel Journal of Mathematics*, 79:129–151, 1992.

[Hru93] E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62:147–166, 1993.

[Ike01] K. Ikeda. Minimal but not strongly minimal structures with arbitrary finite dimension. *Journal of Symbolic Logic*, 66:117–126, 2001.

[Ike05] K. Ikeda. A remark on the stability of saturated generic graphs. *Journal Mathematical Society of Japan*, 57:1229–1234, 2005.

[Kir07] Jonathan Kirby. Definability of quasiminimal excellent classes. http://arxiv.org/PS$_cache/arxiv/pdf$/0707/0707.4496$v3.pdf$, 2007.

[KL92] D.W. Kueker and C. Laskowski. On generic structures. *Notre Dame Journal of Formal Logic*, 33:175–183, 1992.

[Las07] C. Laskowski. A simpler axiomatization of the Shelah-Spencer almost sure theory. *Israel Journal of Mathematics*, 161:157–186, 2007.

[MJDB98] A. Nesin M. J. De Bonis. There are $2^{\aleph_0}$ many almost strongly minimal generalized n-gons that do not interpret an infinite group. 1998.

[Poi99] Bruno Poizat. Le carré de l'egalité. *The Journal of Symbolic Logic*, 64:1339–1356, 1999.

[Poi01] Bruno Poizat. L'egalité au cube. *The Journal of Symbolic Logic*, 66:1647–1676, 2001.

[Poi02] Bruno Poizat. Amalgames de hrushovski. In Katrin Tent, editor, *Tits Buildings and the Theory of Groups Wurzburg Sept 14-17 2000*. Cambridge University Press, 2002.

[Pou00] M. Pourmahdian. *Simple Generic Theories*. PhD thesis, Oxford University, 2000.

[PW06] M. Pourmahdian and F. Wagner. A simple positive Robinson theory with $Lstp \neq Stp$. In Ali Enayat, Iraj Kalantari, and Mojtaba Moniri, editors, *Logic in Tehran Proceedings of the workshop and conference on logic, algebra, and arithmetic, held October 18-22, 2003*, Lecture Notes in Logic, 26. Association of Symbolic Logic, 2006.

[She83a] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part A. *Israel Journal of Mathematics*, 46:3:212–240, 1983. paper 87a.

[She83b] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part B. *Israel Journal of Mathematics*, 46;3:241–271, 1983. paper 87b.

[She00] S. Shelah. 0-1 laws. preprint 550, 200?

[SS88] S. Shelah and J. Spencer. Zero-one laws for sparse random graphs. *Journal of A.M.S.*, 1:97–115, 1988.

[Ten00] Katrin Tent. A note on the model theory of generalized polygons. *Journal of Symbolic Logic*, 65:692–702, 2000.

[Vau61] R.L. Vaught. Denumerable models of complete theories. In *Infinitistic Methods, Proc. Symp. Foundations of Math., Warsaw, 1959*, pages 303–321. Państwowe Wydawnictwo Naukowe, Warsaw, 1961.

[Ver06] V. Verbovskiy. On the elimination of imaginaries for the strongly minimal sets of Hrushovski. In M.M.Erimbetov, editor, *Model Theory in Kazakhstan*, pages 41–55. Eco Study, Almaty, 2006.

[VY03] V.V. Verbovskiy and I. Yoneda. Cm-triviality and relational structures. *Annals of Pure and Applied Logic*, 122:175–194, 2003.

[Wag94] F. Wagner. Relational structures and dimensions. In Kaye Richard and et. al., editors, *Automorphisms of First Order Structures*, pages 153–180. Oxford, Clarendon Press, 1994.

[Zie08] M. Ziegler. Fusions of structures of finite morley rank. In Chatzidakis Z, D. MacPherson, A. Pillay, and A. Wilkie, editors, *Model Theory with Applications to Algebra and Analysis*, London Mathematical Society Lecture Notes, pages 225–248. Cambridge University Press, 2008.

[Zil03] B.I. Zilber. Model theory, geometry and arithmetic of universal covers of a semi-abelian variety. In *Model Theory and Applications*, Quaterna di matematica, pages 427–458, 2003.

[Zil04] B.I. Zilber. Pseudo-exponentiation on algebraically closed fields of characteristic 0. *Annals of Pure and Applied Logic*, 132:67–95, 2004.

[Zil05] B.I. Zilber. A categoricity theorem for quasiminimal excellent classes. In *Logic and its Applications*, Contemporary Mathematics, pages 297–306. AMS, 2005.

[Zil06] B.I. Zilber. Covers of the multiplicative group of an algebraically closed field of characteristic 0. *J. London Math. Soc.*, pages 41–58, 2006.