Math 180
Spring 2015
Final Exam
5/7/2015
Time Limit: 2 Hours

Name (Print): ______________________________

This exam contains 15 pages (including this cover page) and 13 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided. No extra paper will be provided.

Circle your instructor.

- Cabrera
- Cohen
- Groves
- Kobotis
- Lowman
- Shulman
- Steenbergen

TA Name:______________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	15	
11	15	
12	15	
13	10	
Total:	150	

1. (15 points) (a) (5 points) Write the formula for the definition of the derivative of $f(x)$ at $x = a$ (i.e. the limit of a difference quotient).

$$f'(a) =$$

(b) (10 points) Using the definition of the derivative from part (a), find $f'(1)$ if $f(x) = 7x^2+3$.

SOLUTION:

(a)

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

(b)

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{7(1+h)^2 + 3 - (7(1)^2 + 3)}{h} \\ &= \lim_{h\to 0} \frac{7 + 14h + 7h^2 + 3 - 10}{h} = \lim_{h\to 0} \frac{h(14+7h)}{h} \\ &= \lim_{h\to 0} (14 + 7h) = 14. \end{aligned}$$

GRADING RUBRIC: (a)

2 points – If the student has the correct limit OR the correct difference quotient BUT NOT BOTH

5 points – If the student has the correct limit AND difference quotient

(b)

4 points – If the student sets up the definition

7 points – If the student multiplies out the numerator and cancels to $14h + 7h^2$

9 points – If the student cancels the h to $14 + 7h$

10 points – If the student evaluates the limit

-2 points – If the student made a minor algebra error

-3 points – If the student improperly used limit notation or used none at all

2. (10 points) Consider a function $f(x)$ such that $f(3) = 5$, $f'(3) = -2$, and $f''(x) > 0$ for all $3 \leq x \leq 4$.

   (a) (7 points) Using a linear approximation for f at $x = 3$, estimate $f(4)$.

   (b) (3 points) Is your estimate in part (a) an overestimate or underestimate to the actual value of $f(4)$? Justify your answer.

SOLUTION:

(a) The linear approximation at $x = 3$ is

$$L(x) - 5 = -2(x - 3) \Longrightarrow L(x) = -2x + 11.$$

Therefore

$$f(4) \approx L(4) = -2(4) + 11 = 3.$$

(b) Since f is concave up on $3 \leq x \leq 4$, the tangent line lies below the curve so the approximation is an underestimate.

GRADING RUBRIC: (a)

0 points – If the student has 3 as an answer with no work; the directions clearly say to show work

5 points – If the student has the linear approximation

7 points – If the student uses the linear approximation to approximate $f(4)$

Note: If the student even has just $5 + (-2) = 3$, the student should receive full credit

-2 points – If the student switched $x - 3$ with $3 - x$

(b)

0 points – If the student has an answer with no justification

3 points – If the student has "underestimate" with proper justification

3. (10 points) Calculate $\lim_{x \to 0^+} x \ln x$

SOLUTION: Evaluating each piece, we find that the limit has the form $0 \cdot (-\infty)$, so we rewrite the function to use L'Hopital's rule.

$$\begin{aligned} \lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{+\infty} \\ &\stackrel{L'H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0 \end{aligned}$$

GRADING RUBRIC:

2 points – If the student recognizes the original limit is of the form $0 \cdot (-\infty)$

5 points – If the student rewrites the function as $\dfrac{\ln(x)}{\frac{1}{x}}$

7 points – If the student found the derivative of $\ln x$ OR $\dfrac{1}{x}$ BUT NOT BOTH

9 points – If the student found the derivative of $\ln x$ AND $\dfrac{1}{x}$

10 points – If the student has the correct limit

-1 point – If the student does not say that s/he is using L'Hopital's rule at the proper point in the solution

4. (10 points) Find $\frac{d}{dt}\left(\sin^{-1}(e^t)\right)$. Do not simplify your answer.

SOLUTION:

$$\frac{d}{dt}\left(\sin^{-1}(e^t)\right) = \frac{1}{\sqrt{1-(e^t)^2}} \cdot e^t$$

GRADING RUBRIC:

If the student made an attempt at using the Chain Rule, then use the following scale:

4 points – If the student has the derivative of $\sin^{-1}(t)$ without e^t substituted in

7 points – If the student substituted e^t into the derivative of $\sin^{-1}(t)$

10 points – If the student has the correct derivative

-2 points – If the student uses $\frac{d}{dt}\left(\sin^{-1}(t)\right) = \frac{1}{\sqrt{1+t^2}}$ or $\frac{d}{dt}\left(\sin^{-1}(t)\right) = \frac{1}{\sqrt{t^2-1}}$

-1 point – If the student uses x's instead of t's in the final answer

5. (10 points) Write the equation of the tangent line to $y = x^3 \ln x$ at $x = 1$.

SOLUTION: A point on the tangent line is $(1, y(1)) = (1, 0)$. The slope of the tangent line is $y'(1)$, and since

$$y' = 3x^2 \ln x + x^3 \cdot \frac{1}{x}$$

we see that $y'(1) = 1$. Therefore the equation of the tangent line is

$$y - 0 = 1(x - 1) \Longrightarrow y = x - 1.$$

GRADING RUBRIC:

2 points – If the student found the point $(1, 0)$ on the tangent line

6 points – If the student found y'

8 points – If the student found y' and the slope $y'(1) = 1$

10 points – Only if the student found both the point and slope and wrote the equation of the tangent line

If the student incorrectly finds y', the student can earn at most 4 points on the problem.

6. (10 points) Calculate $\displaystyle\int \frac{(\ln x)^{99}}{x}\, dx$

SOLUTION: Using the substitution $u = \ln x$ and $du = \dfrac{1}{x}\, dx$ we have

$$\int \frac{(\ln x)^{99}}{x}\, dx = \int u^{99}\, du = \frac{1}{100}u^{100} + C = \frac{1}{100}(\ln x)^{100} + C.$$

GRADING RUBRIC:

2 points – If the student chooses some u and computes du correctly for his/her choice of u, but it is the wrong u-substitution and the rest of the problem is wrong

4 points – If the student chooses $u = \ln x$ and finds $du = \dfrac{1}{x}\, dx$

6 points – If the student transforms the integral to $\displaystyle\int u^{99}\, du$

8 points – If the student computes $\displaystyle\int u^{99}\, du$

10 points – If the student substitutes back in u and puts $+C$

-1 point – If the student writes $du = \dfrac{1}{x}$ and not $du = \dfrac{1}{x}\, dx$

-1 point – If the student forgets $+C$ or if the student had a small error

-2 points – If the student has a big error, like keeping the integral symbol in the final answer

Note: A student was awarded 0 points if s/he had no work or incorrect work even with the correct answer

7. (10 points) Calculate $\int_0^2 \left(9x^2 + 3x + 8e^{2x}\right)\, dx$

SOLUTION: An antiderivative of $9x^2 + 3x + 8e^{2x}$ is $F(x) = 3x^3 + \frac{3}{2}x^2 + 4e^{2x}$ so

$$\int_0^2 \left(9x^2 + 3x + 8e^{2x}\right)\, dx = F(2) - F(0) = 30 + 4e^4 - 4 = 26 + 4e^4.$$

GRADING RUBRIC:

2 points – If the student finds an antiderivative of $9x^2$

2 points – If the student finds an antiderivative of $3x$

3 points – If the student finds an antiderivative of $8e^{2x}$

3 points – If the student uses the FTC

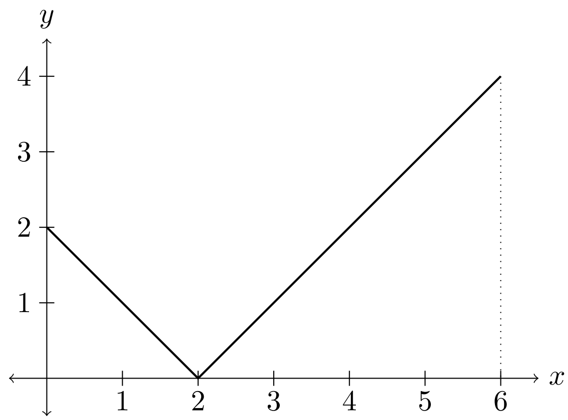
-1 point – If the student commits a small algebra error in evaluation

-2 points – If the student uses the integral symbol after calculating the antiderivative

Note: If the student used a substitution (correctly) on $8e^{2x}$ but did not complete the problem or did not complete the problem correctly, a maximum of 4 points could be earned on the problem

8. (10 points) Calculate $\int_0^6 |x-2|\,dx$

SOLUTION: The graph of $y = |x-2|$ on the interval $0 \le x \le 6$ looks like this:



Using geometry, we see that

$$\int_0^6 |x-2|\,dx = \frac{1}{2}(2)(2) + \frac{1}{2}(4)(4) = 10.$$

GRADING RUBRIC:

2 points – If the student drew ANY graph on $0 \le x \le 6$ (right or wrong)

5 points – If the student drew the correct graph of $|x-2|$ on $0 \le x \le 6$

10 points – If the student used geometry to evaluate the integral (showing his/her work)

3 points – If the student performed a u-substitution and changed the bounds, but went wrong after this part

-1 point – If the student had $+C$ in the final answer.

Note: If the student ignored the absolute values, a maximum of 2 points could be earned on this problem

9. (10 points) Calculate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{4+5\sin x}}\,dx$

Solution: Using the substitution $u = 4 + 5\sin x$, then $du = 5\cos x\ dx$ so $\frac{1}{5}\ du = \cos x\ dx$. The new limits of integration are $u(0) = 4$ to $u(\pi/2) = 9$, so the integral in terms of u is

$$\frac{1}{5}\int_4^9 \frac{1}{\sqrt{u}}\ du = \frac{1}{5}\int_4^9 u^{-1/2}\ du.$$

Using the power rule

$$\frac{1}{5}\int_4^9 u^{-1/2}\ du = \frac{2}{5}u^{1/2}\Big|_4^9 = \frac{2}{5}\sqrt{9} - \frac{2}{5}\sqrt{4} = \frac{2}{5}.$$

Grading Rubric:

If the student chooses to change the limits of integration, the following grading scale will be used:

3 points – If the student chooses $u = 4 + 5\sin x$ and $du = 5\cos x\ dx$

5 points – If the student rewrites the integrand as $\frac{1}{5\sqrt{u}}$ or finds the new limits of integration BUT NOT BOTH

7 points – If the student rewrites the integrand as $\frac{1}{5\sqrt{u}}$ AND finds the new limits of integration

9 points – If the student integrates with respect to u

10 points – If the student has the final answer

If the student chooses not to change the limits of integration, the following grading scale will be used:

3 points – If the student chooses $u = 4 + 5\sin x$ and $du = 5\cos x\ dx$

5 points – If the student rewrites the integrand as $\frac{1}{5\sqrt{u}}$

7 points – If the student integrates with respect to u

10 points – If the student has the final answer

-2 points – If the student changes the limits of integration but goes back to x

10. (15 points) Let $y = xe^{1/x}$.

(a) (2 points) State the domain of y.

(b) (7 points) Find and classify all critical points of y.

(c) (6 points) Find the intervals where y is increasing and the intervals where y is decreasing.

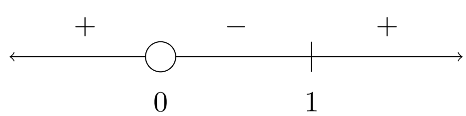
SOLUTION:

(a) The domain of y is $x \neq 0$.

(b) Since $y' = e^{1/x} + xe^{1/x} \cdot -\frac{1}{x^2}$, the critical points come from setting $y' = 0$ since y' is defined for all $x \neq 0$. Then using the fact that $e^{1/x} \neq 0$,

$$\begin{aligned} e^{1/x} + xe^{1/x} \cdot -\frac{1}{x^2} &= 0 \\ 1 - \frac{1}{x} &= 0 \\ x - 1 &= 0 \\ x &= 1. \end{aligned}$$

Now we create a sign diagram.



Therefore $x = 1$ is a local minimum.

(c) y is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$.

GRADING RUBRIC:

(a)

2 points – If the student has the correct domain

(b)

2 points – If the student incorrectly found the derivative of y

4 points – If the student found y'

7 points – If the student found the critical point $x = 1$ and classified it

-1 point – If the student claimed that 0 is a local maximum

(c)

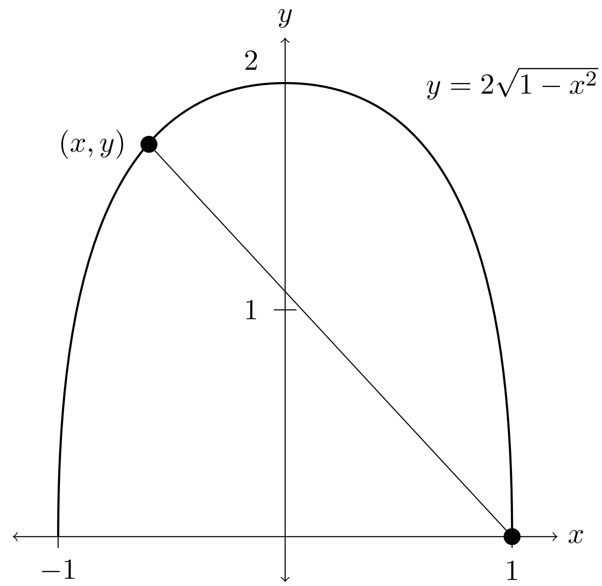
4 points – If the student created the correct sign diagram

6 points – If the student made the correct conclusion about increasing and decreasing

Notes: If the student did not mark off $x = 0$ on the sign diagram, the student can earn at most 3 points for part (c)

If the student has part (b) wrong, but makes the correct conclusion on the sign diagram, a maximum of 2 points can be earned on part (c)

11. (15 points) Consider the function $y = 2\sqrt{1-x^2}$ graphed below. What is the x-coordinate of the point (x, y) on this graph that is a maximum distance from the point $(1, 0)$? Hint: Maximize the square of the distance.



SOLUTION: The square of the distance between (x, y) and $(1, 0)$, which we'll label D is

$$D = (x-1)^2 + (y-0)^2 = (x-1)^2 + \left(2\sqrt{1-x^2}\right)^2 = x^2 - 2x + 1 + 4(1-x^2).$$

Then

$$D' = 2x - 2 + 4(-2x) = 0 \Longrightarrow x = -\frac{1}{3}.$$

This must be a maximum so $x = -\dfrac{1}{3}$ is the x-coordinate we seek.

GRADING RUBRIC:

5 points – If the student had the function $D = (x-1)^2 + (y-0)^2$, or $D = \sqrt{(x-1)^2 + (y-0)^2}$

9 points – If the student substituted $y = 2\sqrt{1-x^2}$ into D

13 points – If the student calculated D'

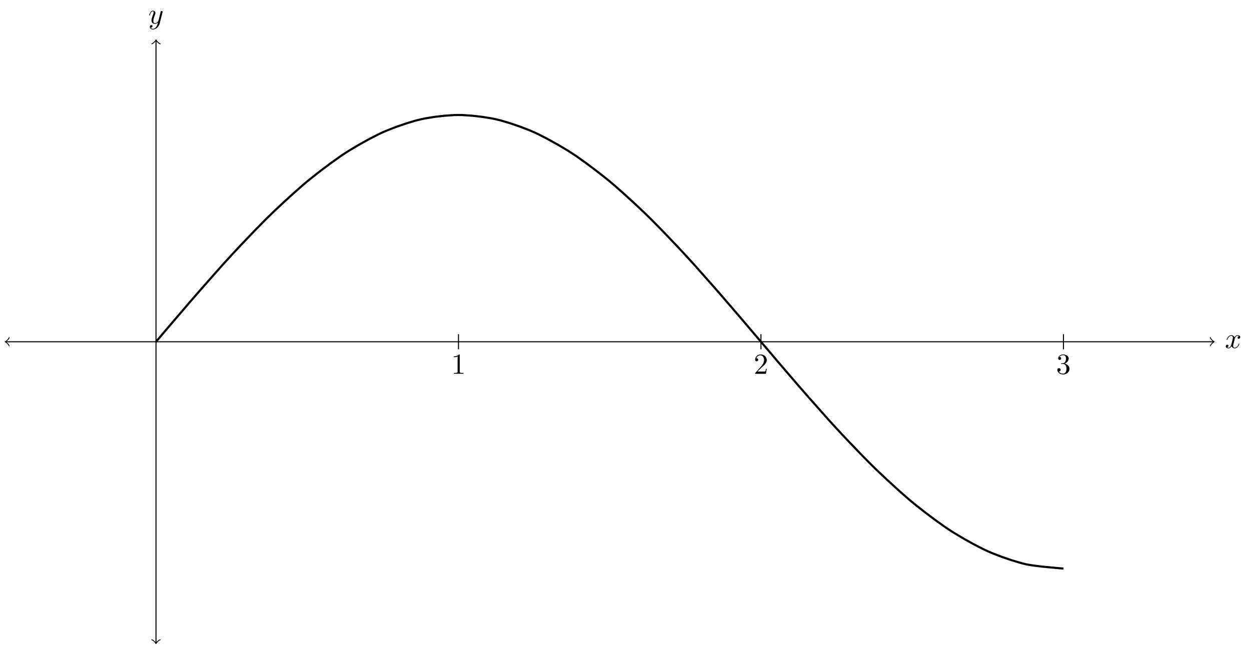
15 points – If the student found $x = -\dfrac{1}{3}$

12. (15 points) Consider a function f that is continuous on $0 \le x \le 3$ with the following information.

x	$0 < x < 1$	$1 < x < 2$	$2 < x < 3$
f	$+$	$+$	$-$
f'	$+$	$-$	$-$
f''	$-$	$-$	$+$

Sketch a possible graph for f that satisfies the information above (recall that f is continuous).

Solution: Here is one possible graph.



Grading Rubric:

Grade the intervals $0 < x < 1$, $1 < x < 2$, and $2 < x < 3$ each using the following scale:

1 point – If the graph satisfies one of the three properties on the given interval

3 points – If the graph satisfies two of the three properties on the given interval

5 points – If the graph satisfies all three of the three properties on the given interval

-4 points – If the student's graph is not continuous

13. (10 points) (a) (7 points) Calculate $\int_0^8 \sqrt{x+1}\,dx$. Write your answer in the form $\frac{a}{b}$ where a and b are integers.

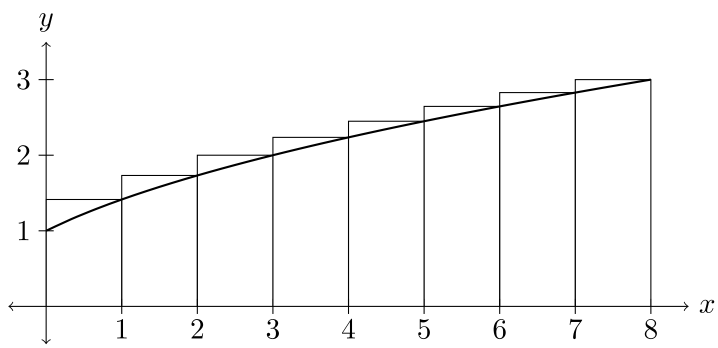
(b) (3 points) If a right hand Riemann sum is used to estimate $\int_0^8 \sqrt{x+1}\,dx$, would the Riemann sum be an underestimate or overestimate to the actual value of the integral? Justify your answer.

SOLUTION:

(a) Using the substitution $u = x + 1$, we have $du = dx$ and the new limits of integration are $u(0) = 1$ to $u(8) = 9$. Therefore the new integral in terms of u is

$$\int_1^9 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^9 = \frac{2}{3}\cdot 9^{3/2} - \frac{2}{3}\cdot 1^{3/2} = \frac{52}{3}.$$

(b) If we graph $y = \sqrt{x+1}$ on $[0, 8]$, we have



Notice that on any interval, since the function is concave down, using the right hand height to generate a rectangle for the Riemann sum is an overestimate to the actual area, so the right hand Riemann sum will be an overestimate.

GRADING RUBRIC:

(a) If the student chose to do a u-substitution, use the following scale:

1 point – If the student chooses $u = x + 1$

2 points – If the student chooses $u = x + 1$ and finds $du = dx$

4 points – If the student rewrites the integral in terms of u

7 points – If the student calculates the integral

-2 points – If the student did not write the answer in the form $\frac{a}{b}$

If the student did not use a u-substitution, use the following scale:

4 points – If the student had the correct antiderivative

7 points – If the student had the correct final answer

-2 points – If the student did not write the answer in the form $\frac{a}{b}$

(b)

0 points – If the student has an answer with no justification

3 points – If the student has “overestimate” with proper justification (a picture with no words would be proper justification)