Solutions

Math 180 **Name (Print):** ______________________

2/10/2015 **NetID:** ______________________
Time Limit: 90 Minutes

This exam contains 8 pages (including this cover page) and 10 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

- You may not open this exam until you are instructed to do so.
- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

TA Name:______________________

Circle your instructor.

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- Evangelos Kobotis
- Bonnie Saunders

1. (8 points) Let $f(x) = \begin{cases} 3x+5 & x<1 \\ 3 & x=1 \\ x^2+x & x>1 \end{cases}$

(1) Compute the following, or say that it does not exist:

(a) (2 points) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (3x+5) = 3+5 = 8$

(b) (2 points) $\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (x^2+x) = 1+1 = 2$

(c) (2 points) $\lim_{x \to 1} f(x)$ DNE because $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

(2) (2 points) Is $f(x)$ continuous at $x = 1$? No, because $\lim_{x \to 1} f(x)$ DNE

f has a jump at $x=1$

2. (5 points) Let $g(2) = 3$, and $g'(2) = 4$. Find the equation of the tangent line to the graph of $y = g(x)$ at $x = 2$.

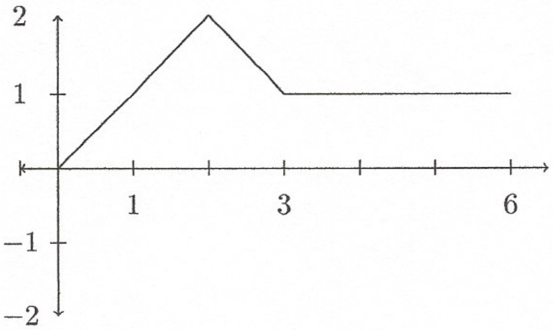
slope $= 4$

point $= (2,3)$

$\Big\}\ y-3 = 4(x-2)$

$\boxed{y = 4x-5}$

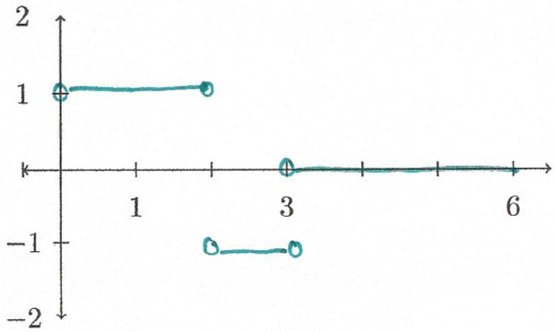
3. (10 points) The graph of the function $y = g(x)$ on the interval $[0, 6]$ is shown below.



(a) At which point(s) x for $0 < x < 6$ is g not differentiable?

@ $x = 2$ and @ $x = 3$

(b) Sketch the graph of the derivative function $y = g'(x)$.



4. (2 points) Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Under what condition(s) is the following statement true?

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

true if $\lim_{x \to a} g(x) \neq 0$

5. (12 points) Use the definition of the derivative as the limit of difference quotients in order to compute the derivative of the function $f(x) = \dfrac{2}{x-3}$. No points will be given if the definition is not used.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2}{x+h-3} - \frac{2}{x-3}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{2(x-3) - 2(x+h-3)}{(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{\cancel{2x} - \cancel{6} - \cancel{2x} - 2h + \cancel{6}}{(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}} \cdot \frac{-2\cancel{h}}{(x+h-3)(x-3)}$$

$$= \frac{-2}{(x-3)(x-3)}$$

$$\Rightarrow \quad f'(x) = \boxed{\frac{-2}{(x-3)^2}}$$

6. (16 points) Evaluate the following limits. Show all your work, and if you are using a theorem then state the name of the theorem.

(a) (8 points) $\lim_{x \to 1} \dfrac{\sqrt{x+8}-3}{x-1}$ of type $\dfrac{\sqrt{9}-3}{1-1} = \dfrac{0}{0}$

multiply times the conjugate

$$= \lim_{x \to 1} \frac{\sqrt{x+8}-3}{x-1} \cdot \frac{\sqrt{x+8}+3}{\sqrt{x+8}+3}$$

$$= \lim_{x \to 1} \frac{x\cancel{+8-9}}{\cancel{x-1}} \cdot \frac{1}{\sqrt{x+8}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$$

(b) (8 points) $\lim_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right)$

use the squeeze theorem:

$-1 \le \cos(\frac{1}{x}) \le +1$ multiply times x^2

$-x^2 \le x^2 \cos(\frac{1}{x}) \le +x^2$

apply the squeeze theorem

$$0 = \lim_{x \to 0}(-x^2) \le \lim_{x \to 0}\left(x^2 \cos\left(\tfrac{1}{x}\right)\right) \le \lim_{x \to 0}(x^2) = 0$$

$$\Rightarrow \lim_{x \to 0}\left(x^2 \cos\left(\tfrac{1}{x}\right)\right) = 0$$

7. (16 points) Let

$$f(x) = \frac{2x^2 - 2x - 12}{x^2 - 2x - 3} = \frac{2(x-3)(x+2)}{(x-3)(x+1)}$$

(a) (12 points) Evaluate the following limits:

1. $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2-2x-12}{x^2-2x-3} = \lim_{x\to-\infty} \frac{2-\frac{2}{x}-\frac{12}{x^2}}{1-\frac{2}{x}-\frac{3}{x^2}} = \frac{2-0-0}{1-0-0} = \boxed{2}$

2. $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2(x+2)}{x+1} = \frac{2\cdot 5}{4} = \boxed{\frac{5}{2}}$

3. $\lim_{x\to -1^-} f(x)$ of type $\frac{2}{0^-}$, 0^- because $x < -1 \Rightarrow x+1 < 0$

   $= \boxed{-\infty}$

4. $\lim_{x\to -1} f(x)$ since $\lim_{x\to -1^+} f(x) = +\infty$ of type $\frac{2}{0^+}$

   $\boxed{\text{DNE}}$

(b) (4 points) Based on your answers in part (a) does f have

1. a horizontal asymptote? If yes, state the horizontal asymptote, and list which of the limits 1-4 in part (a) support this claim.

   $y = 2$ limit 1. supports this claim

2. a vertical asymptote? If yes, state the vertical asymptote, and list which of the limits 1-4 in part (a) support this claim.

   $x = -1$ limit 3. supports this claim

   (note that the limit 4. does not support this claim necessarily)

8. (12 points) Compute the derivatives of the following functions, you do not need to simplify!

(a) (6 points) $f(x) = \dfrac{x^2 - x + 1}{x^2 + 3}$

quotient rule

$$f'(x) = \frac{(2x-1)(x^2+3) - (x^2-x+1)\cdot 2x}{(x^2+3)^2}$$

(b) (6 points) $f(x) = \sqrt{x}\cos x$

product rule

$$f'(x) = \frac{1}{2\sqrt{x}}\cos x + \sqrt{x}\cdot(-\sin x)$$

9. (9 points) Fill in the blanks in the use of the Intermediate Value Theorem to show that there exists a real number c between 0 and $\frac{\pi}{2}$ whose cosine equals twice the number, i.e. we will show that there is a c with $\cos(c) = 2c$ or $\cos(c) - 2c = 0$.

$f(x) = \cos x - 2x$ is a function that is CONTINUOUS on the interval [0 , $\pi/2$].

Since $L =$ 0 is a number that is strictly between $f(0) = 1$ and $f(\frac{\pi}{2}) = -\pi$.

Then by the Intermediate Value Theorem there exists at least one number c in the interval (0 , $\pi/2$) satisfying $f(c) =$ 0 .

In conclusion we found a number c whose cosine equals twice its number.

10. (10 points) A rock is dropped of the edge of a cliff, and its distance s in feet from the ground is given by $s(t) = 96 - 16t^2$, t in seconds.

(a) (5 points) Find the average velocity of the rock over the time interval $[1, 2]$. Include units in your answer.

average velocity $1 \le t \le 2$ $= \frac{s(2) - s(1)}{2 - 1} = \frac{(96 - 64) - (96 - 16)}{1}$

$= -48$ ft/s

(b) (5 points) Find the instantaneous velocity of the rock at time $t = 1$ second. Include units in your answer.

instantaneous velocity @ $t=1$ } $s'(1) = ?$

$s'(t) = -32t$

$s'(1) = -32$ ft/s