Math 180 **Name (Print):** SOLUTIONS

Exam #2 **UIN:** ________________

10/21/2015 **Email:** ________________

Time Limit: 2 Hours

This exam contains 8 pages (including this cover page) and 8 problems. After starting the exam, check to see if any pages are missing. Enter all requested information on the top of this page.

The following rules apply:

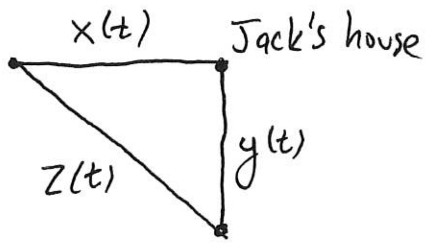
- You may not open this exam until you are instructed to do so.
- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, calculators, or any electronic device including cell phones. Only pencils/pens allowed.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You *must* complete your work in the space provided. We will be scanning your answers into our grading system, so any work you do that is out of place, too close to the page border, or on the wrong page will *not* be graded!

TA Name:________________

Circle your instructor.

- Bode
- Goldbring
- Hachtman
- Riedl
- Sinapova
- Steenbergen @ 11am
- Steenbergen @ 12pm
- Steenbergen @ 2pm

1. (12 points) Jack and Jill start riding their bikes from Jack's house at noon. Jack heads west at 6 miles per hour while Jill heads south at 8 miles per hour. How fast is the distance between them increasing at 2pm? Draw a picture modeling the situation. Don't forget to include the units in your answer.



$x(t)=$ Jack's distance from his house at time t

$y(t)=$ Jill's distance from Jack's house at time t

$z(t)=$ distance between Jack and Jill at time t.

Want: $\left.\frac{dz}{dt}\right|_{t=2}$ given $\frac{dx}{dt}=6\text{mi/hr}$ and $\frac{dy}{dt}=8\text{mi/hr}$.

Know: $z^2=x^2+y^2$

$$2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}.$$

When $t=2$, $x=2(6)=12\text{mi}$, $y=2(8)=16\text{mi}$ and

$$z=\sqrt{12^2+16^2}=20\text{ mi}.$$

So $$\frac{dz}{dt}=\frac{x\frac{dx}{dt}+y\frac{dy}{dt}}{z}$$

$$\left.\frac{dz}{dt}\right|_{t=2}=\frac{(12\text{mi})(6\text{mi/hr})+(16\text{mi})(8\text{mi/hr})}{20\text{ mi}}$$

$$=\frac{72\text{ mi}^2/\text{hr}+128\text{mi}^2/\text{hr}}{20\text{ mi}}=10\text{ mi/hr}.$$

2. (12 points) A water rocket is launched vertically upward from a platform 32 feet above the ground at an initial velocity of 64 feet per second. Its height at time t (seconds) is given by the equation $s(t) = -16t^2 + 64t + 32$ feet.

(a) (4 points) What is the velocity of the rocket after 2 seconds?

$v(t) = -32t + 64$

$v(2) = -32(2) + 64 = 0$ ft/s.

(b) (4 points) What is the acceleration of the rocket after 2 seconds?

$a(t) = -32$

$a(2) = -32$ ft/s^2.

(c) (4 points) What is the highest distance from the ground reached by the rocket?

The highest distance occurs when $v = 0$.
We already saw that $v(2) = 0$, so the highest distance is reached at $t = 2$ seconds.
At this time, the distance is

$$s(2) = -16(2)^2 + 64(2) + 32$$
$$= 96 \text{ ft.}$$

3. (10 points) Find the equation of the tangent line to the curve

$$\sin(xy) = x - \pi$$

at the point $(\pi, 1)$.

$$\cos(xy)\left[1\cdot y + x\frac{dy}{dx}\right] = 1.$$

$$x\cos(xy)\frac{dy}{dx} = 1 - y\cos(xy)$$

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}$$

$$\left.\frac{dy}{dx}\right|_{(\pi,1)} = \frac{1 - 1\cos(\pi\cdot 1)}{\pi\cos(\pi\cdot 1)} = \frac{2}{-\pi}$$

$$\boxed{y - 1 = -\frac{2}{\pi}(x - \pi)}$$

4. (10 points) Let $f(x) = -x^3 + 3x^2 - 5$. Find the absolute maximum and minimum values of $f(x)$ on $[1, 3]$.

$$f'(x) = -3x^2 + 6x = 0$$

$$-3x(x-2) = 0 \quad \text{crit points: } x = 0, 2$$

$x = 0$ doesn't belong to $[1,3]$, so we disregard it.

Compare: $f(1) = -(1)^3 + 3(1)^2 - 5 = -3$

$f(2) = -(2)^3 + 3(2)^2 - 5 = -1$

$f(3) = -(3)^3 + 3(3)^2 - 5 = -5$

So $f(2) = -1$ is the absolute maximum and $f(3) = -5$ is the absolute minimum.

5. (20 points) Find the derivatives of the following functions:

(a) (6 points) $\sqrt{x}\ln(x^3)$ for $x > 0$.

$$\frac{d}{dx}\left(x^{1/2}\cdot \ln(x^3)\right) = \frac{1}{2}x^{-1/2}\cdot \ln(x^3) + x^{1/2}\cdot \frac{1}{x^3}\cdot 3x^2$$

$$= \frac{1}{2}x^{-1/2}\ln(x^3) + 3x^{-1/2}$$

(b) (6 points) $\ln(\tan^{-1}(3^x))$ for $x > 0$.

$$\frac{d}{dx}\left(\ln(\tan^{-1}(3^x))\right) = \frac{1}{\tan^{-1}(3^x)}\cdot \frac{d}{dx}\left(\tan^{-1}(3^x)\right)$$

$$= \frac{1}{\tan^{-1}(3^x)}\cdot \frac{1}{1+(3^x)^2}\cdot \frac{d}{dx}(3^x)$$

$$= \frac{1}{\tan^{-1}(3^x)}\cdot \frac{1}{1+(3^x)^2}\cdot 3^x \ln 3$$

(c) (8 points) $(\cos x)^{x+1}$ for $0 \le x \le \pi/2$.

$$y = (\cos x)^{x+1}$$

$$\ln y = \ln[(\cos x)^{x+1}] = (x+1)\ln(\cos x)$$

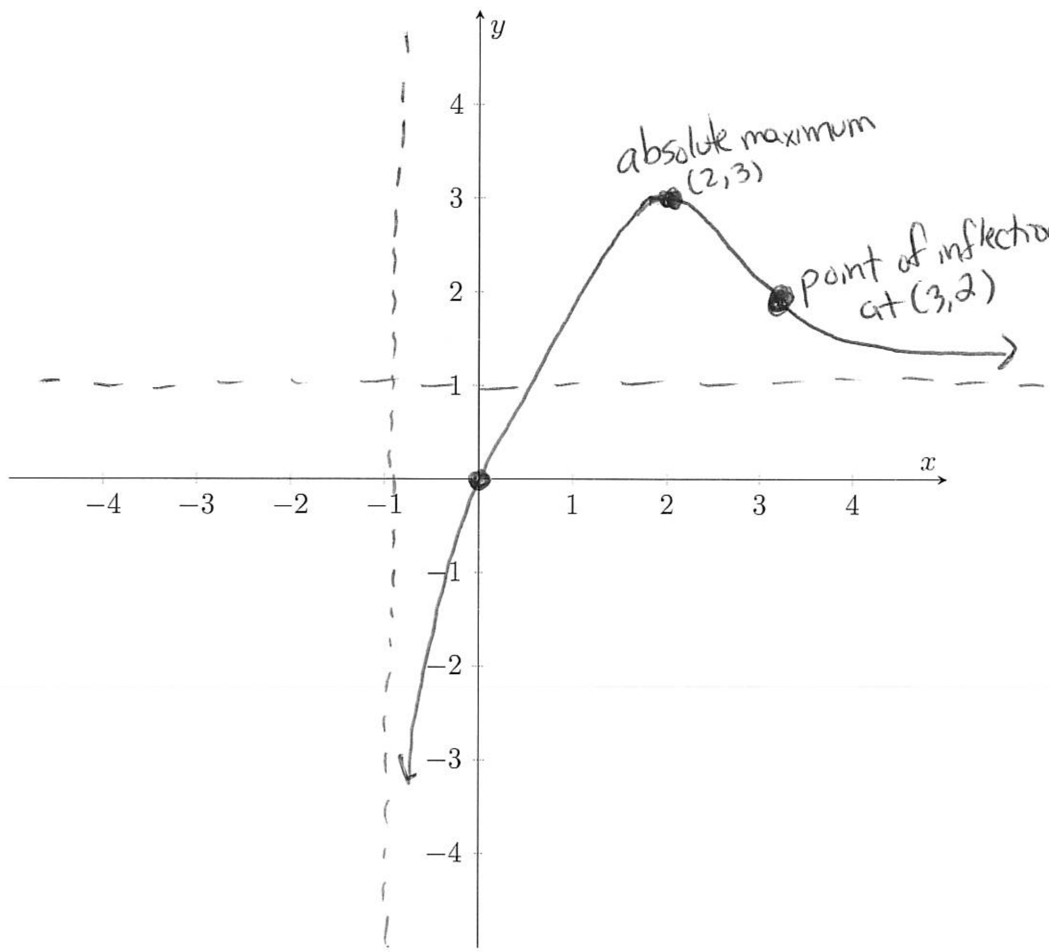
$$\frac{1}{y}\frac{dy}{dx} = 1\cdot \ln(\cos x) + (x+1)\cdot \frac{1}{\cos x}\cdot(-\sin x)$$

$$\frac{dy}{dx} = (\cos x)^{x+1}\left[\ln(\cos x) - (x+1)\cdot \tan x\right]$$

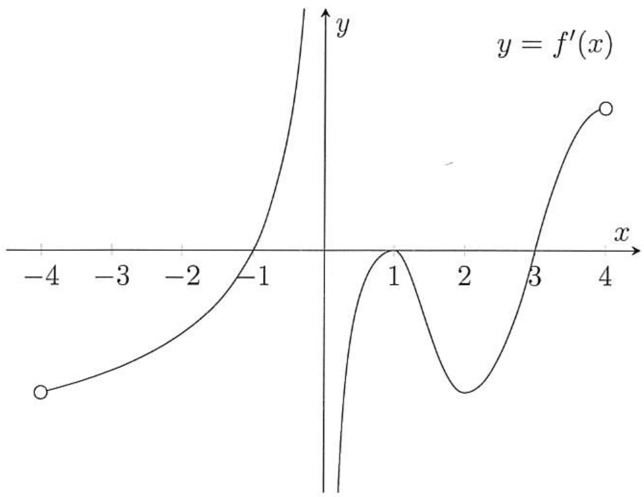
6. (10 points) Suppose f is a function that is continuous and differentiable on the interval $(-1, \infty)$. In addition f has the following properties:

- Domain of f is $(-1, \infty)$
- f has a vertical asymptote at $x = -1$, and a horizontal asymptote at $y = 1$.
- $f(0) = 0$.
- $f'(x) > 0$ on the interval $(-1, 2)$, $f'(2) = 0$, and $f'(x) < 0$ on the interval $(2, \infty)$.
- $f''(x) > 0$ on the interval $(3, \infty)$, $f''(3) = 0$, and $f''(x) < 0$ on the interval $(-1, 3)$.
- $f(2) = 3$ and $f(3) = 2$.

Sketch a possible graph of $y = f(x)$. Label all absolute maximum and minimum points if there are any, and all points of inflection.



7. (14 points) Below is a graph of the *derivative* of a continuous function.



Based on this graph of $y = f'(x)$, answer the following questions about the function $f(x)$.

(a) (4 points) On what intervals is $f(x)$ increasing?

f increasing when $f' > 0$. This happens on $(-1,0) \cup (3,4)$.

(b) (4 points) On what intervals is $f(x)$ concave down?

f concave down when f' decreases. This happens on $(1,2)$

(c) (8 points) For which x does $f(x)$ have a local minimum? a local maximum?

f has a local minimum when f' changes from negative to positive, so when $x=-1$ and $x=3$.

f has a local maximum when f' changes from positive to negative, so when $x=0$.

8. (12 points) Consider the function $g(x) = xe^{-x}, g'(x) = -e^{-x}(x-1), g''(x) = e^{-x}(x-2)$

(a) (4 points) Determine those intervals on which the function $g(x)$ is increasing and those on which it is decreasing.

$g'(x) = 0$ when $x = 1$

g' + | −
g ↗ 1 ↘

g increasing on $(-\infty, 1)$

g decreasing on $(1, +\infty)$

(b) (2 points) Does $g(x)$ have a local maximum? local minimum?

By the first derivative test, $g(1) = 1 \cdot e^{-1} = e^{-1}$ is a local maximum for $g(x)$.

g has no local minimum.

(c) (4 points) Determine those intervals on which the function $g(x)$ is concave up and those on which it is concave down.

$g''(x) = 0$ when $x = 2$

g'' − | +
g ∩ 2 ∪

g concave up on $(2, +\infty)$

g concave down on $(-\infty, 2)$

(d) (2 points) Does $g(x)$ have any points of inflection?

Yes, $x = 2$ is an inflection point as g changes concavity there.