

DO NOT WRITE ABOVE THIS LINE!!

Solutions

MATH 181 Final Exam December 8, 2016

Directions. Fill in each of the lines below. Circle your instructor's name and write your TA's name. Then read the directions that follow before beginning the exam. YOU MAY NOT OPEN THE EXAM UNTIL TOLD TO DO SO BY YOUR INSTRUCTOR. Good luck!

Print Name: _____

University Email: _____

UIN: _____

Circle your instructor's name:

Bode

Lesieutre

TA's Name: _____

- VERY IMPORTANT!!! CHECK THAT THE NUMBER AT THE TOP OF EACH PAGE OF YOUR EXAM IS THE SAME. IT IS THE NUMBER PRECEDED BY A POUND (#) SIGN. IF THEY ARE NOT ALL THE SAME, NOTIFY YOUR INSTRUCTOR OR TA RIGHT AWAY.
- All of your work must fit within the boxes on each page for each question. Nothing outside of the box will be graded! If you write outside of the box, there is a good chance that your exam will not be read and therefore not graded.
- A solution for one problem may not go on another page.
- Make clear to the grader what your final answer is.
- Have your student ID ready to be checked when submitting your exam.

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Question 1. For each of the following questions, circle the correct answer. No explanation is necessary. No partial credit will be given.

1. (6 points) Find the derivative of the function: $g(x) = \int_1^{\tan x} e^{t^2} dt$

(a) $g'(x) = 2xe^{\tan^2(x)}$

$g'(x) = e^{\tan^2 x} \cdot \sec^2 x$

(b) $g'(x) = \sec^2 x e^{\tan^2 x}$

(c) $g'(x) = \sec^2(x) e^{\tan^2 x} \sec x \tan x$

(d) $g'(x) = \sec^2(x) e^{\tan^2 x} - \sec^2(1) e^{\tan^2(1)}$

2. (6 points) Some values for a function $y = f(t)$ are given in the table below. Use Simpson's Rule to estimate the definite integral $\int_0^{24} f(t) dt$.

t	0	6	12	18	24
$f(t)$	1/2	3/4	1/2	3/8	1/2

$\Delta t = 6$
 $n = 4$

$\frac{\Delta t}{3} = \frac{6}{3}$

(a) $\frac{1}{2} \cdot \left(\frac{1}{2} + 2 \cdot \frac{3}{4} + 2 \cdot \frac{1}{2} + 2 \cdot \frac{3}{8} + \frac{1}{2} \right)$

(b) $\frac{6}{3} \cdot \left(\frac{1}{2} + 4 \cdot \frac{3}{4} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{3}{8} + \frac{1}{2} \right)$

(c) $\frac{1}{3} \cdot \left(\frac{1}{2} + 4 \cdot \frac{3}{4} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{3}{8} + \frac{1}{2} \right)$

(d) $\frac{6}{2} \cdot \left(\frac{1}{2} + 4 \cdot \frac{3}{4} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{3}{8} + \frac{1}{2} \right)$

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Question 2. (6 points) Given the matrix $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$ and the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

One of these vectors is an eigenvector, which vector and what is its eigenvalue?

$$A \cdot \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$A \cdot \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \text{eigenvector } \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ with } \lambda = 2$$

$$A \cdot \mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Question 3. (12 points) Use partial fractions to evaluate the integral:

$$I = \int \frac{(5x+1) dx}{x^2+x-2} \quad x^2+x-2 = (x+2)(x-1)$$

$$\frac{5x+1}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$5x+1 = A(x-1) + B(x+2)$$

$$x=1 \quad 6 = 0 + 3B \Rightarrow \boxed{B=2}$$

$$x=-2 \quad -9 = -3A \Rightarrow \boxed{A=3}$$

$$\Rightarrow I = \int \left(\frac{3}{x+2} + \frac{2}{x-1} \right) dx = 3 \ln|x+2| + 2 \ln|x-1| + c$$

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Question 4. (20 points) Evaluate the following integrals.

(a) (10 points) $\int_0^{\sqrt{\pi}} 6\theta \sin(\theta^2 + \pi) d\theta$

u -substitution

$$\boxed{\begin{array}{l} u = \theta^2 + \pi \\ du = 2\theta d\theta \end{array}}$$

$$= \int_{\pi}^{2\pi} 3 \sin u du$$

$$= -3 \cos u \Big|_{\pi}^{2\pi} = -3 \underbrace{\cos(2\pi)}_{+1} + 3 \underbrace{\cos(\pi)}_{-1} = -3 + 3 = \boxed{0}$$

(b) (10 points) $\int (2x+1) \ln x dx$

by parts

$$u = \ln x \quad dv = (2x+1) dx$$

$$du = \frac{dx}{x} \quad v = x^2 + x$$

$$= (x^2+x) \ln x - \int (x+1) dx$$

$$= (x^2+x) \ln x - \frac{x^2}{2} - x + C$$

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Question 5. (12 points) Rewrite the following improper integral as limit and check whether or not the integral converges or diverges. If it converges, evaluate the integral.

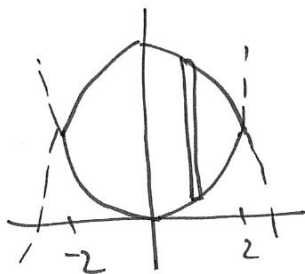
$$\begin{aligned} & \int_0^4 \frac{4}{x-4} dx \\ &= \lim_{b \rightarrow 4^-} \int_0^b \frac{4}{x-4} dx \\ &= \lim_{b \rightarrow 4^-} 4 \ln|x-4| \Big|_0^b \\ &= \lim_{b \rightarrow 4^-} 4 \ln|b-4| - 4 \ln(4) \\ &= -\infty \end{aligned}$$

$\Rightarrow$ the integral diverges

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Question 6. (16 points) Let R be the region enclosed by the parabolas $y = x^2$ and $y = 8 - x^2$.

(a) (8 points) Sketch the region and set up the integral for the area of this region.



$$x^2 = 8 - x^2$$

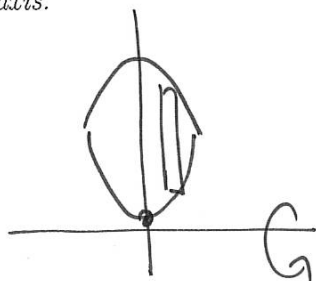
$$2x^2 = 8$$

$$x^2 = 4$$

$$x = \pm 2$$

$$\begin{aligned} \Rightarrow \text{Area} &= \int_{-2}^2 ((8 - x^2) - x^2) dx \\ &= \int_{-2}^2 (8 - 2x^2) dx \end{aligned}$$

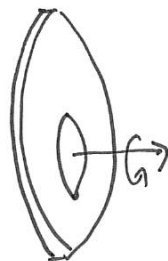
(b) (8 points) Set up the integral for the volume of the solid obtained by rotating the region R about the x -axis.



$$R = 8 - x^2 \text{ outer radius}$$

$$r = x^2 \text{ inner radius}$$

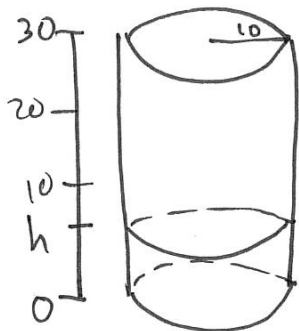
$$\Rightarrow \int_{-2}^2 \pi \cdot ((8 - x^2)^2 - (x^2)^2) dx$$



washer

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Question 7. (10 points) A cylindrical tank measures 30 ft high and 20 ft in diameter. It is one third filled with kerosene weighing approximately 50 lb/ft³. Set up the integral for the work done to pump all the kerosene out over the top of the tank.



typical slice at height h :



$$0 \leq h \leq 10$$

$$\begin{aligned} \text{force of slice} &= (\pi \cdot (10)^2 \Delta h) \cdot (50) = 5000\pi \Delta h \\ \text{distance} &= 30 - h \end{aligned}$$

$\Rightarrow$ Work done

$$= \int_0^{10} (30 - h) (5000\pi) dh$$

Question 8. (10 points) Find the Taylor polynomial of degree 3 centered about $a = \pi$ for $f(x) = \sin x$.

$f(x) = \sin x$	$f(\pi) = \sin(\pi) = 0$
$f'(x) = \cos x$	$f'(\pi) = \cos(\pi) = -1$
$f''(x) = -\sin x$	$f''(\pi) = -\sin(\pi) = 0$
$f'''(x) = -\cos x$	$f'''(\pi) = -\cos(\pi) = -(-1) = +1$

$$\Rightarrow T_3(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3$$

$$\Rightarrow T_3(x) = 0 - (x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3$$

$$T_3(x) = -(x-\pi) + \frac{1}{6}(x-\pi)^3$$

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Question 9. (20 points) Test the following series for convergence or divergence. Justify your answer!

(a) (8 points) $\sum_{n=1}^{\infty} \frac{n}{n^3+2}$ compare to $\sum \frac{n}{n^3} = \sum \frac{1}{n^2}$ $p=2$, p -test ~~diverges~~ converges

use direct comparison

$$0 \leq \frac{n}{n^3+2} \leq \frac{n}{n^3} = \frac{1}{n^2}$$

since $\sum \frac{1}{n^2}$ converges, so does $\sum \frac{n}{n^3+2}$.

(b) (8 points) $\sum_{n=1}^{\infty} \frac{4}{3^{n-1}}$ geometric series with $a=4$ & $r=\frac{1}{3}$

converges since $|r| < 1$

$$\left(\text{and converges to } \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = \cancel{4} \cdot \frac{3}{2} \right) = 6$$

(c) ~~(8)~~⁴ points) $\sum_{n=2}^{\infty} \frac{n+1}{n-1}$ diverges since $\lim_{n \rightarrow \infty} \frac{n+1}{n-1} = 1 \neq 0$

by divergence test.

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Question 10. (10 points) Does the following series converge absolutely, conditionally, or does it diverge? Justify your answer!

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

(a) series does ~~not~~ converge absolutely since

$$\sum \frac{1}{\sqrt{n}} \text{ diverges, } p=1/2, \text{ } p\text{-test}$$

(b) series does converge, use ~~absolute~~ series test:

$$(i) \ b_n = \frac{1}{\sqrt{n}} \geq 0 \quad \checkmark$$

$$(ii) \ b_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = b_n \Rightarrow b_n \text{ decreasing } \checkmark$$

$$(iii) \ \lim b_n = 0 \quad \checkmark$$

$\Rightarrow$ by AST series converges

$$\Rightarrow \sum \frac{(-1)^n}{\sqrt{n}} \text{ converges conditionally}$$

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Question 11. (16 points) Find the radius and the interval of convergence for the following power series. Justify your answer!

$$\sum_{k=1}^{\infty} \frac{(x-3)^k}{k \cdot 6^k}$$

ratio test:

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(x-3)^{k+1}}{(k+1)6^{k+1}} \cdot \frac{k \cdot 6^k}{(x-3)^k} \right|$$

$$= \lim_{k \rightarrow \infty} \frac{|x-3|}{6} \cdot \frac{k}{k+1}$$

$$= \frac{|x-3|}{6} \Rightarrow \text{converges for } \frac{|x-3|}{6} < 1$$

$$\Rightarrow |x-3| < 6 \text{ or } -3 < x < 9$$

check endpoints

@ $x = -3$ $\sum \frac{(-6)^k}{k \cdot 6^k} = \sum \frac{(-1)^k}{k}$ alternating harmonic series
converges

@ $x = 9$ $\sum \frac{(6)^k}{k \cdot 6^k} = \sum \frac{1}{k}$ harmonic series or $p=1$, p -test
diverges

$$\Rightarrow \text{Interval of Convergence} = [-3, 9)$$

$$\text{Radius of Convergence} = 6$$

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Question 12. (20 points)

(a) (10 points) Find a power series representation and the interval of convergence for the function:

$$f(x) = \frac{1}{1-x^4}$$

(You do not need to find a summation formula, just write out the addition or subtraction of the first 5 non-zero terms.)

geometric series with $a=1$ & $r=x^4$

$$\Rightarrow f(x) = 1 + x^4 + x^8 + x^{12} + x^{16} + \dots$$

(b) (6 points) Use the methods or results from part (a) to find a power series representation for the function:

$$g(x) = \frac{x}{1-x^4}$$

$$\frac{x}{1-x^4} = x + x^5 + x^9 + x^{13} + x^{17} + \dots$$

(c) (4 points) Use the first three non-zero terms of the series you found in part (b) to approximate the indefinite integral:

$$\int g(x) dx$$

$$= \int x + x^5 + x^9 + x^{13} + x^{17} + \dots dx$$

$$\approx \frac{x^2}{2} + \frac{x^6}{6} + \frac{x^{10}}{10} + C$$

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Question 13. (10 points) For the parametric curve given by $x = 2t$ and $y = 4t^2 + 1$

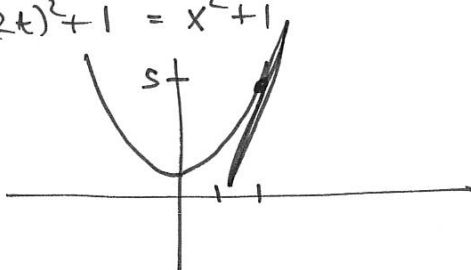
(a) (5 points) Determine dy/dx in terms of t and find an equation of the tangent line at $t = 1$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{8t}{2} = 4t \quad \text{At } t=1 \text{ slope} = 4 \text{ \& point } \begin{matrix} x=2 \\ y=5 \end{matrix} \Rightarrow y-5=4(x-2)$$

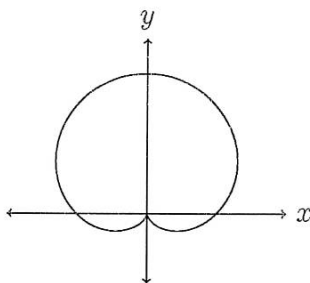
(b) (5 points) Eliminate the parameter t and make a sketch of the curve showing the tangent line at the point corresponding to $t = 1$.

$$x = 2t \Rightarrow y = (2t)^2 + 1 = x^2 + 1$$

$$y = x^2 + 1$$



Question 14. (10 points) Consider the polar curve $r = 1 + \sin \theta$ graphed below.



(a) (5 points) Find the rectangular coordinates x and y for the point on the graph with $\theta = \pi/4$.

$$x = r \cos \theta \Rightarrow \theta = \pi/4 \quad r = 1 + \sin \pi/4 = 1 + \frac{\sqrt{2}}{2} \Rightarrow x = \left(1 + \frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}+1}{2}$$

$$y = r \sin \theta \Rightarrow y = \left(1 + \frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \frac{2}{4} = \frac{\sqrt{2}+1}{2} \Rightarrow \left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2}\right)$$

(b) (5 points) Set up the integral for the area bounded by this curve in quadrant 1.

in quadrant 1 $0 \leq \theta \leq \pi/2$

$$\Rightarrow \text{Area} = \frac{1}{2} \int_0^{\pi/2} (1 + \sin \theta)^2 d\theta$$

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Question 15. (16 points) Consider the linear system:

$$\begin{aligned}x + y &= 4 \\x - y &= 2\end{aligned}$$

(a) (4 points) Write down the augmented matrix for the linear system.

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array} \right]$$

(b) (3 points) Translate this system to the format: $A \cdot \mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

(c) (4 points) Find the inverse matrix A^{-1} to A .

$$\begin{aligned}\det A &= (-1) - 1 = -2 \\ \Rightarrow A^{-1} &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\end{aligned}$$

(d) (4 points) Use the inverse matrix from part (b) to solve for $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

$$\begin{aligned}\Rightarrow \vec{x} &= A^{-1} \cdot \vec{b} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \boxed{x=3 \text{ \& } y=1}\end{aligned}$$

(e) (1 point) Double check your answer by plugging in x and y to the linear system above.

$$\begin{aligned}3+1 &\stackrel{!}{=} 4 \\ 3-1 &\stackrel{!}{=} 2\end{aligned}$$