MATH 210 - Final Exam, Fall 2005
Answers

1. (a) The velocity and acceleration vectors are $\vec{\mathbf{v}}(t) = \langle 3\cos t, 3\sin t, 4\rangle$ and $\vec{\mathbf{a}}(t) = \langle -3\sin t, 3\cos t, 0\rangle$.

   (b) The length of the curve is

$$L = \int_0^\pi \left|\left|\vec{\mathbf{v}}(t)\right|\right| \, dt = \int_0^\pi 5\, dt = 5\pi$$

2. The partial derivatives of $f(x, y, z)$ are

$$f_x = 2(y + \sin(z + xy^2)) \cdot \cos(z + xy^2) \cdot y^2$$
$$f_y = 2(y + \sin(z + xy^2)) \cdot \left[1 + \cos(z + xy^2) \cdot 2xy\right]$$
$$f_z = 2(y + \sin(z + xy^2)) \cdot \cos(z + xy^2)$$

   At the point $(0, 1, 0)$, values of the partial derivatives are $f_x(0, 1, 0) = 2$, $f_y(0, 1, 0) = 2$, and $f_z(0, 1, 0) = 2$. Thus, the gradient vector is $\vec{\nabla} f(0, 1, 0) = \langle 2, 2, 2\rangle$. The direction vector is a unit vector pointing in the direction of the vector $\vec{\mathbf{u}} = \langle 4 - 0, 0 - 1, 3 - 0\rangle = \langle 4, -1, 3\rangle$. This unit vector is $\hat{\mathbf{u}} = \frac{1}{||\vec{\mathbf{u}}||}\vec{\mathbf{u}} = \left\langle \frac{4}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}\right\rangle$. Thus, the directional derivative is $D_{\mathbf{u}}f(0, 1, 0) = \vec{\nabla} f(0, 1, 0) \cdot \hat{\mathbf{u}} = \frac{12}{\sqrt{26}}$.

4. We use the method of Lagrange multipliers. The system of equations to be solved is

$$4x = \lambda(2x), \quad 6y = \lambda(2y), \quad x^2 + y^2 = 1$$

   The first equation tells us that either $x = 0$ or $\lambda = 2$. If $x = 0$ then the third equation gives us $y = \pm 1$. If $\lambda = 2$ then the second equation gives us $y = 0$. The third equation then gives us $x = \pm 1$. So the critical points are $(0, 1)$, $(0, -1)$, $(1, 0)$, and $(-1, 0)$. Plugging these points into $f(x, y)$ we find that $f(0, 1) = f(0, -1) = 3$ and $f(1, 0) = f(-1, 0) = 2$. Thus, the absolute maximum of f on the unit circle is 3 and the absolute minimum is 2.

5. The region of integration is bounded above by the line $y = 2x$ and below by the parabola $y = x^2$. These curves intersect at $x = 0$ and $x = 2$. Changing the order of integration we get

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x, y)\, dx\, dy$$

6. The volume of E, using cylindrical coordinates, is

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_{4r^2}^4 r\, dz\, dr\, d\theta = 2\pi$$

7. Using spherical coordinates, the value of the integral is

$$\iiint_R \frac{dV}{x^2+y^2+z^2} = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}{\rho^2} = 4\pi$$

8. A parametrization of C is $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$. The value of the line integral is then

$$\int_C y\,dx - x\,dy = \int_0^{\pi} (\sin t)(-\sin t\,dt) - (\cos t)(\cos t\,dt) = -\int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt = -\pi$$

9. (a) Let $F_1 = 1 + 2ye^{2x}$ and $F_2 = 2y + e^{2x}$. Since $\frac{\partial F_1}{\partial y} = 2e^{2x} = \frac{\partial F_2}{\partial x}$, we know that $\overrightarrow{\mathbf{F}}$ is conservative.

   (b) The function $f(x,y) = x + ye^{2x} + y^2$ has the property that $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla} f$.

   (c) The integral is path independent because $\overrightarrow{\mathbf{F}}$ is conservative. The endpoints of the curve C are $Q = \overrightarrow{\mathbf{r}}(1) = (1,0)$ and $P = \overrightarrow{\mathbf{r}}(0) = (0,2)$. Thus, the value of the integral is

$$\int_C \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = f(Q) - f(P) = f(1,0) - f(0,2) = 1 - 6 = -5$$

10. Let $P = 3x + 2y$ and $Q = y^2 - x^2$. Then $\frac{\partial Q}{\partial x} = -2x$ and $\frac{\partial P}{\partial y} = 2$. Using Green's Theorem, the value of the integral is

$$\oint_C (3x+2y)\,dx + (y^2 - x^2)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (-2x-2)\ dA$$

where D is the region on and inside the triangle. Using the order of integration $dx\,dy$ we get

$$\iint_D (-2x-2)\,dA = \int_0^1 \int_{y-1}^{1-y} (-2x-2)\,dx\,dy = -2$$