

MATH 210 - Final Exam, Fall 2007
Answers

1. (a) A parametrization for the circle is $\vec{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$. Then

$$\vec{F}(\vec{c}(t)) = \langle -\sin t, \cos t \rangle, \quad \vec{c}'(t) = \langle -\sin t, \cos t \rangle$$

The value of the integral is

$$\int_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t) dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi$$

- (b) \vec{F} is not conservative. If it were, the integral in part (a) would be 0 because the path is closed.

2. (a) $\vec{r}'(t) = \langle 1, e^t, 2t \rangle$

(b) $\|\vec{r}'(t)\| = \sqrt{1 + e^{2t} + 4t^2}$

(c) $\vec{r}''(t) = \langle 0, e^t, 2 \rangle$

(d) $D = \int_0^3 \sqrt{1 + e^{2t} + 4t^2} dt$

3. (a) Let $f(x, y, z) = ze^{x^2-y^2}$. Then

$$\vec{\nabla} f = \langle 2xze^{x^2-y^2}, -2yze^{x^2-y^2}, e^{x^2-y^2} \rangle$$

A vector normal to the surface is $\vec{n} = \vec{\nabla} f(1, -1, 2) = \langle 4, 4, 1 \rangle$. An equation for the tangent plane at $(1, -1, 2)$ is

$$4(x - 1) + 4(y + 1) + (z - 2) = 0$$

- (b) The directional derivative of f at $(1, -1, 2)$ in the direction of $\langle 2, 2, 1 \rangle$ is

$$D_{\mathbf{u}}f(1, -1, 2) = \vec{\nabla} f(1, -1, 2) \cdot \hat{\mathbf{u}} = \langle 4, 4, 1 \rangle \cdot \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle = \frac{17}{3}$$

4. The critical points must satisfy the equations

$$f_x = y - x = 0, \quad f_y = x + y^2 - 2 = 0$$

From the first equation we have $y = x$. Substituting this into the second equation we have $x^2 + x - 2 = 0$. The two solutions are $x = 1$ and $x = -2$. The corresponding

y -values are $y = 1$ and $y = -2$. The second derivatives are $f_{xx} = -1$, $f_{yy} = 2y$, and $f_{xy} = 1$. So,

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -2y - 1$$

At the critical points we have

$$\begin{aligned} D(1, 1) &= -3 < 0 \implies \text{saddle point} \\ D(-2, -2) &= 3 > 0, \quad f_{xx}(-2, -2) = -1 < 0 \implies \text{local maximum} \end{aligned}$$

5. We will use the order of integration $dx dy$ so that

$$\iint_D e^{y^3} dA = \int_0^1 \int_0^{y^2} e^{y^3} dx dy = \frac{1}{3}(e - 1)$$

6. (a) $\overrightarrow{AB} = \langle 1, 2 \rangle$, $\overrightarrow{BC} = \langle -1, 1 \rangle$. The area of the parallelogram is $\left\| \overrightarrow{AB} \times \overrightarrow{BC} \right\| = 3$.

(b) We recognize that, for the vector field $\overrightarrow{\mathbf{F}} = \langle -y, x \rangle$,

$$\oint_C -y dx + x dy = 2 \times \text{Area enclosed by } C = 2(3) = 6$$

7. Using Cylindrical Coordinates, the integral is

$$\iiint_{\mathcal{B}} x^2 dV = \int_0^{2\pi} \int_0^1 \int_0^{r^2} (r \cos \theta)^2 r dz dr d\theta = \frac{\pi}{6}$$

8. (a) $\mathbf{Curl}(\overrightarrow{\mathbf{F}}) = \overrightarrow{\nabla} \times \overrightarrow{\mathbf{F}} = \langle -y, 0, -x^2 \rangle$

(b) $\overrightarrow{\mathbf{F}}$ is not conservative because $\mathbf{Curl}(\overrightarrow{\mathbf{F}}) \neq \overrightarrow{\mathbf{0}}$

(c) $\mathbf{Div}(\overrightarrow{\mathbf{F}}) = 2xy + z + 3z^2$

(d) $\mathbf{Div}(\mathbf{Curl}(\overrightarrow{\mathbf{F}})) = 0$

9. Let $\Phi(u, v) = (u, v, 5 - 2u + 2v)$, $0 \leq u \leq 1$, $0 \leq v \leq 1$ be a parametrization of S . Then,

$$\overrightarrow{\mathbf{T}}_u = \langle 1, 0, -2 \rangle, \quad \overrightarrow{\mathbf{T}}_v = \langle 0, 1, 2 \rangle, \quad \overrightarrow{\mathbf{n}}(u, v) = \overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v = \langle 2, -2, 1 \rangle$$

The surface integral is

$$\begin{aligned} \iint_S \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}} &= \iint_D \overrightarrow{\mathbf{F}}(\Phi(u, v)) \cdot \overrightarrow{\mathbf{n}}(u, v) dA = \int_0^1 \int_0^1 \langle u, v, 5 - 2u + 2v \rangle \cdot \langle 2, -2, 1 \rangle du dv \\ &= \int_0^1 \int_0^1 (2u - 2v + 5 - 2u + 2v) du dv = 5 \end{aligned}$$