MATH 210 - Final Exam, Spring 2008
Answers

1. (a) Let $F_1 = y^2$ and $F_2 = 2xy + 2y$. Since $\frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}$, we know that $\overrightarrow{\mathbf{F}}$ is conservative.

   (b) The function $\varphi(x,y) = xy^2 + y^2$ has the property that $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\varphi$.

   (c) The integral is path independent because $\overrightarrow{\mathbf{F}}$ is conservative. Therefore,

$$\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}} = \varphi(3,0) - \varphi(-1,2) = 0 - 0 = 0$$

2. (a) The velocity and acceleration are $\overrightarrow{\mathbf{v}}(t) = \langle \pi\cos(\pi t), 2t, 1\rangle$, $\overrightarrow{\mathbf{a}}(t) = \langle -\pi^2\sin(\pi t), 2, 0\rangle$. At $t = 2$, we have $\overrightarrow{\mathbf{v}}(2) = \langle \pi, 4, 1\rangle$ and $\overrightarrow{\mathbf{a}}(2) = \langle 0, 2, 0\rangle$. The speed at $t = 2$ is $||\overrightarrow{\mathbf{v}}(2)|| = \sqrt{\pi^2 + 17}$.

   (b) The gradient of f is $\overrightarrow{\nabla} f = \langle e^{x+y}(\sin(xy) + y\cos(xy)), e^{x+y}(\sin(xy) + x\cos(xy))\rangle$. At the point $(\pi, 1)$ we have $\overrightarrow{\nabla} f(\pi,1) = \langle -e^{\pi+1}, -\pi e^{\pi+1}\rangle$. The directional derivative in the direction of $\overrightarrow{\mathbf{v}} = \langle 4, 0\rangle$ is $D_{\mathbf{u}} f = \overrightarrow{\nabla} f(\pi,1)\cdot\frac{1}{||\overrightarrow{\mathbf{v}}||}\overrightarrow{\mathbf{v}} = \langle -e^{\pi+1}, -\pi e^{\pi+1}\rangle \cdot \langle 1, 0\rangle = -e^{\pi+1}$.

3. The volume of the region, using cylindrical coordinates, is

$$V = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta = \frac{2\pi}{3}(8 - 3\sqrt{3})$$

4. To find the critical points we solve the system $f_x = 2x + 2xy = 0$ and $f_y = 2y + x^2 = 0$. From the first equation we have either $x = 0$ or $y = -1$. If $x = 0$ then the second equation gives us $y = 0$. If $y = -1$ then the second equation gives us $x = \pm\sqrt{2}$. So the critical points are $(0,0)$, $(\sqrt{2}, -1)$, and $(-\sqrt{2}, -1)$.

   The discriminant function is $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (2+2y)(2) - (2x)^2 = 4 + 4y - 4x^2$. At the critical points we have $D(0,0) = 4$, $D(\sqrt{2}, -1) = -8$, and $D(-\sqrt{2}, -1) = -8$. Since $D(\pm\sqrt{2}, -1) < 0$, these are saddle points. Since $D(0,0) > 0$ and $f_{xx}(0,0) = 2 > 0$, $(0,0)$ corresponds to a local minimum.

5. The region $\mathcal{D}$ is bounded above by the parabola $y = 1 - x^2$ and below by the line $y = 1 - x$. The curves intersect at $x = 0$ and $x = 1$. Thus, the value of the integral of $f(x,y) = x + 3$ is

$$\iint_{\mathcal{D}} (x+3)\, dA = \int_0^1 \int_{1-x}^{1-x^2} (x+3)\, dy\, dx = \frac{7}{12}$$

6. Start with the parametrization $\Phi(u, v) = (u, v, 6-u-v)$ with domain $\mathcal{D} = \{(x, y) \mid 0 \leq u \leq 2,\ 1 \leq v \leq 3\}$. The tangent vectors are $\overrightarrow{\mathbf{T}}_u = \frac{\partial \Phi}{\partial u} = \langle 1, 0, -1 \rangle$ and $\overrightarrow{\mathbf{T}}_v = \frac{\partial \Phi}{\partial v} = \langle 0, 1, -1 \rangle$. Not surprisingly, we get the normal vector $\overrightarrow{\mathbf{n}}(u, v) = \overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v = \langle 1, 1, 1 \rangle$. Note that this vector points upward since the z-component is positive. Writing $\overrightarrow{\mathbf{F}}$ in terms of u and v we have $\overrightarrow{\mathbf{F}}(\Phi(u, v)) = \langle u, v, 6-u-v \rangle$. Thus, the value of the integral is

$$\iint_{\mathcal{S}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}} = \int_0^3 \int_0^2 \langle u, v, 6-u-v \rangle \cdot \langle 1, 1, 1 \rangle \, du \, dv = 36$$

7. Begin by forming two vectors in the plane. Let $\overrightarrow{\mathbf{u}} = \langle -3-1, 0-2, 1-1 \rangle = \langle -4, -2, 0 \rangle$ and $\overrightarrow{\mathbf{v}} = \langle 2-1, 2-2, 0-1 \rangle = \langle 1, 0, -1 \rangle$. Then $\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = \langle 2, -4, 2 \rangle$ is perpendicular to the plane containing the three points. Using this vector and the point $(1, 2, 1)$ we have $2(x-1) - 4(y-2) + 2(z-1) = 0$ as an equation for the plane.

8. First note that $\mathcal{C}$ is oriented CCW. Now let $P = e^{2x} + y$ and $Q = 4x + \sin(y^2)$. Then $\frac{\partial Q}{\partial x} = 4$ and $\frac{\partial P}{\partial y} = 1$. So Green's Theorem tells us that

$$\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{\mathcal{D}} (4-1) \, dA = 3 \iint_{\mathcal{D}} dA$$

In other words, the value of the line integral is 3 times the area of $\mathcal{D}$. The region is comprised of 64 boxes each of which has an area of $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$. So the area of $\mathcal{D}$ is $64 \cdot \frac{1}{16} = 4$ and the value of the integral is $\oint_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}} = 3(4) = 12$.