MATH 210
Sample exam problems for the final exam
Fall 2009
"Answers"

1. (a) $\overrightarrow{AB} = \langle 0, -4, 3\rangle$, $\overrightarrow{AC} = \langle 1, -2, 2\rangle \quad \Longrightarrow \quad \cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\left|\overrightarrow{AB}\right|\right|\ \left|\left|\overrightarrow{AC}\right|\right|} = \dfrac{14}{(5)(3)} = \dfrac{14}{15}$

   (b) A normal vector is $\overrightarrow{\mathbf{n}} = \overrightarrow{AB}\times\overrightarrow{AC} = \langle -2, 3, 4\rangle$. Using A as a point on the plane, an equation for the plane containing the triangle is

$$-2(x-1)+3(y-1)+4(z-1)=0$$

2. The critical points must satisfy the equations

$$f_x = 2x+2xy = 0, \quad f_y = 2y + x^2 = 0$$

From the first equation we have $x(1+y)=0$. So we have either $x=0$ or $y=-1$. When $x=0$, the second equation gives us $y=0$. When $y=-1$, the second equation gives us $x=\pm\sqrt{2}$. The critical points are $(0,0)$, $(\sqrt{2},-1)$, and $(-\sqrt{2},-1)$.

The second derivatives are $f_{xx} = 2+2y$, $f_{yy}=2$, and $f_{xy}=2x$. So,

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4+4y-4x^2$$

At the critical points we have

$$\begin{aligned} D(0,0) = 4 > 0, \quad f_{xx}(0,0) = 2 > 0 \quad &\Longrightarrow \quad \text{local minimum} \\ D(\sqrt{2},-1) = -8 < 0 \quad &\Longrightarrow \quad \text{saddle point} \\ D(-\sqrt{2},-1) = -8 < 0 \quad &\Longrightarrow \quad \text{saddle point} \end{aligned}$$

3. The gradient of f is $\overrightarrow{\nabla} f = \langle e^x\sin(xy)+ye^x\cos(xy), xe^x\cos(xy)\rangle$. At the point $(0,\pi)$ we have $\overrightarrow{\nabla} f(0,\pi) = \langle \pi, 0\rangle$. The directional derivative in the direction of $\overrightarrow{\mathbf{v}} = \langle 1, 0\rangle$ is

$$D_{\mathbf{v}}f(0,\pi) = \overrightarrow{\nabla} f(0,\pi)\cdot\overrightarrow{\mathbf{v}} = \pi$$

The direction of maximal increase of f at $(0,\pi)$ is $\hat{\mathbf{u}} = \dfrac{\overrightarrow{\nabla} f(0,\pi)}{\left|\left|\overrightarrow{\nabla} f(0,\pi)\right|\right|} = \langle 1, 0\rangle$.

4. We have $\overrightarrow{\mathbf{r}}'(t) = \langle -\pi\sin(\pi t), 2t, 0\rangle$ and $\overrightarrow{\mathbf{r}}''(t) = \langle -\pi^2\cos(\pi t), 2, 0\rangle$. At $t=2$ we have $\overrightarrow{\mathbf{r}}'(2) = \langle 0, 4, 0\rangle$ and $\overrightarrow{\mathbf{r}}''(2) = \langle -\pi^2, 2, 0\rangle$. The unit tangent vector is

$$\overrightarrow{\mathbf{T}}(2) = \frac{\overrightarrow{\mathbf{r}}'(2)}{\left|\left|\overrightarrow{\mathbf{r}}'(2)\right|\right|} = \frac{\langle 0, 4, 0\rangle}{4} = \langle 0, 1, 0\rangle$$

The curvature is $\kappa(2) = \dfrac{\left|\left|\vec{\mathbf{r}}\,'(2) \times \vec{\mathbf{r}}\,''(2)\right|\right|}{\left|\left|\vec{\mathbf{r}}\,'(2)\right|\right|^3} = \dfrac{||\langle 0,0,4\pi^2\rangle||}{||\langle 0,4,0\rangle||^3} = \dfrac{4\pi^2}{4^3} = \dfrac{\pi^2}{16}$.

5. Using polar coordinates we have

$$\iint_D e^{-(x^2+y^2)}\,dA = \int_0^{\pi/2}\int_0^1 e^{-r^2} r\,dr\,d\theta = \frac{\pi}{4}\left(1-e^{-1}\right)$$

6. The vector from $(1,1,0)$ to $(2,0,-1)$ is $\vec{\mathbf{v}} = \langle 1,-1,-1\rangle$ and is parallel to the line. Using $(1,1,0)$ as a point on the line, a set of parametric equations for the line is $x = 1+t$, $y = 1-t$, and $z = -t$. A parametrization of the line segment and its derivative are

$$\begin{aligned}\vec{\mathbf{c}}\,(t) &= (1+t, 1-t, -t), \quad 0 \le t \le 1\\ \vec{\mathbf{c}}\,'(t) &= \langle 1,-1,-1\rangle\end{aligned}$$

The vector $\vec{\mathbf{F}}$ written in terms of t is $\vec{\mathbf{F}}\left(\vec{\mathbf{c}}\,(t)\right) = \langle 1-2t, 1, 2\rangle$. The line integral is then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_0^1 \vec{\mathbf{F}}\left(\vec{\mathbf{c}}\,(t)\right) \cdot \vec{\mathbf{c}}\,'(t)\,dt = \int_0^1 (-2t-2)\,dt = -3$$

7. (a) Let $f(x,y) = 4-x^2-y^2$. Then $f_x = -2x$ and $f_y = -2y$. When $x = 1$ and $y = 2$ we have $f_x(1,2) = -2$ and $f_y(1,2) = -4$. An equation for the tangent plane is

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \quad \Longrightarrow \quad z = -1-2(x-1)-4(y-2)$$

(b) Using Cylindrical Coordinates, the volume is

$$V = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta = 8\pi$$

8. (a) Since $\dfrac{\partial Q}{\partial x} = 2x$ and $\dfrac{\partial P}{\partial y} = Bx$, we need $B = 2$. A potential function is then $\varphi(x,y) = x^2y + x + y^2$.

(b) Since $\vec{\mathbf{F}} = \vec{\nabla}\varphi$, we know that the vector field is conservative. Therefore,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \varphi(-1,0) - \varphi(1,0) = -2$$

9. (a) The partial derivatives are

$$f_x = x\cos(x+2y) + \sin(x+2y), \quad f_y = 2x\cos(x+2y)$$

$$f_{xx} = 2\cos(x+2y) - x\sin(x+2y), \quad f_{yy} = -4x\sin(x+2y)$$
$$f_{xy} = 2\cos(x+2y) - 2x\sin(x+2y)$$

(b) The partials f_s and f_t are

$$\begin{aligned} f_s &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2s\left(x\cos(x+2y) + \sin(x+2y)\right) + 2\left(2x\cos(x+2y)\right) \\ f_t &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \left(x\cos(x+2y) + \sin(x+2y)\right) + 3t^2\left(2x\cos(x+2y)\right) \end{aligned}$$

10. Let $f(x,y) = x^2 + y^2$ and $g(x,y) = x^2 + xy + y^2$. Then, using the method of Lagrange multipliers we must solve the equations:

$$2x = \lambda(2x+y), \quad 2y = \lambda(x+2y), \quad x^2 + xy + y^2 = 9$$

From the first two equations we have $\lambda = \dfrac{2x}{2x+y}$ and $\lambda = \dfrac{2y}{x+2y}$. Setting these equal to each other and simplifying we get $x^2 = y^2 \implies x = y,\ x = -y$. We then use the third equation as follows:

$$\begin{aligned} x = y &\quad\implies\quad y^2 + y^2 + y^2 = 9 \quad\implies\quad y = \pm\sqrt{3} \\ x = -y &\quad\implies\quad y^2 - y^2 + y^2 = 9 \quad\implies\quad y = \pm 3 \end{aligned}$$

The critical points are $(\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3})$, $(3,3)$, and $(-3,-3)$. The function values are

$$f(\pm\sqrt{3}, \pm\sqrt{3}) = 6 \ \ \text{(minimum)}, \quad f(\pm 3, \pm 3) = 18 \ \ \text{(maximum)}$$

11. Changing the order of integration and evaluating we have

$$\int_0^1 \int_y^1 e^{-x^2}\, dx\, dy = \int_0^1 \int_0^x e^{-x^2}\, dy\, dx = \frac{1}{2}(1 - e^{-1})$$

12. We have $f\left(\overrightarrow{\mathbf{c}}(t)\right) = 3t\sqrt{(4\cos t)^2 + (4\sin t)^2} = 12t$ and $\left|\left|\overrightarrow{\mathbf{c}}'(t)\right|\right| = 5$. Then line integral is then

$$\int_{\mathcal{C}} f(x,y,z)\, ds = \int_0^{2\pi} f\left(\overrightarrow{\mathbf{c}}(t)\right) \left|\left|\overrightarrow{\mathbf{c}}'(t)\right|\right|\, dt = \int_0^{2\pi} 60t\, dt = 120\pi^2$$

13. (a) We need $\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{w}} = 1 + 2 + a = 0 \implies a = -3$.

(b) The area of the parallelogram is $\left|\left|\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}}\right|\right| = \sqrt{(2-a)^2 + (a-1)^2 + (-1)^2} = \sqrt{6}$. So, $(2-a)^2 + (a-1)^2 + 1 = 6$. The two solutions are $a = 0$ and $a = 3$.

14. (a) $\overrightarrow{\mathbf{r}}'(t) = \langle \pi\cos(\pi t), 2t, 1\rangle$, $\overrightarrow{\mathbf{r}}''(t) = \langle -\pi^2\sin(\pi t), 2, 0\rangle$

(b) The distance traveled is $D = \displaystyle\int_0^4 \left|\left|\overrightarrow{\mathbf{r}}'(t)\right|\right| \, dt = \int_0^4 \sqrt{\pi^2\cos^2(\pi t) + 4t^2 + 1}\, dt$.

15. Using Cylindrical Coordinates, the volume is

$$V = \int_0^{2\pi}\int_0^2\int_0^{4 - r\sin\theta} r\,dz\,dr\,d\theta = 16\pi$$

16. Using Green's Theorem, the integral is

$$\oint_{\mathcal{C}} xy\,dx + y^5\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \int_0^2\int_0^{x/2} (0 - x)\;dy\,dx = -\frac{4}{3}$$

17. (a) Let $\overrightarrow{AB} = \langle 1, 1, 1\rangle$ and $\overrightarrow{AC} = \langle 0, 0, 2\rangle$. Then $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 2, -2, 0\rangle$ is perpendicular to P. Turning this into a unit vector we have $\hat{\mathbf{u}} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\rangle$

(b) Let $\overrightarrow{\mathbf{n}} = \langle 2, -2, 0\rangle$ be normal to the plane. Then an equation for the plane is $2(x-1) - 2(y-0) + 0(z-0) = 0 \implies x - y = 1$. A set of parametric equations for the line perpendicular to the plane is $x = 1 + 2t$, $y = 1 - 2t$, and $z = 1$. Plugging these into the equation for the plane and solving for t we get

$$x - y = 1 \quad\implies\quad 1 + 2t - 1 + 2t = 1 \quad\implies\quad t = \frac{1}{4}$$

At $t = 1/4$, we have

$$x = 1 + 2\left(\frac{1}{4}\right) = \frac{3}{2}, \quad y = 1 - 2\left(\frac{1}{4}\right) = \frac{1}{2}, \quad z = 1$$

18. Using Cylindrical Coordinates, the volume is

$$V = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = 2\pi\left(\frac{8}{3} - \sqrt{3}\right)$$

19. (a) A parametrization is $\Phi(u, v) = (u\cos v, u\sin v, u)$, $0 \le u \le 4$, $0 \le v \le 2\pi$.

(b) The tangent vectors are

$$\overrightarrow{\mathbf{T}}_u = \frac{\partial\Phi}{\partial u} = \langle \cos v, \sin v, 1\rangle\,, \quad \overrightarrow{\mathbf{T}}_v = \frac{\partial\Phi}{\partial v} = \langle -u\sin v, u\cos v, 0\rangle$$

The normal vector is $\overrightarrow{\mathbf{n}}(u,v) = \overrightarrow{\mathbf{T}}_u \times \overrightarrow{\mathbf{T}}_v = \langle -u\cos v, -u\sin v, u\rangle$ and its magnitude is $\left|\left|\overrightarrow{\mathbf{n}}(u,v)\right|\right| = u\sqrt{2}$. The surface area is

$$SA = \iint_{\mathcal{S}} dS = \iint_{\mathcal{D}} \left|\left|\overrightarrow{\mathbf{n}}(u,v)\right|\right| \,du\,dv = \int_0^{2\pi}\int_0^4 u\sqrt{2}\,du\,dv = 16\pi\sqrt{2}$$

20. Since $x = t$, $y = t^2$, and $z = t^3$ we have $dx = dt$, $dy = 2t\,dt$, and $dz = 3t^2\,dt$. The line integral is then

$$\int_{\mathcal{C}} y\,dx + (x+z)\,dy + y\,dz = \int_0^1 t^2\,dt + (t+t^3)(2t\,dt) + t^2(3t^2\,dt) = \int_0^1 (5t^4 + 3t^2)\,dt = 2$$