

1. The position vector

$$\vec{\mathbf{r}}(t) = t^3 \hat{\mathbf{i}} + 3\sqrt{2}t \hat{\mathbf{j}} + 3t^{-1} \hat{\mathbf{k}}, \quad 1 \leq t \leq 2$$

describes the motion of a particle.

- (a) Find the position at time $t = 2$.
 (b) Find the velocity at time $t = 2$.
 (c) Find the acceleration at time $t = 2$.
 (d) Find the length of the path traveled by the particle during the time $1 \leq t \leq 2$.

Solution:

(a) $\vec{\mathbf{r}}(2) = 8 \hat{\mathbf{i}} + 6\sqrt{2} \hat{\mathbf{j}} + \frac{3}{2} \hat{\mathbf{k}}$

- (b) The velocity is

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}'(t) = 3t^2 \hat{\mathbf{i}} + 3\sqrt{2} \hat{\mathbf{j}} - 3t^{-2} \hat{\mathbf{k}}$$

At $t = 2$ we have

$$\vec{\mathbf{v}}(2) = 12 \hat{\mathbf{i}} + 3\sqrt{2} \hat{\mathbf{j}} - \frac{3}{4} \hat{\mathbf{k}}$$

- (c) The acceleration is

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}''(t) = 6t \hat{\mathbf{i}} + 6t^{-3} \hat{\mathbf{k}}$$

At $t = 2$ we have

$$\vec{\mathbf{a}}(2) = 12 \hat{\mathbf{i}} + \frac{3}{4} \hat{\mathbf{k}}$$

- (d) The length of the path traveled is given by the equation

$$L = \int_a^b \|\vec{\mathbf{r}}'(t)\| dt$$

The speed $\|\vec{\mathbf{r}}'(t)\|$ is given by

$$\begin{aligned} \|\vec{\mathbf{r}}'(t)\| &= \sqrt{(3t^2)^2 + (3\sqrt{2})^2 + (-3t^{-2})^2} \\ &= \sqrt{9t^4 + 18 + 9t^{-4}} \\ &= 3\sqrt{t^4 + 2 + t^{-4}} \\ &= 3\sqrt{(t^2 + t^{-2})^2} \\ &= 3(t^2 + t^{-2}) \end{aligned}$$

Therefore, the length of the path traveled is

$$\begin{aligned} L &= \int_1^2 3(t^2 + t^{-2}) dt \\ &= 3 \left[\frac{1}{3}t^3 - \frac{1}{t} \right]_1^2 \\ &= 3 \left[\left(\frac{1}{3} \cdot 2^3 - \frac{1}{2} \right) - \left(\frac{1}{3} \cdot 1^3 - \frac{1}{1} \right) \right] \\ &= \boxed{\frac{17}{2}} \end{aligned}$$

2. (a) For $f(x, y) = e^{(x+1)y}$, find the derivatives:

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y^2}$$

- (b) Find the gradient of f at the point $(2, 3)$.

Solution:

- (a) The derivatives are:

$$\begin{aligned} \frac{\partial f}{\partial x} &= ye^{(x+1)y}, & \frac{\partial f}{\partial y} &= (x+1)e^{(x+1)y} \\ \frac{\partial^2 f}{\partial x^2} &= y^2e^{(x+1)y}, & \frac{\partial^2 f}{\partial x \partial y} &= y(x+1)e^{(x+1)y}, & \frac{\partial^2 f}{\partial y^2} &= (x+1)^2e^{(x+1)y} \end{aligned}$$

- (b) At the point $(2, 3)$ we have

$$\vec{\nabla} f(2, 3) = \left\langle \frac{\partial f}{\partial x}(2, 3), \frac{\partial f}{\partial y}(2, 3) \right\rangle = \left\langle 3e^{(2+1) \cdot 3}, (2+1)e^{(2+1) \cdot 3} \right\rangle = \boxed{\langle 3e^9, 3e^9 \rangle}$$

3. (a) Find a potential function for the vector field

$$\vec{\mathbf{F}}(x, y, z) = (1-z)\hat{\mathbf{i}} + y\hat{\mathbf{j}} - x\hat{\mathbf{k}}$$

- (b) Integrate $\vec{\mathbf{F}}$ over the straight line from $(1, 0, 1)$ to $(0, 1, 2)$.

Solution:

- (a) One can show that a potential function is

$$\boxed{\varphi(x, y, z) = x(1-z) + \frac{1}{2}y^2}$$

To verify, we take the gradient of φ :

$$\vec{\nabla} \varphi = \left\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right\rangle = \langle 1-z, y, -x \rangle = \vec{\mathbf{F}}$$

(b) The line integral of $\vec{\mathbf{F}}$ is then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \varphi(0, 1, 2) - \varphi(1, 0, 1) = \left[0 \cdot (1 - 2) + \frac{1}{2} \cdot 1^2 \right] - \left[1 \cdot (1 - 1) + \frac{1}{2} \cdot 0^2 \right] = \boxed{\frac{1}{2}}$$

4. (a) Find the critical points of the function $f(x, y) = x^3 - 3x - y^2$.
(b) Use the second derivative test to classify each critical point as a local maximum, a local minimum, or a saddle point.

Solution:

(a) The critical points are solutions to $f_x = 0$ and $f_y = 0$.

$$\begin{aligned} f_x &= 3x^2 - 3 = 3(x - 1)(x + 1) = 0 \\ f_y &= -2y = 0 \end{aligned}$$

The solutions are $\boxed{(1, 0), (-1, 0)}$.

(b) The second derivatives of $f(x, y)$ are

$$f_{xx} = 6x, \quad f_{yy} = -3, \quad f_{xy} = 0$$

Therefore, the discriminant function is

$$D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = -18x$$

The values of D at the critical points are:

$$D(1, 0) = -18, \quad D(-1, 0) = 18$$

Since $D < 0$ at $(1, 0)$, the point corresponds to a $\boxed{\text{saddle point}}$. Since $D > 0$ and $f_{xx} = -6 < 0$ at $(-1, 0)$, the point corresponds to a $\boxed{\text{local maximum}}$.

5. Find the maximum and minimum of the function $f(x, y) = (x - 1)^2 + y^2$ subject to the constraint

$$g(x, y) = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

Solution: We will use the method of Lagrange multipliers to solve the problem. The equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g(x, y) = 1$ will give us the system of equations:

$$\begin{aligned} 2(x - 1) &= \lambda \left(\frac{2}{9}x\right) \\ 2y &= \lambda \left(\frac{1}{2}y\right) \\ \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 &= 1 \end{aligned}$$

The second equation has two solutions: $y = 0$ or $\lambda = 4$. When $y = 0$, the third equation gives us $x = \pm 3$. Therefore, two critical points are:

$$(3, 0), \quad (-3, 0)$$

When $\lambda = 4$, the first equation gives us

$$2(x - 1) = 4 \left(\frac{2}{9}x \right) \iff x = \frac{9}{5}$$

The third equation then gives us

$$\left(\frac{9}{5} \right)^2 + \left(\frac{y}{2} \right)^2 = 1 \iff y = \pm \frac{14}{15}$$

Therefore, the other two critical points are:

$$\left(\frac{9}{5}, \frac{14}{15} \right), \quad \left(\frac{9}{5}, -\frac{14}{15} \right)$$

The values of f at the critical points are:

$$f(3, 0) = 4, \quad f(-3, 0) = 16, \quad f\left(\frac{9}{5}, \pm \frac{14}{15}\right) = \frac{16}{5}$$

Therefore, the minimum value of f is $\boxed{\frac{13}{5}}$ and the maximum value of f is $\boxed{16}$.

6. Compute the integral

$$\iint_{\mathcal{R}} xy \, dx \, dy$$

over the quarter circle $\mathcal{R} = \{(x, y) : 0 \leq x, 0 \leq y, x^2 + y^2 \leq 1\}$.

Solution: Using $x = r \cos \theta$, $y = r \sin \theta$, and $dx \, dy = r \, dr \, d\theta$ we have:

$$\begin{aligned} \iint_{\mathcal{R}} xy \, dx \, dy &= \int_0^{\pi/2} \int_0^1 (r \cos \theta)(r \sin \theta)r \, dr \, d\theta \\ &= \int_0^{\pi/2} \int_0^1 r^3 \sin \theta \cos \theta \, dr \, d\theta \\ &= \int_0^{\pi/2} \sin \theta \cos \theta \left[\frac{1}{4}r^4 \right]_0^1 \, d\theta \\ &= \frac{1}{4} \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta \\ &= \frac{1}{4} \left[\frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} \\ &= \boxed{\frac{1}{8}} \end{aligned}$$

7. Compute the integral

$$\iiint_{\mathcal{W}} 1 \, dx \, dy \, dz$$

over the tetrahedron

$$\mathcal{W} = \{(x, y, z) : 0 \leq x, 0 \leq y, 0 \leq z, x/3 + y/5 + z/7 \leq 1\}$$

Solution: The value of the integral is

$$\begin{aligned} \iiint_{\mathcal{W}} 1 \, dx \, dy \, dz &= \int_0^5 \int_0^{(15-5x)/3} \int_0^{(105-21y-35x)/15} 1 \, dz \, dy \, dx \\ &= \int_0^5 \int_0^{(15-5x)/3} \frac{105 - 21y - 35x}{15} \, dy \, dx \\ &= \frac{1}{15} \int_0^5 \left[105y - \frac{21}{2}y^2 - 35xy \right]_0^{(15-5x)/3} \, dx \\ &= \frac{1}{15} \int_0^5 \left[105 \left(\frac{15-5x}{3} \right) - \frac{21}{2} \left(\frac{15-5x}{3} \right)^2 - 35x \left(\frac{15-5x}{3} \right) \right] \, dx \\ &= 35 \int_0^5 \left(\frac{1}{18}x^2 - \frac{1}{3}x + \frac{1}{2} \right) \, dx \\ &= 35 \left[\frac{1}{54}x^3 - \frac{1}{6}x^2 + \frac{1}{2}x \right]_0^5 \\ &= 35 \left(\frac{1}{54} \cdot 5^3 - \frac{1}{6} \cdot 5^2 + \frac{1}{2} \cdot 5 \right) \\ &= \boxed{\frac{1225}{54}} \end{aligned}$$

8. Find an equation for the tangent plane to the surface defined by $xy^2 + 2z^2 = 12$ at the point $(1, 2, 2)$.

Solution: Let $F(x, y, z) = xy^2 + 2z^2$. The gradient of F is

$$\vec{\nabla} F = \langle y^2, 2xy, 4z \rangle$$

At the point $(1, 2, 2)$ we have

$$\vec{\mathbf{n}} = \vec{\nabla} F(1, 2, 2) = \langle 4, 4, 8 \rangle$$

This vector is perpendicular to the tangent plane. Using this vector and the point $(1, 2, 2)$, an equation for the tangent plane is

$$\boxed{4(x-1) + 8(y-2) + 8(z-2) = 0}$$

9. Compute the integral

$$\oint (3x^2 + y) \, dx + (x^2 + y^3) \, dy$$

over the counterclockwise boundary of the rectangle

$$\mathcal{R} = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$$

Solution: Green's Theorem is appropriate here. First, we recognize that $P = 3x^2 + y$ and $Q = x^2 + y^3$. Then,

$$\frac{\partial Q}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 1$$

The value of the integral is

$$\begin{aligned} \oint (3x^2 + y) dx + (x^2 + y^3) dy &= \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \int_0^2 \int_0^3 (2x - 1) dy dx \\ &= \int_0^2 3(2x - 1) dx \\ &= \int_0^2 (6x - 3) dx \\ &= 3x^2 - 3x \Big|_0^2 \\ &= \boxed{6} \end{aligned}$$