MATH 210

A collection of extra problems

Prepared by the Math 210 instructors of Spring 2012

The goal of the present collection of problems is to extend the student's understanding and appreciation of the material covered in Math 210. It is not to serve as a study guide for the midterm or the final exams. It is our hope that the student will find in these pages some interesting examples that further illustrate the concepts that are taught in class.

The problems are provided with full solutions but the interested student is invited to try them first without looking at the solution.

Problem 1: Find a non-zero vector that is perpendicular to both $\langle 1, 2, 1\rangle$ and $\langle 0, 2, 1\rangle$.

Solution: Let us denote this unknown vector by $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Recall that two non-zero vectors are perpendicular if they are orthogonal, that is, if their dot product is zero.

Therefore, we are looking for a vector $\mathbf{v}$ such that $\mathbf{v} \cdot \langle 1, 1, 1\rangle = 0$ and $\mathbf{v} \cdot \langle 0, 1, 1\rangle = 0$. By the theorem from chapter 11.3,

$$\mathbf{v} \cdot \langle a_1, a_2, a_3\rangle = v_1 a_1 + v_2 a_2 + v_3 a_3.$$

Thus, we can rewrite the previous two equation as

$$\mathbf{v} \cdot \langle 1, 1, 1\rangle = v_1 + 2v_2 + v_3 = 0$$

and

$$\mathbf{v} \cdot \langle 0, 1, 1\rangle = 2v_2 + v_3 = 0$$

There are infinitely many solutions to this pair of equations. We just have to find *one* non-zero solution.

The second equation yields $v_3 = -2v_2$, and by substituting in the first equation we get $v_1 = 0$. Therefore, we may take $v_1 = 0, v_2 = 1, v_3 = -2$. It is easy to check that $\mathbf{v} = \langle 0, 1, -2\rangle$ is indeed orthogonal to the two given vectors.

Problem 2: A 600-pound boat sits on a ramp inclined at 30°. What force is required to keep the boat from rolling down the ramp?

Solution: Since the force due to gravity is vertical and downward, we represent the gravitational force by the vector

$$\mathbf{F} = -600\mathbf{j}.$$

To find the force required to keep the boat from rolling down the ramp, we project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$\mathbf{v} = \cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is given by

$$\begin{aligned} \text{proj}_{\mathbf{v}}\mathbf{F} &= \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})\,\mathbf{v} \\ &= -300\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right). \end{aligned}$$

The magnitude of this force is 300, and therefore a force of 300 pounds is required to keep the boat from rolling down the ramp.

Problem 3: Find real numbers a and b so that the vector $\langle a-11, 2, b\rangle$ is parallel to the vector $\langle b, 1, 3a\rangle$.

Solution: Let us first recall that two nonzero vectors **u** and **v** are parallel if there exists a number λ such that:

$$\mathbf{u} = \lambda \mathbf{v}$$

This is because one can always scale a nonzero vector in order to get any vector which is parallel to it. This means that the vector $\langle a-11, 2, b\rangle$ will be parallel to the vector $\langle b, 1, 3a\rangle$ if we can find a real number λ such that:

$$\langle a-11, 2, b\rangle = \lambda\langle b, 1, 3a\rangle$$

or equivalently

$$\langle a-11, 2, b\rangle = \langle \lambda b, \lambda, 3\lambda a\rangle$$

Since equal vectors must have the same coordinates, we must have $2 = \lambda$. This means that the above equation can be rewritten as:

$$\langle a-11, 2, b\rangle = \langle 2b, 2, 6a\rangle$$

By comparing the coordinates one by one we get:

$$a - 11 = 2b \quad \text{and} \quad b = 6a$$

By plugging the second equation into the first, we get

$$a - 11 = 12a$$

which yields $a = -1$. Once we know this, we easily find $b = -6$.

Problem 4: Find a unit vector that is orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$.

Solution: Two vectors are orthogonal is their dot product is 0. So we're looking for a unit vector $\mathbf{v}$ such that

$$\begin{aligned} \mathbf{v}\cdot(\mathbf{i}+\mathbf{j}) = \mathbf{v}\cdot\langle 1,1,0\rangle &= 0 \\ \mathbf{v}\cdot(\mathbf{i}+\mathbf{k}) = \mathbf{v}\cdot\langle 1,0,1\rangle &= 0. \end{aligned}$$

Denote by $\mathbf{v}=\langle a,b,c\rangle$. Then we need to find a,b,c such that

$$\begin{aligned} a+b &= 0 \\ a+c &= 0. \end{aligned}$$

Note that not all of a,b,c can be 0 because we want $\mathbf{v}$ to be a unit vector. Solving the system above, we find that the vector

$$\langle -c,c,c\rangle,$$

is orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$ for any real number c. We want the vector $\mathbf{v}$ to be a unit vector, so let

$$\mathbf{v}=\frac{\langle -c,c,c\rangle}{\sqrt{3c^2}}=\frac{\langle -1,1,1\rangle}{\sqrt{3}},$$

which is the vector we seek.

Problem 5: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ for some constants a and b find $\mathbf{w} \cdot \mathbf{u}$.

Solution:

$$\begin{aligned}
\mathbf{w} \cdot \mathbf{v} &= (a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{u} \\
&= a(\mathbf{u} \cdot \mathbf{u}) + b(\mathbf{v} \cdot \mathbf{u}), \qquad \text{(using the distributive property of the dot product)} \\
&= a|\mathbf{u}|^2 + 0, \qquad \text{(using } \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \text{ and the orthogonality of } \mathbf{u} \text{ and } \mathbf{v}\text{)} \\
&= a. \qquad \text{(since } \mathbf{u} \text{ is a unit vector)}
\end{aligned}$$

Problem 6: Consider nonzero vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ then does $\mathbf{v} = \mathbf{w}$? Justify your answer.

Solution: No, $\mathbf{v}$ need not necessarily equal $\mathbf{w}$.

As a counterexample take $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j}$. Then clearly $\mathbf{v} \neq \mathbf{w}$ but

$$\mathbf{i} \times (\mathbf{i} + \mathbf{j}) = (\mathbf{i} \times \mathbf{i}) + (\mathbf{i} \times \mathbf{j}) = \mathbf{k}.$$

and

$$\mathbf{i} \times (-\mathbf{i} + \mathbf{j}) = (\mathbf{i} \times (-\mathbf{i})) + (\mathbf{i} \times \mathbf{j}) = \mathbf{k}.$$

Problem 7: Find the limit.

$$\lim_{t\to 0}(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k})$$

Solution: We begin by using the fact that

$$\lim_{t\to 0}(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}) = \lim_{t\to 0} t^2\mathbf{i} + \lim_{t\to 0} 3t\mathbf{j} + \lim_{t\to 0}\frac{1-\cos t}{t}\mathbf{k}.$$

It is easy to see that

$$\lim_{t\to 0} t^2\mathbf{i} = 0$$

and

$$\lim_{t\to 0} 3t\mathbf{j} = 0.$$

However, at $t = 0$, the final term $\frac{1-\cos t}{t}\mathbf{k}$ is of the form $\frac{0}{0}$. Hence we use l'Hopital's rule to compute

$$\lim_{t\to 0}\frac{1-\cos t}{t}\mathbf{k} = \lim_{t\to 0}\frac{\sin t}{1}\mathbf{k} = 0.$$

Hence

$$\lim_{t\to 0}(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}) = \mathbf{0}.$$

Problem 8: Find all unit vectors that are perpendicular to the plane containing the points $(1, 1, 2)$, $(3, 0, -1)$ and $(2, 1, 0)$.

Solution: For the sake of clarity, let us assign letters to the points:

$$A(1, 1, 2) \qquad B(3, 0, -1) \qquad C(2, 1, 0)$$

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ belong to the plane. The idea is that any vector which is orthogonal to these two vectors is perpendicular to the plane that contains A, B and C. Also any two such vectors are parallel to each other. This means that there are going to be two unit vectors perpendicular to the given plane.

We will begin by taking the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This will provide us with a vector which is perpendicular to the plane. We first compute:

$$\overrightarrow{AB} = \langle 2, -1, -3 \rangle \qquad \overrightarrow{AC} = \langle 1, 0, -2 \rangle$$

Their cross product is:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -3 \\ 1 & 0 & -2 \end{vmatrix} = \langle 2, 1, 1 \rangle$$

According to what we explained above, the vector $\langle 2, 1, 1 \rangle$ is perpendicular to the plane containing A, B, C. Its length is $\sqrt{6}$. Thus, the only two *unit* vectors that are perpendicular to the given plane are:

$$\pm \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

Problem 9: Let $P = (3, 1, 0)$ and $Q = (4, 1, 3)$. Find all points on the line passing through P and Q whose distance from P is twice the distance from Q.

Solution: Let us denote by $\mathbf{v}$ the vector pointing from P to Q. Then

$$\mathbf{v} = \overrightarrow{PQ} = \langle 4-3, 1-1, 3-0\rangle = \langle 1, 0, 3\rangle.$$

We will denote by $\mathbf{r}_0$ the position vector pointing to P from the origin. Hence $\mathbf{r}_0 = \overrightarrow{OP} = \langle 3, 1, 0\rangle$. Recall that the line passing through P and Q is described by the position vectors

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}.$$

Moreover, we have seen (by an simple calculation) that $\mathbf{r}(1) = \mathbf{r}_0 + \mathbf{v} = \overrightarrow{OQ}$.

Let us denote by R a point on the line whose distance from P is twice its distance from Q (assuming such point exists). Let $\mathbf{r}(t)$ be a position vector pointing to R. The distance between P and R is given by the norm of the vector $\overrightarrow{PR}$. By the triangle rule

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \mathbf{r}(t) - \mathbf{r}_0 = t\mathbf{v}.$$

Hence, the distance between P and R is

$$|\vec{P}R| = |t\mathbf{v}| = |t||\mathbf{v}|.$$

Similarly,

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \mathbf{r}(t) - (\mathbf{r}_0 + \mathbf{v}) = (t-1)\mathbf{v},$$

and the distance between Q and R is

$$|\overrightarrow{QR}| = |t-1||\mathbf{v}|.$$

The condition that the distance from P to R is twice its distance from Q to R can now be written as the equation

$$|t||\mathbf{v}| = 2|t-1||\mathbf{v}|.$$

Since $|\mathbf{v}| \neq 0$, we can divide both sides and obtain the equation

$$|t| = 2|t-1|.$$

There are two solution to this equation: $t = 2$ and $t = 2/3$. Thus the only two points that satisfy the condition are those corresponding to $\mathbf{r}(2)$ and $\mathbf{r}(\frac{2}{3})$. Plugging these values into the definition of $\mathbf{r}$, we find that these points are $(5, 1, 6)$ and $(3\frac{2}{3}, 1, 2)$.

Credits

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