 Name: ____________________

Final Exam
Duration: 2 hours
Total: 100 points

The following rules apply:

- You are expected to abide by the University's rules concerning Academic Honesty.
- You may *not* use your books, notes, or any electronic device including cell phones.
- You must show all of your work. An answer, right or wrong, without the proper justification will receive little to no credit.
- You must complete your work in the space provided.

Check next to your instructor:

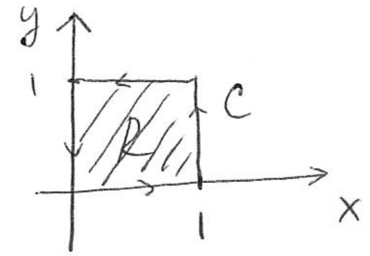
Kobotis	
Lukina @ 11am	
Lukina @ 2pm	
Cole	
Levine	
Steenbergen @ noon	
Steenbergen @ 2pm	
Xie	
Shvydkoy	

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

(10 pts) **1.** The square C with vertices at $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ is oriented counterclockwise. Compute the circulation of the vector field

$$F = \langle -1 + e^{x^2}, x^2y^5 + \cos(y^2) \rangle$$

around the square.



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

$$f = -1 + e^{x^2} \qquad \frac{\partial f}{\partial y} = 0$$

$$g = x^2y^5 + \cos y^2 \qquad \frac{\partial g}{\partial x} = 2xy^5$$

$$\iint_R 2xy^5 \, dA = \int_0^1 \int_0^1 2xy^5 \, dx\, dy = \int_0^1 2 \frac{x^2}{2} y^5 \Big|_0^1 dy =$$

$$\int_0^1 y^5 \, dy = \frac{y^6}{6} \Big|_0^1 = \frac{1}{6}$$

(10 pts) **2.** Given two vectors in space

$$\mathbf{u} = \langle 1, -1, 0 \rangle, \quad \mathbf{v} = \langle 1, 0, 1 \rangle$$

find the angle between them; compute the area of the parallelogram formed by the vectors.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

$$\vec{u} \cdot \vec{v} = \langle 1, -1, 0 \rangle \cdot \langle 1, 0, 1 \rangle = 1$$

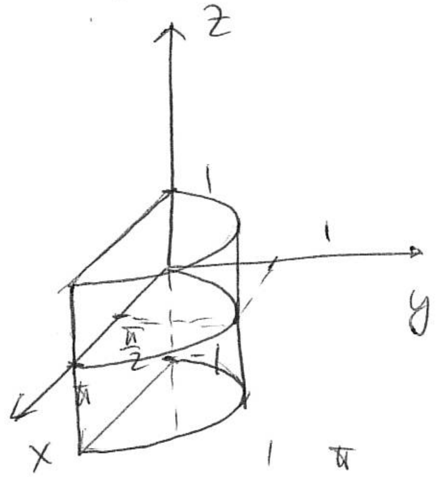
$$|\vec{u}| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2} \qquad |\vec{v}| = \sqrt{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \qquad \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \vec{i}(-1) - \vec{j} \cdot 1 + \vec{k}(-(-1)) = \langle -1, -1, 1 \rangle$$

$$\text{area} = |\vec{u} \times \vec{v}| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}$$

(10 pts) **3.** Find the volume of the region bounded by the cylinder $y = \sin x$ and restricted by $-1 \leq z \leq 1$, $0 \leq x \leq \pi$, $y \geq 0$.



$$D = \{(x,y,z) : -1 \leq z \leq 1,\ 0 \leq x \leq \pi,\ 0 \leq y \leq \sin x\}$$

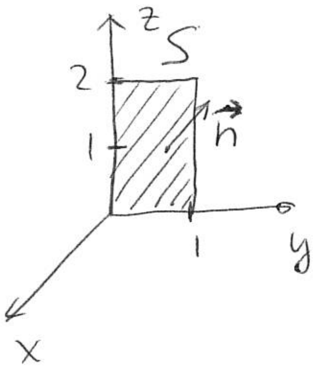
$$\iiint_D dV = \int_{-1}^{1}\int_{0}^{\pi}\int_{0}^{\sin x} dy\,dx\,dz =$$

$$\int_{-1}^{1}\int_{0}^{\pi} y\Big|_{0}^{\sin x} dx\,dz = \int_{-1}^{1}\int_{0}^{\pi} \sin x\,dx\,dz =$$

$$\int_{-1}^{1} -\cos x\Big|_{0}^{\pi} dz = \int_{-1}^{1} (-\cos\pi + \cos 0)\,dz =$$

$$2\int_{-1}^{1} dz = 2z\Big|_{-1}^{1} = 2-(-2) = 4.$$

(10 pts) **5.** A rectangular window given by $0 \le y \le 1$, $0 \le z \le 2$, $x = 0$, stands in the wind blowing with velocity field $F = \langle -1, 1, 0\rangle$. Calculate the flux of the wind through the window relative to the incoming direction (negative direction of the x-axis).



$u = y, \quad v = z$

$\vec{r}(u,v) = \langle 0, u, v\rangle,$

$0 \le u \le 1, \quad 0 \le v \le 2.$

$\vec{t}_u = \langle 0, 1, 0\rangle \qquad \vec{t}_v = \langle 0, 0, 1\rangle$

$\vec{t}_u \times \vec{t}_v = \langle 1, 0, 0\rangle$, opposite orientation to $\vec{n}$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^2 \langle -1, 1, 0\rangle \cdot \langle -1, 0, 0\rangle\,dv\,du = \int_0^1\int_0^2 dv\,du = \int_0^1 v\Big|_0^2 du = 2\int_0^1 du = 2u\Big|_0^1 = 2.$$

(10 pts) **4.** A surface S is given by equation

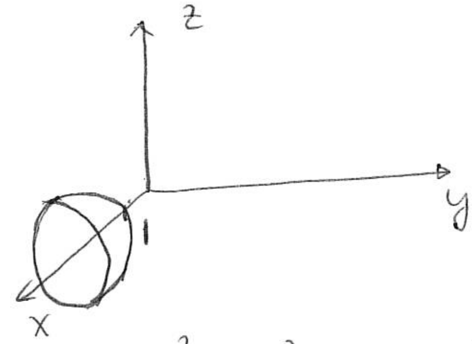
$$x = y^2 + z^2 + 1$$

(a) Describe the surface.

(b) Verify that the point $P(3, 1, 1)$ belongs to the surface, and find an equation of the tangent plane to S at point P.

(c) Find all points on the surface where the tangent plane is parallel to the yz-plane.

a)



an elliptic paraboloid

b) $1^2 + 1^2 + 1 = 3$, yes, $P(3,1,1)$ belongs to S.

$F(x,y,z) = x - y^2 - z^2 - 1 = 0$

$\nabla F(x,y,z) = \langle 1, 2y, -2z \rangle$ $\nabla F(3,1,1) = \langle 1, 2, -2 \rangle$

eqn. of the tangent plane at $(3,1,1)$:

$1(x-3) + 2(y-1) - 2(z-1) = 0$

$x + 2y - 2z = 3 + 2 - 2 = 3$

c) When the gradient vector is parallel to the x-axis, i.e. $2y = 0$, $-2z = 0$, then $(1, 0, 0)$ is such a point.

(10 pts) **7.** Find the absolute maximum and minimum of the function $f(x, y) = 1 - 2x^2 - y^2$ over the unit disk $x^2 + y^2 \leq 1$.

$f_x = -4x = 0$
$f_y = -2y = 0$ $(x, y) = (0,0)$ is a critical point.

$f(0,0) = 1$

investigate the boundary: use Lagrange multipliers: $\nabla f = \langle -4x, -2y \rangle$,

$g(x,y) = x^2 + y^2 - 1 = 0$,

$\nabla g = \langle 2x, 2y \rangle$

$-4x = \lambda 2x$
$-2y = \lambda 2y \Rightarrow \lambda = -1$ or $y = 0$
$x^2 + y^2 = 1$

if $\lambda = -1$: $-4x = -2x \Rightarrow x = 0$, then
$y = \pm\sqrt{1 - 0^2} = \pm 1$

if $y = 0$: $x = \pm\sqrt{1 - 0^2} = \pm 1$

points: $(1,0)$, $(-1,0)$, $(0,1)$, $(-0,-1)$.

$f(1,0) = 1 - 2 = -1$ $f(-1,0) = 1 - 2 = -1$
$f(0,1) = 1 - 1 = 0$ $f(0,-1) = 1 - 1 = 0$

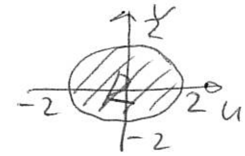
abs. max $f(0,0) = 1$,
abs. min $f(1,0) = f(-1,0) = -1$.

(10 pts) **6.** Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ above the plane $z = 0$. Find the surface area of S.

$x = u, \quad y = v$

$\vec{r}(u,v) = \langle u, v, 4 - u^2 - v^2 \rangle,$

$u^2 + v^2 \le 4.$



$\vec{t}_u = \langle 1, 0, -2u \rangle, \quad \vec{t}_v = \langle 0, 1, -2v \rangle$

$\vec{t}_u \times \vec{t}_v = \langle 2u, 2v, 1 \rangle, \quad |\vec{t}_u \times \vec{t}_v| = \sqrt{1 + 4u^2 + 4v^2}$

surface area $= \iint_S dS = \iint_R \sqrt{1 + 4(u^2+v^2)}\, dA$

Change to polar coordinates:

$u = r\cos\theta, \quad v = r\sin\theta, \quad 0 \le \theta \le 2\pi, \; 0 \le r \le 2.$

$$\iint_R \sqrt{1+4(u^2+v^2)}\, dA = \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$a = 1 + 4r^2 \qquad da = 8r\,dr$

if $r = 0$, then $a = 1$

$r = 2 \qquad a = 17$

$$\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\; r\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi}\int_1^{17} \sqrt{a}\, da\, d\theta = \frac{1}{8}\int_0^{2\pi} \left.\frac{a^{3/2}}{3/2}\right|_1^{17} d\theta =$$

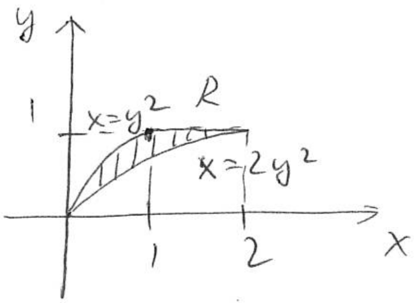
$$\frac{1}{8}\int_0^{2\pi} \frac{2}{3}\left(17^{3/2} - 1\right) d\theta = \frac{17^{3/2} - 1}{12}\, \theta\Big|_0^{2\pi} =$$

$$\frac{\left(17^{3/2} - 1\right)\pi}{6}$$

(10 pts) **8.** Compute the following double integral

$$\iint_R y dA$$

where R is the region confined between two parabolas $x = y^2$, $x = 2y^2$, and the lines $y = 0$, $y = 1$.



$R = \{(x,y) : 0 \le y \le 1, \ y^2 \le x \le 2y^2\}$

$$\iint_R y\,dA = \int_0^1 \int_{y^2}^{2y^2} y\,dx\,dy =$$

$$\int_0^1 yx \Big|_{y^2}^{2y^2} dy = \int_0^1 (2y^3 - y^3)dy =$$

$$\int_0^1 y^3\,dy = \frac{y^4}{4}\Big|_0^1 = \frac{1}{4}$$

(10 pts) **9.** A force field is given by

$$F = \langle y\cos(xy), x\cos(xy) - 2y \rangle.$$

Examine whether this field is conservative or not. If it is conservative find a potential function. Compute the work done by the force in moving an object from point $(0, 0)$ to point $(\pi/2, 1)$ along a straight line.

$f = y \cos(xy)$, $\frac{\partial f}{\partial y} = \cos xy + y(-\sin(xy)) \cdot x$

$g = x \cos(xy) - 2y$, $\frac{\partial g}{\partial x} = \cos xy + x(-\sin(xy)) \cdot y$

yes, conservative, $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

$\varphi = \int \varphi_x \, dx = \int f \, dx = \int y \cos(xy) \, dx = \sin(xy) + C(y)$

$\varphi_y = x\cos(xy) + C'(y) = x \cos(xy) - 2y$,

so $C'(y) = -2y$

$C(y) = \int (-2y) \, dy = -2 \frac{y^2}{2} + K = -y^2 + K$,

take $K = 0$.

$\varphi = \sin(xy) - y^2$

$\int_C \vec{F} \cdot d\vec{r} = \varphi(\frac{\pi}{2}, 1) - \varphi(0,0) =$

$(\sin \frac{\pi}{2} - 1) - (\sin 0 - 0) = 0.$

(10 pts) **10.** A function is given by

$$h(x, y) = (1 + y)\arctan(x).$$

(a) Compute the rate of change in the direction of vector $\mathbf{u} = \langle -1, 1 \rangle$ at the origin.

(b) Determine the direction of the maximum rate of increase at the origin.

(c) Determine the maximal rate of increase itself at the origin.

a) $$\frac{\vec{u}}{|\vec{u}|} = \frac{\langle -1, 1 \rangle}{\sqrt{(-1)^2 + 1^2}} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\nabla h(x, y) = \left\langle \frac{1+y}{1+x^2}, \tan^{-1} x \right\rangle$$

$$\nabla h(0,0) = \langle 1, 0 \rangle$$

$$D_{\frac{\vec{u}}{|\vec{u}|}} h(0,0) = \nabla h(0,0) \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 1, 0 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = -\frac{1}{\sqrt{2}}$$

b) the direction of the max rate of increase: $\nabla h(0,0) = \langle 1, 0 \rangle$

c) max rate of increase:

$$D_{\langle 1, 0 \rangle} h(0,0) = |h(0,0)| = 1$$