DO NOT WRITE ABOVE THIS LINE!!

1. **(15 pt)** Given the function $z = f(x, y) = x^2 + xy - e^y$,

   (a) Compute the directional derivative in the direction of the vector $\mathbf{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ at $(1, 0)$.

   (b) Find the direction and the rate of the steepest ascent at $(1, 0)$.

a) $\nabla f(x,y) = \langle 2x+y,\ x - e^y \rangle$

$\nabla f(1,0) = \langle 2,\ 1-1 \rangle = \langle 2, 0 \rangle$

$D_{\vec{u}} f(1,0) = \langle 2, 0 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = 1$

b) the direction: $\nabla f(1,0) = \langle 2, 0 \rangle$

the rate: $|\nabla f(1,0)| = 2$

DO NOT WRITE ABOVE THIS LINE!!

2. **(10 pt)** Let T be a triangle with vertices $A(6,1,2)$, $B(3,6,3)$ and $C(7,0,5)$.

(a) Find the area of the triangle.

(b) Find an equation of the plane, containing the triangle.

a) $\vec{AB} = \langle 3-6, 6-1, 3-2\rangle = \langle -3, 5, 1\rangle$

$\vec{AC} = \langle 7-6, 0-1, 5-2\rangle = \langle 1, -1, 3\rangle$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 5 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \vec{i}(15+1) - \vec{j}(-9-1) + \vec{k}(3-5) = 16\vec{i} + 10\vec{j} - 2\vec{k}$$

area $= \frac{1}{2}\sqrt{16^2+10^2+(-2)^2}$

b) $\vec{n} = \vec{AB}\times\vec{AC} = \langle 16, 10, -2\rangle$

$16(x-6)+10(y-1)-2(z-2)=0$

$16x+10y-2z = 96+10-4 = 102$

DO NOT WRITE ABOVE THIS LINE!!

3. **(10 pt)** Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y) = 2x + y^2$ subject to the constraint $x^2 + 7y^2 = 64$.

$\nabla f(x,y) = \langle 2, 2y \rangle$

$g(x,y) = x^2 + 7y^2 - 64 = 0$

$\nabla g(x,y) = \langle 2x, 14y \rangle$

$2 = \lambda 2x$
$2y = \lambda 14y$
$x^2 + 7y^2 = 64$
$\Rightarrow$
$1 = \lambda x$
$y = \lambda 7y$
$x^2 + 7y^2 = 64$
$\Rightarrow$ $y(1 - 7\lambda) = 0$
$y = 0$ or $\lambda = \frac{1}{7}$

$\lambda = \frac{1}{7}$: $1 = \frac{1}{7}x$
$x = 7$

$49 + 7y^2 = 64$
$7y^2 = 15$
$y = \pm\sqrt{\frac{15}{7}}$

points $\left(7, -\sqrt{\frac{15}{7}}\right)$
$\left(7, \sqrt{\frac{15}{7}}\right)$

$y = 0$: $x^2 = 64$
$x = \pm\sqrt{64} = \pm 8$,

points $(-8, 0)$
$(8, 0)$

$f\left(7, -\sqrt{\frac{15}{7}}\right) = 14 + \frac{15}{7} = \frac{113}{7} \approx 16.14$

$f\left(7, \sqrt{\frac{15}{7}}\right) = \frac{113}{7} \approx 16.14$

$f(-8, 0) = -8 \cdot 2 = -16$

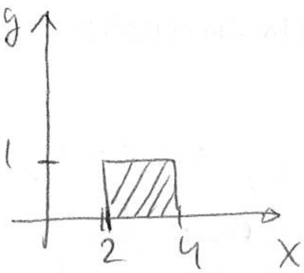
$f(8, 0) = 8 \cdot 2 = 16$

max $f\left(7, \pm\sqrt{\frac{15}{7}}\right) = \frac{113}{7}$

min $f(-8, 0) = -16$

DO NOT WRITE ABOVE THIS LINE!!

4. **(10pt)** Find the average value of the function $f(x, y) = xy$ on the rectangle $\{(x, y) \mid 2 \leq x \leq 4, 0 \leq y \leq 1\}$.



$$\iint_R f(x,y)\,dA = \int_2^4 \int_0^1 xy\,dy\,dx =$$

$$\int_2^4 x \frac{y^2}{2}\Big|_0^1 dx = \frac{1}{2}\int_2^4 x\,dx =$$

$$\frac{1}{2}\,\frac{x^2}{2}\Big|_2^4 = \frac{1}{4}(16-4) = \frac{12}{4} = 3$$

area of $R = 2 \cdot 1 = 2$

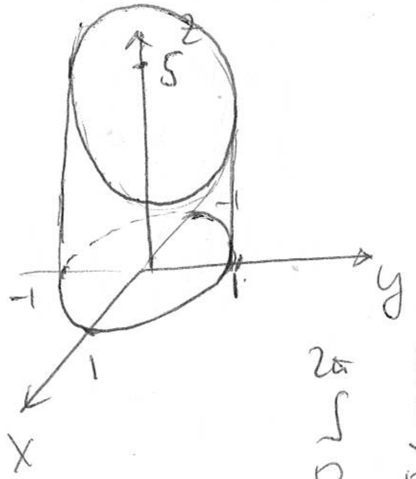
$$\bar{f} = \frac{1}{\text{area of } R}\iint_R f(x,y)\,dA = \frac{3}{2}$$

DO NOT WRITE ABOVE THIS LINE!!

5. **(15 pt)** Evaluate the integral

$$\iiint_D x^2 + y^2 \, d\cancel{A}V,$$

where D is the solid bounded by the cylinder with the circle base given by the equation $x^2 + y^2 = 1$ and the planes $z = 0$ and $x + y + z = 5$.



$D = \{ (r, \theta, z) \mid 0 \le \theta \le 2\pi, \ 0 \le r \le 1, \ 0 \le z \le 5 - r\cos\theta - r\sin\theta \}$

$$\iiint_D x^2 + y^2 \, dV = \int_0^{2\pi} \int_0^1 \int_0^{5 - r(\cos\theta + \sin\theta)} r^2 \, r \, dz\,dr\,d\theta =$$

$$\int_0^{2\pi} \int_0^1 r^3 z \Big|_0^{5 - r(\cos\theta + \sin\theta)} dr\,d\theta =$$

$$\int_0^{2\pi} \int_0^1 5r^3 - r^4(\cos\theta + \sin\theta)\, dr\,d\theta =$$

$$\int_0^{2\pi} \left(5\frac{r^4}{4} - \frac{r^5}{5}(\cos\theta + \sin\theta)\right)\Big|_0^1 d\theta =$$

$$\int_0^{2\pi} \frac{5}{4} - \frac{1}{5}(\cos\theta + \sin\theta)\, d\theta = \frac{5}{4}\theta -$$

$$\frac{1}{5}(\sin\theta - \cos\theta)\Big|_0^{2\pi} = \frac{5\pi}{2} - \frac{1}{5}(\sin 2\pi -$$

$$\cos 2\pi) + \frac{1}{5}(\sin 0 - \cos 0) = \frac{5\pi}{2}$$

DO NOT WRITE ABOVE THIS LINE!!

6. **(10 pt)** Compute the curl and the divergence of the vector field

$$\mathbf{F}(x, y, z) = \langle xy + z, x - y, z - x^2 y \rangle.$$

$\operatorname{div} \vec{F} = y - 1 + 1 = y$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy+z & x-y & z-x^2y \end{vmatrix} = \vec{i}(-x^2 - 0) - \vec{j}(-2xy - 1) + \vec{k}(1 - x) =$$

$\langle -x^2, 2xy + 1, 1 - x \rangle$

DO NOT WRITE ABOVE THIS LINE!!

7. **(10 pt)** The vector field $\mathbf{F}(x, y, z) = \langle 2xy + yz, x^2 + xz, xy \rangle$ is conservative.

(a) Find a potential function for $\mathbf{F}$.

(b) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a curve originating at $(1, 1, 0)$ and terminating at $(2, -1, 1)$.

a) $\varphi = \int (2xy + yz)\,dx = 2y\,\frac{x^2}{2} + xyz + C(y,z) = x^2y + xyz + C(y,z)$

$\varphi_y = x^2 + xz + C_y(y,z) = x^2 + xz$,

so $C_y(y,z) = 0$, $C(y,z) = d(z)$

$\varphi = x^2y + xyz + d(z)$

$\varphi_z = xy + d'(z) = xy$, so $d'(z) = 0$,

$d(z) = k$, take $k = 0$

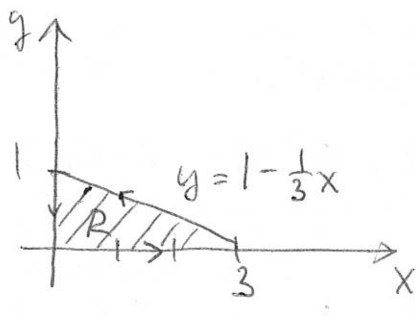
$\boxed{\varphi = x^2y + xyz}$

b) $\int_C \vec{F} \cdot d\vec{r} = \varphi(2,-1,1) - \varphi(1,1,0) =$

$4 \cdot (-1) + 2(-1) \cdot 1 - (1^2 \cdot 1) = -4 - 2 - 1 = -7$

DO NOT WRITE ABOVE THIS LINE!!

8. **(10pt)** Find the circulation of the vector field $\mathbf{F}(x,y) = \langle 2x+3y, yx+e^{\cos y}\rangle$ on the triangular curve with vertices $(0,0)$, $(0,1)$ and $(3,0)$ oriented counterclockwise. (Hint: use Green's theorem.)



$f = 2x+3y \qquad \frac{\partial f}{\partial y} = 3$

$g = yx + e^{\cos y} \qquad \frac{\partial g}{\partial x} = y$

$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (y-3)\,dA =$

$\int_0^3 \int_0^{1-\frac{1}{3}x} (y-3)\,dy\,dx = \int_0^3 \left(\frac{y^2}{2} - 3y\right)\Big|_0^{1-\frac{1}{3}x} dx =$

$\int_0^3 \left(\frac{1}{2}\left(1 - \frac{2x}{3} + \frac{x^2}{9}\right) - 3 + x\right)dx =$

$\int_0^3 \left(\frac{1}{2} - \frac{x}{3} + \frac{x^2}{18} - 3 + x\right)dx = \int_0^3 \left(\frac{x^2}{18} + \frac{2x}{3} - \frac{5}{2}\right)dx =$

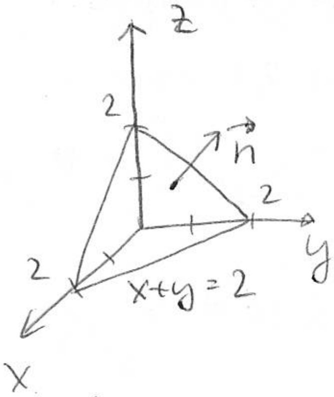
$\left(\frac{1}{18}\frac{x^3}{3} + \frac{2}{3}\frac{x^2}{2} - \frac{5}{2}x\right)\Big|_0^3 =$

$\frac{1}{2\cdot 9\cdot 3}3^3 + \frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}\cdot 9 - \frac{5}{2}\cdot 3 =$

$\frac{1}{2} + 3 - \frac{15}{2} = 3 - \frac{14}{2} = -4$

DO NOT WRITE ABOVE THIS LINE!!

9. **(10pt)** Compute the flux of the vector field $\mathbf{F} = \langle 0, -z, 3y \rangle$ across the surface in the first octant given by the equation $x + y + z = 2$, in the direction of the positive z-semiaxis.



$u = x, \quad v = y,$

$\vec{r}(u,v) = \langle u, v, 2-u-v \rangle$

$R = \{(u,v) \mid 0 \le u \le 2,\ 0 \le v \le 2-u \rangle$

$\vec{t}_u = \langle 1, 0, -1 \rangle, \quad \vec{t}_v = \langle 0, 1, -1 \rangle$

$$\vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i}(1) - \vec{j}(-1) + \vec{k}\cdot 1 = \langle 1, 1, 1 \rangle$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^2 \int_0^{2-u} \langle 0, -2+u+v, 3v \rangle \cdot \langle 1,1,1 \rangle \, dv\,du =$$

$$\int_0^2 \int_0^{2-u} (-2+u+v+3v)\,dv\,du = \int_0^2 \int_0^{2-u} (-2+4v+u)\,dv\,du =$$

$$\int_0^2 \left(-2v + 4\frac{v^2}{2} + uv\right)\Big|_0^{2-u} du = \int_0^2 \left(-2(2-u) + 2(2-u)^2 + u(2-u)\right) du =$$

$$\int_0^2 \left(-4 + 2u + 2(4 - 4u + u^2) + 2u - u^2\right) du =$$

$$\int_0^2 (u^2 - 4u + 4)\,du = \left(\frac{u^3}{3} - 4\frac{u^2}{2} + 4u\right)\Big|_0^2 =$$

$$\frac{8}{3} - 2\cdot 4 + 4\cdot 2 = \frac{8}{3}$$