(10 pts) **1.**(a) Find an equation of the plane passing through the point $P(0,1,2)$ and containing the line

$$r(t) = < 3 - t, 2t, 4 + 3t > .$$

(b) Find an equation of the plane passing through the points $(-1,1,1)$, $(0,0,2)$ and $(3,-1,-2)$.

(10 pts) **2.** Evaluate the following limits, if the limit exists; explain why if the limit does not exist.

(a)

$$\lim_{(x,y)\to(\pi,0)} \frac{\cos xy + \sin xy}{3x}$$

$-u$

(b) $1-6=-5$

$$\lim_{(x,y)\to(2,14)} \frac{\sqrt{x+y}-4}{x+y-16}$$

(c)

$$\lim_{(x,y)\to(0,0)} \frac{y^4 - 2x^2}{y^4 + x^2}$$

(10 pts) **3.**(a) Write the Chain Rule formula for w_s and w_t where $w = w(x,y)$, and $x = x(s,t)$ and $y = y(s,t)$. Given $w = \frac{x-y}{x+y}$, $x = s + t$, $y = st$. Find w_s, and w_t.

(b) The function $y = y(x)$ is given implicitly by $ye^{xy} + x - 2 = 0$. Find $\frac{dy}{dx}$.

(10 pts) **4.** Given function $f(x,y) = x^2 + 4xy - y^2$, and point $P(2,1)$. $4+8-1 = 11$

(a) Find the unit vectors that give the direction of steepest ascent and steepest descent at P.

(b) Find at least one unit vector that points in the direction of zero rate of change of the function at P.

(c) Write an equation of the tangent plain to the graph of f at point P.

(10 pts) **5.** The surface area of a torus with an inner radius r and an outer radius $R > r$ is $S = 4\pi^2(R^2 - r^2)$. Estimate the change in the surface area of the torus when (r,R) changes from $(3,4)$ to $(3.1, 4.2)$ (you may leave any π's in your answer unevaluated).

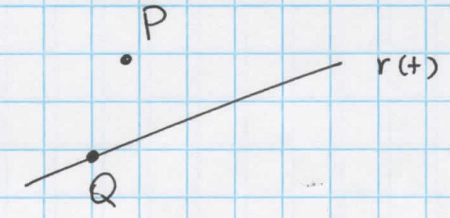
(10 pts) **6.** Find and classify all critical points of the function $f(x,y) = y^2 + xy + x$ in $\mathbb{R}^2$.

(10 pts) **7.** For the same function $f(x,y) = y^2 + xy + x$ find its absolute maximum and minimum values on the triangle $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1 - x\}$.

(10 pts) **8.** Use the Method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x + 2y$ subject to the constraint $\frac{x^2}{2} + y^2 = 1$.

Midterm 2 solves.

#1 a)



Let $Q = r(0) = \langle 3, 0, 4 \rangle$. Two vectors that lie on the plane are $\overline{PQ}$ and direction of r, $v = \langle -1, 2, 3 \rangle$.

The vector product is normal

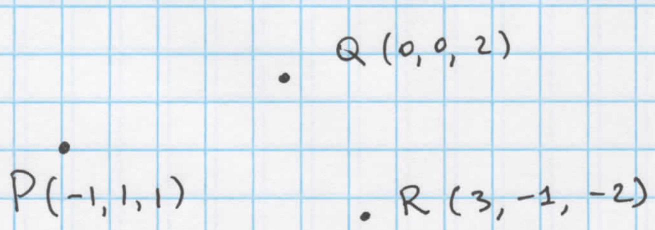
$$n = v \times \overline{PQ} = \begin{vmatrix} i & j & k \\ -1 & 2 & 3 \\ 3 & -1 & 2 \end{vmatrix} =$$

$$= 7i + 11j - 5k$$

So, $7(x-0) + 11(y-1) - 5(z-2) = 0$

$$\boxed{7x + 11y - 5z = 1}$$

b)



$u = \overline{PQ} = \langle 1, -1, 1 \rangle$
$v = \overline{PR} = \langle 4, -2, -3 \rangle$

$u \times v = \langle 5, 7, 2 \rangle$

$$5x + 7y + 2(z-2) = 0$$

$$\boxed{5x + 7y + 2z = 4}$$

#2 a) The denominator does not vanish.

So, $\lim = \dfrac{\cos(0) + \sin(0)}{3\pi} = \dfrac{1}{3\pi}$

b) $\dfrac{\sqrt{x+y} - 4}{x+y-16} = \dfrac{\sqrt{x+y} - 4}{(\sqrt{x+y}-4)(\sqrt{x+y}+4)} =$

$= \dfrac{1}{\sqrt{x+y}+4}$

So, $\lim = \dfrac{1}{\sqrt{2+14}+4} = \dfrac{1}{4+4} = \dfrac{1}{8}$

c) Same degrees on top and bottom. So, we try the 2 path Test.

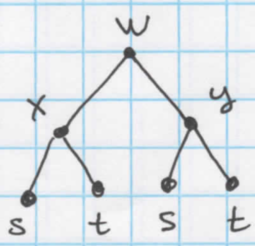
Try axes:

$x=0, \quad y \to 0 \quad , \qquad \dfrac{y^4 - 2x^2}{y^4 + x^2} = \dfrac{y^4}{y^4} = \underline{1}$

$y=0, \quad x \to 0 \quad , \qquad$ —"— $= -\dfrac{2x^2}{x^2} = -2$

Two different limits obtained. So, the limit does not exist.

#3 a)



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

For $w = \dfrac{x-y}{x+y}$ we find

$$\frac{\partial w}{\partial x} = \frac{\underline{1} \cdot (x+y) - \underline{1} \cdot (x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-(x+y) - (x-y)}{(x+y)^2} = -\frac{2x}{(x+y)^2}$$

$$\frac{\partial x}{\partial s} = 1 \qquad \frac{\partial x}{\partial t} = 1$$

$$\frac{\partial y}{\partial s} = t \qquad \frac{\partial y}{\partial t} = s$$

So, $$\frac{\partial W}{\partial s} = \frac{2st}{(s+t+st)^2} - \frac{2(s+t)}{(s+t+st)^2}\,t$$

$$= -\frac{2t^2}{(s+t+st)^2}$$

$$\frac{\partial w}{\partial t} = \frac{2st}{(s+t+st)^2} - \frac{2(s+t)}{(s+t+st)^2}\,s$$

$$= -\frac{2s^2}{(s+t+st)^2}$$

b) $$y\,e^{xy} + x - 2 = 0$$

Differentiate in x and use Chain Rule:

$$y'\,e^{xy} + y\,e^{xy}\,(y + xy') + 1 = 0$$

$$y'\,e^{xy} + y'\,yx\,e^{xy} + y^2 e^{xy} + 1 = 0$$

$$y' = -\frac{1 + y^2 e^{xy}}{e^{xy}(1+xy)}.$$

#4. a) Ascent : $\nabla f(P)$

$$\nabla f = \langle 2x + 4y,\ 4x - 2y \rangle$$

$$\nabla f(2,1) = \langle 6, 6 \rangle$$

Unit : $\boxed{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}$

Descent : $\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

b) $\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ or $\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

c) $z = f_x(p)(x-a) + f_y(p)(y-b) + f(p)$

$$\boxed{z = 6(x-2) + 6(y-1) + 11}$$

#5. $S_R = 8\pi^2 R = 32\pi^2$

$S_r = -8\pi^2 r = -24\pi^2$

$dS \sim S_R\, dR + S_r\, dr = 32\pi^2\, 0.2 - 24\pi^2\, 0.1$

$= (6.4 - 2.4)\pi^2 = 4\pi^2$

#6. $f_x = y + 1 = 0$ $\qquad y = -1$

$f_y = 2y + x = 0$ $\qquad x = 2$

$(2, -1)$ is a critical point

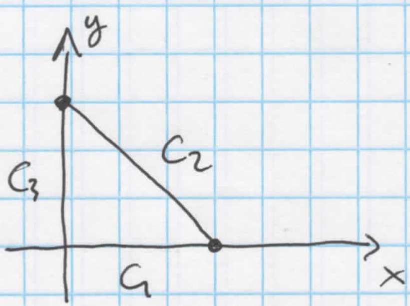
$f_{xx} = 0$

$f_{xy} = 1$ $\qquad D = f_{xx} f_{yy} - f_{xy}^2 = -1 < 0$

$f_{yy} = 2$

$\therefore$ $(2, -1)$ is a saddle.

#7.



Notice that $(2, -1)$ is outside of the triangle, so it won't be considered.

Parametrize

C_1 : $y=0$, $0 \le x \le 1$

$f = x$, $f' = 1 \neq 0$.

C_2 : $x+y=1$

$f = y(y+x) + x = y+x = 1$

$f' = 0$ so, all points are critical.

But f is 1 identically, so we will compare it to other values of f.

C_3 : $x=0$, $0 \le y \le 1$

$f = y^2$

$f' = 2y = 0$, $y=0$

Corners : $(0,0)$; $(1,0)$; $(0,1)$

$f = 1$ on C_2

$f(0,0)=0$, $f(1,0) = 1$, $f(0,1) = 1$.

$\therefore$ 0 – min
1 – max

#8. $g = \frac{x^2}{2} + y^2$

$$\nabla f = \lambda \nabla g$$

$$\begin{cases} 1 = \lambda x \\ 2 = 2\lambda y \quad ; \quad 1 = \lambda y \\ \frac{x^2}{2} + y^2 = 1 \end{cases}$$

$\lambda \neq 0$, so, $x = \frac{1}{\lambda}$; $y = \frac{1}{\lambda}$

$$\frac{1}{2\lambda^2} + \frac{1}{\lambda^2} = 1$$

$$\frac{3}{2} \frac{1}{\lambda^2} = 1$$

$$\lambda = \pm \sqrt{3/2}$$

Two points: $P\left(\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$; $Q\left(-\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$

$f(P) = 3\sqrt{\frac{2}{3}} = \sqrt{6}$ – max

$f(Q) = -\sqrt{6}$ – min.