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1. **(10 pt)** Find an equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 1$$

at the point $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2})$.

$$F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$$

$$\nabla F(x,y,z) = \langle 2x, 2y, 2z \rangle$$

$$\nabla F\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \sqrt{3} \right\rangle$$

the equation:

$$\frac{\sqrt{2}}{2}\left(x - \frac{\sqrt{2}}{4}\right) + \frac{\sqrt{2}}{2}\left(y - \frac{\sqrt{2}}{4}\right) + \sqrt{3}\left(z - \frac{\sqrt{3}}{2}\right) = 0$$

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2. **(10 pt)** Consider the function

$$f(x, y) = x^2 + 4y^2.$$

(a) Find the equation of the plane tangent to the graph of the function at the point $(2, 1, 8)$.

(b) Use a linear approximation to the function $f(x, y)$ to estimate $f(1.9, 1.1)$. Your answer should be a single number in decimal form, or written as a reduced fraction.

a) $\nabla f(x,y) = \langle 2x, 8y \rangle$

$\nabla f(2,1) = \langle 4, 8 \rangle$

$z = 4(x-2) + 8(y-1) + 8$ – the equation of the tangent plane at $(2,1,8)$

b) $L(x,y) = 4(x-2) + 8(y-1) + 8$

$L(1.9, 1.1) = 4(1.9-2) + 8(1.1-1) + 8 =$

$4 \cdot (-0.1) + 8 \cdot 0.1 + 8 = 8.4$

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3. **(15 pt)** Find all critical points of the function

$$f(x, y) = x^3 + 2xy + y^2 - x$$

and classify them using the Second Derivative Test.

$f_x = 3x^2 + 2y - 1 = 0$

$f_y = 2x + 2y = 0 \qquad y = -x$

$3x^2 - 2x - 1 = 0$

$D = (-2)^2 - 4 \cdot 3 \cdot (-1) = 4 + 12 = 16$

$x = \frac{2 \pm \sqrt{16}}{2 \cdot 3} \qquad x_1 = \frac{2-4}{6} = -\frac{1}{3}, \quad x_2 = \frac{2+4}{6} = 1$

$\left(-\frac{1}{3}, \frac{1}{3}\right), (1, -1)$ are the critical points

$f_{xx} = 6x \qquad f_{xy} = 2 \qquad f_{yy} = 2$

$D(x, y) = 12x - 4$

$D\left(-\frac{1}{3}, \frac{1}{3}\right) = 12 \cdot \left(-\frac{1}{3}\right) - 4 = -8 < 0$ saddle point

$D(1, -1) = 12 \cdot 1 - 4 > 0$

$f_{xx}(1, -1) = 6 > 0$ local minimum

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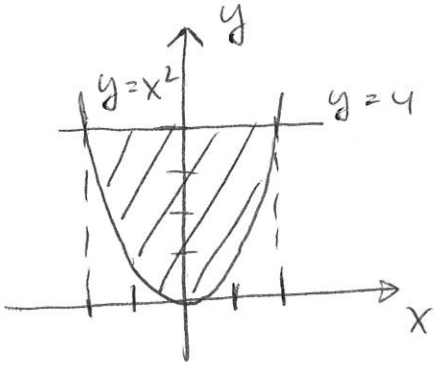
4. **(15 pt)** Let R be the region enclosed by the line $y = 4$ and the parabola $y = x^2$. Find the absolute maximum and minimum values attained by $f(x, y) = 2x^2 - xy$ on R, and the points where these values occur.



Critical points:

$f_x = 4x - y = 0$

$f_y = -x = 0$, $(0,0)$ is in the boundary of R

the boundary:

$y=4$, $-2 \le x \le 2$: $f(x) = 2x^2 - 4x$,

$f'(x) = 4x - 4 = 0$, so $x = 1$, $(1,4)$ is a critical point in the boundary.

$f(1,4) = 2\cdot 1 - 4 \cdot 1 = \boxed{-2}$

the endpoints: $f(-2,4) = 2\cdot 4 + 2 \cdot 4 = \boxed{16}$

$f(2,4) = 2\cdot 4 - 2\cdot 4 = \boxed{0}$

$y = x^2$, $-2 \le x \le 2$:

$f(x) = 2x^2 - x \cdot x^2 = 2x^2 - x^3$

$f'(x) = 4x - 3x^2 = 0$

$x(4-3x) = 0$, $x = 0$ or $x = \frac{4}{3}$

$f(0,0) = \boxed{0,}$

$f\left(\frac{4}{3}, \frac{16}{9}\right) = 2 \cdot \frac{16}{9} - \frac{4}{3} \cdot \frac{16}{9} = \frac{2}{3} \cdot \frac{16}{9} = \boxed{\frac{32}{27}}$

abs. max $f(-2,4) = 16$

abs. min $f(1,4) = -2$

5. **(15 pt)** Use the method of Lagrange multipliers to find the extreme values of $f(x,y) = x + y$ subject to the constraint $2x^2 + y^2 = 2$.

$\nabla f(x,y) = \langle 1, 1 \rangle$

$g(x,y) = 2x^2 + y^2 - 2 = 0 \qquad \nabla g(x,y) = \langle 4x, 2y \rangle$

$1 = 4\lambda x \qquad x = \frac{1}{4\lambda}$

$1 = 2\lambda y \qquad y = \frac{1}{2\lambda} = \frac{2}{4\lambda}$

$2\left(\frac{1}{4\lambda}\right)^2 + \left(\frac{2}{4\lambda}\right)^2 = 2.$

$2 + 4 = 2 \cdot 16\lambda^2$

$\lambda^2 = \frac{6}{32} = \frac{3}{16}, \quad \lambda = \pm\frac{\sqrt{3}}{4}$

$\lambda = -\frac{\sqrt{3}}{4}: \qquad x = \frac{1}{4\left(-\frac{\sqrt{3}}{4}\right)} = -\frac{1}{\sqrt{3}}, \quad y = \frac{2}{4\left(\frac{\sqrt{3}}{4}\right)} = -\frac{2}{\sqrt{3}}$

$\lambda = \frac{\sqrt{3}}{4}: \qquad x = \frac{1}{4 \cdot \sqrt{3}/4} = \frac{1}{\sqrt{3}}, \quad y = \frac{2}{\sqrt{3}}$

$f\left(-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

$f\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

max $f\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = \sqrt{3}$

min $f\left(-\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = -\sqrt{3}$

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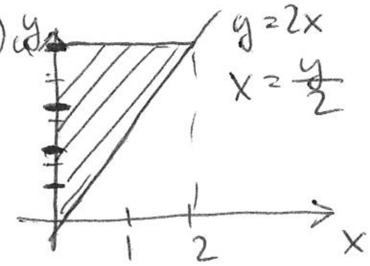
6. **(15 pt)** Consider the integral

$$\int_0^2 \int_{2x}^4 2xy\, dy\, dx.$$

(a) Sketch the region of integration.

(b) Reverse the order of integration, by rewriting the bounds and converting it to an integral $dx\, dy$.

(c) Evaluate the integral (you can use any order of integration).

a) 

$R = \{ (x,y) \mid 0 \le x \le 2,\ 2x \le y \le 4 \}$

b) $\displaystyle\int_0^2 \int_{2x}^4 2xy\, dy \cdot dx = \int_0^4 \int_0^{\frac{y}{2}} 2xy\, dx\, dy$

c) $\displaystyle\int_0^4 \int_0^{\frac{y}{2}} 2xy\, dx\, dy = \int_0^4 \left(2y \frac{x^2}{2} \Big|_0^{\frac{y}{2}} \right) dy =$

$\displaystyle\int_0^4 y\left(\frac{y^2}{4} - 0 \right) dy = \frac{1}{4} \int_0^4 y^3\, dy = \frac{1}{4} \frac{y^4}{4} \Big|_0^4 =$

$\displaystyle\frac{1}{16}(16 \cdot 16 - 0) = 16$

or: $\displaystyle\int_0^2 \int_{2x}^4 2xy\, dy\, dx = \int_0^2 2x \left(\frac{y^2}{2} \Big|_{2x}^4 \right) dx =$

$\displaystyle\int_0^2 x(16 - 4x^2)\, dx = \int_0^2 (16x - 4x^3)\, dx =$

$\displaystyle 16 \frac{x^2}{2} \Big|_0^2 - 4 \frac{x^4}{4} \Big|_0^2 = 8(4-0) - (16-0) =$

$32 - 16 = 16$

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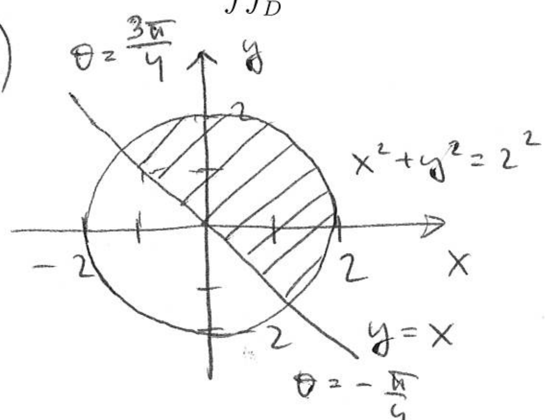
7. **(10 pt)** Let $D = \{(x, y) : x^2 + y^2 \leq 4 \text{ and } y \geq -x\}$.

(a) Sketch the region D.

(b) Express D in polar coordinates via $D = \{(r, \theta) : a \leq r \leq b;\ \alpha \leq \theta \leq \beta\}$.

(c) Evaluate $\iint_D e^{x^2+y^2}\, dA$.

a)



b) $D = \{(r,\theta) : -\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4},\ 0 \leq r \leq 2\}$

c) $\iint_D e^{x^2+y^2}\, dA =$

$\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^2 e^{r^2}\, r\, dr\, d\theta =$

$u = r^2 \quad du = 2r\,dr \quad$ if $r=0$ then $u=0$
$r=2 \quad u=4$

$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^4 e^u\, du\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} e^u \Big|_0^4\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (e^4 - e^0)\, d\theta =$

$\frac{1}{2}(e^4-1)\,\theta \Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{1}{2}(e^4-1)\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) =$

$\frac{\pi}{2}(e^4-1)$

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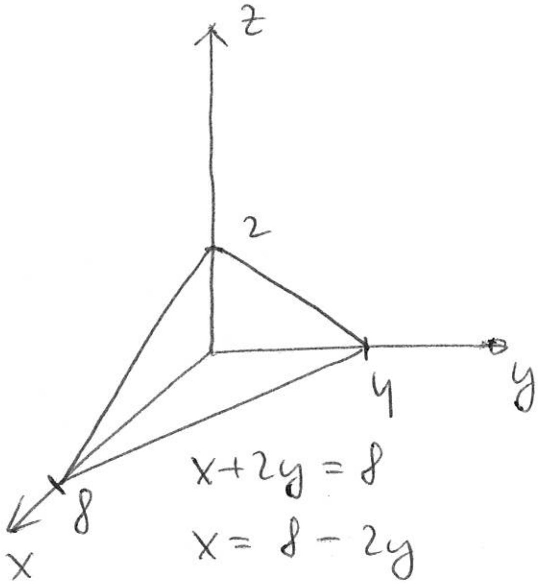
8. **(10 pt)** A tetrahedron is bounded by the xy-plane, yz-plane, xz-plane and the plane

$$x + 2y + 4z = 8.$$

Write down an iterated triple integral that expresses the volume of the tetrahedron. Do not evaluate the integral.



$x + 2y = 8$

$x = 8 - 2y$

$$D = \{ (x, y, z) \mid 0 \le y \le 4,$$
$$0 \le x \le 8 - 2y,$$
$$0 \le z \le \tfrac{1}{4}(8 - x - 2y) \}$$

$$V = \iiint_D dV = \int_0^4 \int_0^{8-2y} \int_0^{\frac{1}{4}(8-x-2y)} dz\,dx\,dy$$