MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago
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Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (30 points) Solve the following first order ordinary differential equations.

   (a) (10 points)
   $$\frac{dy}{dx} = \frac{x\cos(x^2)}{y}$$

   (b) (10 points)
   $$\frac{dy}{dx} = \frac{-3x^2y + e^x}{x^3+1}$$

   (c) (10 points)
   $$\frac{dy}{dx} = \frac{1 - y^2\sin(xy^2)}{2xy\sin(xy^2)}$$

2. (15 points) Consider the initial value problem:
   $$y' = x - y^2, \quad y(0) = 1.$$

   (a) (5 points) Use the Euler method with *one* step to approximate $y(0.2)$.

   (b) (10 points) Use the Euler method with *two* steps to approximate $y(0.2)$.

3. (20 points) Let $p(t)$ be the population of some species, which satisfies the ordinary differential equation
   $$p'(t) = 3p(t) - p^2(t).$$

   (a) (5 points) If $p(0) = 0$ what is $p(t)$ for $t > 0$?

   (b) (5 points) If $p(0) = 3$ what is $p(t)$ for $t > 0$?

   (c) (10 points) Solve for $p(t)$ for the initial condition $p(0) = 1$, and then evaluate the limit of $p(t)$ as $t \to \infty$.

4. (30 points) Solve the following initial value problems.

   (a) (15 points)
   $$y''(t) + 2y'(t) + 2y(t) = 2t + 5e^t, \quad y(0) = 4, \quad y'(0) = 5.$$

   (b) (15 points)
   $$2t^2y''(t) - ty'(t) - 2y(t) = 0, \quad y(1) = 1, \quad y'(1) = -1.$$

5. (20 points) Find the general solution to the following system of ordinary differential equations (your answer should involve only *two* arbitrary constants).

$$\begin{aligned} x'(t) &= 4x(t) - y(t) \\ y'(t) &= 9x(t) - 2y(t). \end{aligned}$$

6. (20 points) Find the inverse Laplace transform of the following function.

$$F(s) = \frac{3s^2 - 7s}{(s^2 - 4s + 5)(s+1)}.$$

7. (20 points) Solve the following initial value problem using the *method of Laplace transforms* (other methods will receive zero points).

$$y''(t) + 4y'(t) + 4y(t) = \delta(t-3), \quad y(0) = 0, \quad y'(0) = 1.$$

8. (25 points) Find the real values of λ (eigenvalues) for which the following problem has a nontrivial solution. Also, determine the nontrivial solutions (eigenfunctions).

$$\begin{aligned} &y''(x) + \lambda y(x) = 0, \quad 0 < x < \pi \\ &y'(0) = 0, \quad y(\pi) = 0. \end{aligned}$$

9. (20 points) Consider the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi/4 \\ 1, & \pi/4 < x < \pi/2 \\ 0, & \pi/2 < x < 3\pi/4 \\ -1, & 3\pi/4 < x < \pi \end{cases}.$$

   (a) (16 points) Find the Fourier sine series for f.

   (b) (2 points) To what value does this series converge at $x = \pi/2$? Why?

   (c) (2 points) To what value does this series converge at $x = \pi$? Why?

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$

2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$

4. $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad s > 0$

5. $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0$

6. $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$

7. $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \quad s > a$

8. $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}, \quad s > a$

9. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$

10. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$

11. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$

12. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

13. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$

14. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$

15. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$

16. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$

17. $\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$

18. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$

19. If f has period T then

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

20. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

List of PDE Formulae

1. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

2. The solution of the homogeneous heat equation $u_t = \beta^2 u_{xx}$ with Neumann boundary conditions is:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\beta n\pi/L)^2 t} \cos\left(\frac{n\pi}{L}x\right).$$

3. The inhomogeneous heat equation has a solution of the form $u(x,t) = v(x) + w(x,t)$, where v is the steady–state solution and w solves a homogeneous heat equation.

4. The solution of the homogeneous wave equation $u_{tt} = \alpha^2 u_{xx}$ with Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\alpha\frac{n\pi}{L}t\right) + b_n \sin\left(\alpha\frac{n\pi}{L}t\right)\right\} \sin\left(\frac{n\pi}{L}x\right).$$