Applied Mathematics Masters Examination
Spring 2011, March 29, 1–4 pm.

Each of the twelve numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of your scores on your eight best questions.

Please observe the following:

• DO NOT answer two or more questions on the same sheet (not even on both sides of the same sheet).

• DO NOT write your name on any of your answer sheets.

You will be given separate instructions on the use of these answer sheets. When you have completed a question, place it in the large envelope provided.

Linear Algebra

Problem 1. Let

\[
A = \begin{pmatrix}
1 & 0 & \frac{2}{3} \\
0 & 1 & \frac{1}{3} \\
1 & 1 & 1
\end{pmatrix}.
\]

(1) Find the kernel and the rank of \( A \).

(2) Calculate the eigenvalues and the eigenspaces of \( A \).

(3) Is \( A \) diagonalizable? If so, find a diagonal matrix conjugate to \( A \).

Problem 2. Define the inner product of two real valued continuous functions \( f, g \) on \([-\frac{1}{2}, \frac{1}{2}]\) by

\[
(f, g) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)g(x) \, dx.
\]

Find an orthonormal basis for the subspace spanned by the functions \( 1, x, x^2 \) with respect to this inner product.
Advanced Calculus

Problem 3. Find the absolute maxima and minima of the function
\[ f(x, y) = xy - y + x - 1 \]
in the domain
\[ A = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}. \]

Problem 4. Let \( F(x, y, z) \) be given by
\[ F(x, y, z) = x^2yi + z^8j - 2xyzk. \]
Evaluate the integral of \( F(x, y, z) \) over the surface of the unit cube, spanned by the unit basis vectors \( i, j, k \).

Complex Analysis

Problem 5. Consider the function
\[ f(z) = \frac{\sin z}{z(z - 1)^2}. \]
(1) Find all singular points.
(2) Classify the singular points as essential, removable, or poles.
(3) Find the residues of all singular points.

Problem 6. Evaluate the improper integral
\[ \int_0^\infty \frac{\cos(2x)}{x^2 + 1} \, dx. \]
Ordinary Differential Equations

Problem 7. Find the general solution to the linear system
\[ t \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} t^2 \\ t \end{pmatrix}. \]

Problem 8. Find the general solution to the following ODE for \( y(x) \):
\[ xy''(x) + 2y'(x) + xy(x) = 0. \]
One possible approach is to express the solutions as Taylor or Frobenius series about \( x = 0 \), and then sum the series to obtain a closed form.

Problem 9. For \( 0 \leq x \leq \pi/2 \), consider the boundary value problem
\[ y''(x) + y(x) = A + \sin(x), \quad y(0) = 0, \quad y'(\pi/2) = 0. \]
(1) For which value(s) of \( A \) does the problem have a solution?
(2) Use elementary methods to find the solution when it exists. Is it unique?
(3) Solve the problem by an eigenfunction expansion, using the solutions of
\[ -\phi''(x) = \lambda \phi(x), \quad \phi(0) = 0, \quad \phi'(\pi/2) = 0. \]

Partial Differential Equations

Problem 10. In the domain \( 0 < x < 2, \ t > 0 \), solve the heat equation
\[ \frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2} + Q(x,t) \]
subject to the initial condition \( u(x,0) = f(x) \) and boundary conditions \( u(0,t) = 0, \ u(2,t) = B \), where \( k > 0 \) and \( B \) are real constants.

Problem 11. Inside a circle of radius \( a \), consider Poisson’s equation
\[ \Delta u(r,\theta) = f(r,\theta) \]
subject to the mixed boundary condition
\[ u(a,\theta) = g(\theta), \quad 0 < \theta < \pi \]
\[ \frac{\partial u}{\partial r}(a,\theta) = h(\theta), \quad \pi < \theta < 2\pi. \]
Represent \( u(r,\theta) \) in terms of the Green’s function, which is assumed to be known. Do not compute the Green’s function explicitly.

Problem 12. Using the Method of Characteristics solve
\[ \frac{\partial u(x,t)}{\partial t} + \sin(t) \frac{\partial u(x,t)}{\partial x} = tu(x,t), \quad u(x,0) = f(x). \]
Solutions

**Problem 1.** Apply Gauss-Jordan elimination to the matrix $A$ to obtain

$$
\begin{pmatrix}
1 & 0 & \frac{2}{3} \\
0 & 1 & \frac{1}{3} \\
0 & 0 & 0
\end{pmatrix}.
$$

We conclude that the rank is 2 and the kernel is the subspace of $\mathbb{R}^3$ of the form

$$
c \begin{pmatrix}
-2 \\
-1 \\
3
\end{pmatrix},
$$

for a scalar $c$.

The characteristic polynomial of $A$ is given by

$$
\det \begin{pmatrix}
1 - \lambda & 0 & \frac{2}{3} \\
0 & 1 - \lambda & \frac{1}{3} \\
1 & 1 & 1 - \lambda
\end{pmatrix} = \lambda(1 - \lambda)(\lambda - 2).
$$

Therefore, the eigenvalues are 0, 1, 2. The eigenspace corresponding to 0 is the kernel and has already been computed. The eigenspace corresponding to 1 is the kernel of the matrix

$$
\begin{pmatrix}
0 & 0 & \frac{2}{3} \\
0 & 0 & \frac{1}{3} \\
1 & 1 & 0
\end{pmatrix}
$$

and is spanned by the vector $(-1, 1, 0)^T$. The eigenspace corresponding to 2 is the kernel of the matrix

$$
\begin{pmatrix}
-1 & 0 & \frac{2}{3} \\
0 & -1 & \frac{1}{3} \\
1 & 1 & -1
\end{pmatrix}
$$

and is spanned by the vector $(2, 1, 3)^T$.

Since $A$ has three distinct eigenvalues, $A$ is diagonalizable. The diagonal matrix $D(0, 1, 2)$ is conjugate to $A$.

**Problem 2.** We apply the Gram-Schmidt Algorithm to obtain an orthonormal basis.

$$
\int_{-1/2}^{1/2} 1 \, dx = 1. \text{ Hence } w_1 = 1 \text{ is a unit vector. } \int_{-1/2}^{1/2} x \, dx = 0. \text{ Hence } 1 \text{ and } x \text{ are orthogonal. } \int_{-1/2}^{1/2} x^2 \, dx = \frac{1}{12}. \text{ Hence, } w_2 = x/\sqrt{12} \text{ is a unit vector orthogonal to } 1. \text{ Let }
$$

$$
v = x^2 - (x^2, 1)1 - (x^2, \frac{x}{\sqrt{12}})\frac{x}{\sqrt{12}}.
$$

We have that $(x^2, 1) = \int_{-1/2}^{1/2} x^2 \, dx = \frac{1}{12}$ and $(x^2, x) = \int_{-1/2}^{1/2} x^3 \, dx = 0$. Hence, $v = x^2 - \frac{1}{12}$ and $w_1, w_2$ give an orthogonal basis of the vector space. We have that $(v, v) = \int_{-1/2}^{1/2} (x^2 - \frac{1}{12})^2 \, dx = \frac{1}{180}$. Hence

$$
1, \frac{x}{\sqrt{12}}, \frac{1}{6\sqrt{5}}(x^2 - 12)
$$

give an orthonormal basis of the subspace.
Problem 3. We start by searching for critical points inside the open unit disk

\[ D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \]

On this set, we require \( \nabla f(x, y) = (0, 0) \), which is equivalent to \( y + 1 = 0 \) and \( x - 1 = 0 \). So the only putative critical point is \((1, -1)\). But \( 1^2 + (-1)^2 = 2 > 1 \), which means that \((1, -1) \notin D_r \), and so \( f \) has no critical points inside the domain \( D \).

So one has to concentrate on the critical points on the boundary

\[ \partial A = \{(x, y) \in \mathbb{R} \mid x^2 + y^2 = 1\} = A \setminus D, \]

which is a level set of the function \( g(x, y) = x^2 + y^2 - 1 = 0 \). By Lagrange multiplier theory, we are searching for \( x, y \) and \( \lambda \), all real, such that

\[
\begin{aligned}
\frac{\partial f}{\partial x}(x, y) &= \lambda \frac{\partial g}{\partial x}(x, y) \\
\frac{\partial f}{\partial y}(x, y) &= \lambda \frac{\partial g}{\partial y}(x, y) \\
g(x, y) &= 0
\end{aligned}
\]

\[ \iff \begin{cases} y + 1 = 2\lambda x \\ x - 1 = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \]

This implies that \( y = 2\lambda x - 1 \) which leads to

\( (4\lambda^2 - 1)x = 2\lambda - 1. \)

If \( 2\lambda - 1 = 0 \), we obtain that \( x = y + 1 \) which, together with the constraint, leads to the critical points

\[ P_1 = (x_1, y_1) = (0, 1) \quad \text{and} \quad P_2 = (x_2, y_2) = (-1, 0). \]

If \( 2\lambda - 1 \neq 0 \), then \( x = \frac{1}{2\lambda + 1} \) and so \( y = \frac{-1}{2\lambda + 1} \). Again using the constraint we get two more critical points, namely

\[ P_3 = (x_3, y_3) = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \quad \text{and} \quad P_4 = (x_4, y_4) = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \]

Finally, comparing the values of \( f \) at the four critical points we find that the absolute maximum is achieved at both \( P_1 \) and \( P_2 \), where the value of \( f \) is 0, while the absolute minimum is achieved at \( P_4 \), where \( f \) takes the value \(-\sqrt{2} - \frac{3}{2}\).

Problem 4. By Gauss’ Theorem, the integral we must compute can be found by evaluating the integral of \( \nabla \cdot F \) on the interior of the cube:

\[
\iint_{\partial D} F \cdot dS = \iiint_D (\nabla \cdot F) \, dV,
\]

where \( D = [0, 1]^3 \subset \mathbb{R}^3 \). In our case

\[
\nabla \cdot F(x, y, z) = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(z^8) + \frac{\partial}{\partial z}(-2xyz) = 2xy - 2xy = 0.
\]

Therefore, we can easily estimate the integral as

\[
\iint_{\partial D} F \cdot dS = \iiint_D (\nabla \cdot F) \, dV = \int_0^1 \int_0^1 \int_0^1 0 \, dx \, dy \, dz = 0.
\]

Problem 5. Solution:

\( z = 0 \) is a removable singular point, \( z = 1 \) is a double pole.

\[
\text{Res}_{z=1} f(z) = \frac{d}{dz} \sin(z) \bigg|_{z=1} = \cos 1 - \sin 1.
\]
Problem 6.
\[
\int_0^\infty \frac{\cos(2x)}{x^2 + 1}\, dx = \frac{1}{2} \int_0^\infty \frac{e^{2ix}}{x^2 + 1}\, dx
\]
\[
= \pi i \text{Res}_{z=i} \left[ \frac{e^{2iz}}{z^2 + 1} \right] - \frac{1}{2} \int_{C_R} \frac{e^{2iz}}{z^2 + 1}\, dz,
\]
where \( C_R \) is the upper half of the circle \( |z| = R \) from \( z = R \) to \( z = -R \). Note that
\[
\text{Res}_{z=i} \left[ \frac{e^{2iz}}{z^2 + 1} \right] = e^{-2}\frac{1}{2i}
\]
and
\[
\left| \int_{C_R} \frac{e^{2iz}}{z^2 + 1}\, dz \right| \leq \pi R \cdot \frac{1}{R^2} \to 0, \quad \text{as} \quad R \to \infty.
\]
Therefore we obtain
\[
\int_0^\infty \frac{\cos(2x)}{x^2 + 1}\, dx = \frac{\pi e^{-2}}{2}.
\]

Problem 7. Homogeneous solution has the form \( \vec{x}_H = t^r \vec{c}, rI \vec{c} = A \vec{c} \), so that \( r \) is the eigenvalue of \( A \), and \( \vec{c} \) is the eigenvector of \( A \). Thus, \( r = 0, 2 \) and \( \vec{c} = (1, -1)^T, (1, 1)^T \), and the homogeneous solution is
\[
\vec{x}_H(t) = C_1 t^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]
A particular solution for \( t(0, 1)^T \) has the form \( \vec{x}_p = t \vec{a} \), where \( \vec{a} = A \vec{a} + (0, 1)^T \), which results in \( \vec{a} = (-1, 0)^T \). A particular solution for \( t^2(1, 0)^T \) has the form \( \vec{x}_p = t^2 \log t \vec{a} + \vec{\beta} \), from where it can be found that \( A \vec{a} = 2 \vec{a} \) and \( A \vec{\beta} = 2 \vec{\beta} + \vec{a} - (1, 0)^T \). We set \( \vec{a} = k(1, 1)^T \) the eigenvector of \( A \), then
\[
\begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} k - 1 \\ k \end{pmatrix},
\]
from which it follows that \( k = 1/2 \). In the end,
\[
\vec{x}(t) = C_1 t^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{t^2 \log t}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

Problem 8.

- By change of variables: \( z(x) = xy(x) \), then obtain \( z'' + z = 0 \), \( z(x) = C_1 \sin x + C_2 \cos x \), and, therefore,
\[
y(x) = C_1 \frac{\sin x}{x} + C_2 \frac{\cos x}{x}.
\]

- By Taylor/Frobenius expansion:
\[
y(x) = x^r \sum_{n=0}^{\infty} a_n x^n,
\]
where \( r(r - 1) + 2r = 0 \), and, therefore \( r = 0, -1 \). For \( r = 0 \) we have
\[
\sum_{n=0}^{\infty} a_{n+2}(n + 2)(n + 1)x^n + 2 \sum_{n=1}^{\infty} na_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0.
\]
Here \( a_1 = 0 \) and
\[
a_{n+2} = -\frac{a_n}{(n+2)(n+3)},
\]
and thus one obtains
\[
a_n = a_0 \left( 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots \right) = a_0 \frac{\sin x}{x}.
\]
For \( r = -1 \) we obtain similarly
\[
b_n = b_0 \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right) = b_0 \frac{\cos x}{x}.
\]

**Problem 9.** Homogeneous problem has a solution \( y_H(x) = \sin x \). Then,
\[
\int_0^{\pi/2} \sin x (y'' + y) \, dx = \int_0^{\pi/2} \sin x (A + \sin x) \, dx
\]
\[
(\sin x)y' - (\cos x)y|^{\pi/2}_0 = A + \pi/4,
\]
thus, \( A = -\pi/4 \) needed. General solution to the problem is
\[
y(x) = C_1 \cos x + C_2 \sin x + A - \frac{x}{2} \cos x.
\]
From boundary conditions obtain \( C_1 = -A \) and \( -C_1 + (\pi/4) \sin(\pi/2) \), thus \( C_1 = \pi/4, A = -\pi/4 \). The solution is not unique since \( C_2 \) is arbitrary.

Eigenfunction expansion: for \( \phi''(x) = \lambda \phi(x) \) and the given boundary conditions we have \( \phi_n(x) = \sin((2n+1)x), \lambda_n = (2n+1)^2, n = 0,1,2,\ldots \). Then we have
\[
\sin x = \sum_{n=0}^{\infty} \delta_{0n} \phi_n(x), \quad A = \sum_{n=0}^{\infty} d_n \phi_n(x),
\]
\[
d_n = A \frac{\phi_n(x)}{\int_0^{\pi/2} \phi_n^2(x) \, dx} = -\frac{\pi}{4} \frac{1}{2n+1} = - \frac{1}{2n+1},
\]
\[
y''(x) = -\sum_{n=0}^{\infty} \lambda_n c_n \phi_n(x),
\]
\[
(1 - \lambda_n) c_n = -\frac{1}{2n+1} + \delta_{0n}.
\]
If \( n = 0 \) \( c_0 \) is free, for \( n > 0 \) we have
\[
c_n = -\frac{1}{2n+1} 1 - \lambda_n = -\frac{1}{4n(n+1)(2n+1)},
\]
hence
\[
y(x) = c_0 \sin x + \sum_{n=1}^{\infty} \frac{\sin((2n+1)x)}{4n(n+1)(2n+1)},
\]
Since
\[
y'(0) = C_2 - \frac{1}{2} = c_0 + \sum_{n=1}^{\infty} \frac{1}{4n(n+1)} = c_0 + \frac{1}{4},
\]
it follows that
\[
C_2 = c_0 + \frac{3}{4}.
\]
Problem 10. Express

\[ u(x, t) = T(t)\phi(x) + \frac{B}{2}x, \]

with \( \phi'' + \lambda^2 \phi = 0 \). Hence \( \phi = \sin \lambda x \), with \( \sin 2\lambda = 0 \), so \( \lambda_n = \pi n/2 \). Thus, the solution has the form

\[ u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \lambda_n x + \frac{B}{2}x, \]

this takes care of BC and second derivative (it becomes multiplication by \(-\lambda^2\)). For \( T_n(t) \) we obtain

\[ T'_n = -\lambda_n^2 kT_n + q_n(t), \quad q_n(t) = \int_0^2 Q(x, t) \sin \lambda_n x \, dx \]

(the basis functions turn out to be orthonormal on \([0, 2]\)). The solution to the ODE above is obviously

\[ T_n(t) = T_n(0)e^{-\lambda_n^2 kt} + \int_0^t e^{-\lambda_n^2 k(t-s)} q_n(s) ds, \]

with \( T_n(0) \) given by

\[ T_n(0) = \int_0^2 \left( f(x) - \frac{B}{2}x \right) \sin \lambda_n x \, dx. \]

Problem 11. The Green’s function \( G : D \to \mathbb{R} \) solves the equation \( \Delta G(\vec{x}) = \delta(\vec{x}) \) on the same domain, with trivial boundary conditions of the same type. Then, the solution is written as

\[ u(\vec{x}) = \int_{\partial D} \left[ u(\vec{y}) \frac{\partial}{\partial n} G(\vec{x} - \vec{y}) - G(\vec{x} - \vec{y}) \frac{\partial}{\partial n} u(\vec{y}) \right] \, ds + \int_D G(\vec{x} - \vec{y}) f(\vec{y}) \, d\vec{y}. \]

For the given domain, it is

\[ u(r, \theta) = \int_0^{2\pi} \int_0^a G(r - s, \theta - \eta) f(s, \eta) \, ds \, d\eta + a \int_0^{2\pi} g(\eta) \frac{\partial}{\partial r} G(r - a, \theta - \eta) \, d\eta - a \int_0^{2\pi} h(\eta) G(r - a, \theta - \eta) \, d\eta. \]

Problem 12. First, we find a family of characteristic curves \( \phi^t y \), such that \( \partial \phi^t y / \partial t = \sin(\phi^t y) \) and \( \phi^0 y = y \), which is given by \( \phi^t y = y + 1 - \cos t \). Then, on the characteristic curve we have

\[ \frac{\partial}{\partial t} u(\phi^t y, t) = tu(\phi^t y, t), \quad u(\phi^t y, t) = u(y, 0)e^{t^2/2}, \]

or

\[ u(y + 1 - \cos t, t) = f(y)e^{t^2/2}. \]

Replacing \( y \) with \( \phi^{-t} x \), we find

\[ u(x, t) = f(x + \cos t - 1)e^{t^2/2}. \]