MATHEMATICS MASTERS EXAMINATION

OPTION 4
COMPUTER SCIENCE

August 19, 2003
2–5 PM

NOTE: Any student whose answers require clarification may be required to submit to an oral examination.

Each of the fourteen numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of your scores on your eight best questions.

Use separate sheets for each question, and do NOT place your name on any of your answer sheets. You will be given separate instructions on the use of these answer sheets. When you have completed a question, place it in the large envelope provided.
Computer Algorithms

1. A directed graph has vertices $A, B, C, D, E, F, G, H, I, J$. In a depth-first traversal of the graph, each vertex is labelled by a pair $(d, f)$, where $d$ is the time at which the vertex is preprocessed (first discovered), and $f$ is the time at which it postprocessed (last backed up from). The vertex labels are

$$
\begin{align*}
A & \quad (1, 20) \\
B & \quad (2, 9) \\
C & \quad (10, 11) \\
D & \quad (12, 19) \\
E & \quad (3, 8) \\
F & \quad (13, 16) \\
G & \quad (17, 18) \\
H & \quad (4, 5) \\
I & \quad (6, 7) \\
J & \quad (14, 15)
\end{align*}
$$

(a) Sketch the depth-first search tree/forest associated with the graph (i.e., the tree or forest formed by all tree edges).

(b) Could the graph have an edge from $J$ to $D$? If so, what type of edge (tree, descendent, cross, or back)?

(c) Could the graph have an edge from $E$ to $D$? If so, what type of edge (tree, descendent, cross, or back)?

(d) What is the maximum number of edges leading out of $I$ that could be present in the graph?

2. (a) Give the exact solution of the following recurrence:

$$
T(n) = 2T\left(\frac{n}{3}\right) + 2, \quad T(2) = 1.
$$

(b) Consider the following problem: "Given an array of $n$ numbers, find the smallest and the largest". Give a divide-and-conquer algorithm for this problem, and determine its worst-case behavior using (a).

We assume that $n$ is a positive power of 2.
Programming Language Design

3. Some languages (e.g., Java) are translated into an intermediate code which is then interpreted, rather than being executed directly.

   (a) State two advantages for taking this approach.

   (b) State one disadvantage for doing so. Does this disadvantage become more or less important as technology (rapidly) improves the performance of computers? Explain your answer.

Combinatorics

4. Find the number of $n$-letter words made from the letters A,B,C,D,E in which the total number of A's and B's is even and the total number of C's and D's and E's is also even. Just to be sure, 0 is an even number.

5. Using generating functions, solve the recurrence

   $$a_n = 2a_{n-2} + n + 1, \quad n \geq 2$$

   $$(a_0, a_1) = (0, 0).$$

   Here is a useful identity that could save you time:

   $$\frac{3x^2 - 2x^3}{(1-x^2)(1-2x^2)} = \frac{-4}{1-x} + \frac{-1}{(1-x)^2} + \frac{6x + 5}{1-2x^2}.$$
Graph Theory

6. Consider the following network:

![Network Diagram]

Assume that we want to find a maximum flow, and we already constructed the following flow:

![Flow Diagram]

Describe the steps of the maximum flow algorithm after this stage until a maximum flow is found. Determine a minimum cut in the network.

Theory of Computation

7. Given a language $L$, let $\text{Suff}(L)$ denote the set of all suffixes of strings in $L$, i.e.,

$$\text{Suff}(L) = \{ u : \text{there is a } v \text{ such that } uv \in L \}.$$ 

(a) Now let $L = \{ a^n b^n : n \geq 0 \}$. Determine $\text{Suff}(L)$.

(b) Show that if $L$ is regular, then $\text{Suff}(L)$ is also regular.

8. Describe an algorithm that decides, given any context-free grammar $G = (V, \Sigma, R, S)$ and any $w \in \Sigma^*$, whether $w \in L(G)$ or not. Explain why the algorithm is correct.

Symbolic Computation

9. Explain the difference between symbolic and automatic differentiation.
Illustrate with an example the difference between the two and give the two Maple commands you need.
Numerical Analysis

10. Using Newton's Method, find a numerical approximation to the Intersection of $e^{x/4}$ and $5.187/x$ on $[2.0, 3.0]$ starting with the endpoint and $k = 1$ having the smallest value of $|f|$, keeping track of the number of ALL function evaluations $k_f$, and tabulating

| k | $k_f$ | $x_k$ | $f_k$ | $f_k'$ | $x_{k+1}$ | $|\Delta x_k|$ |
|---|---|---|---|---|---|---|
| 1 | | | | | | |

until $|\Delta x_k| = |x_{k+1} - x_k| < 5.0 \times 10^{-4}$ and $|f_k| < 5.0 \times 4$. Use 4-Digit Exam Precision: Round to 4 significant digits only when you record an intermediate or final answer in your exam booklet; and continue calculations with these rounded, recorded numbers.

11. Using Forward Gaussian Elimination with Virtual Partial Pivoting, Virtual Row Scaling, and Back Substitution, solve the following algebraic system for the vector $\vec{x} = [x]_{3 \times 1}$,

\[
\begin{align*}
0.5148 & \times x_1 + 7.149 & \times x_2 - 2.426 & \times x_3 = 49.39 \\
7.035 & \times x_1 + 3.353 & \times x_2 + 3.535 & \times x_3 = 69.83 \\
-6.059 & \times x_1 - 10.78 & \times x_2 + 6.846 & \times x_3 = 42.38
\end{align*}
\]

Record Augmented Matrices with Marked Multipliers, Pivot Vectors, Scale Vectors and Scaled Pivots, i.e.,

\[ [A \mid \vec{b}] || \vec{P} \mid || \vec{S} \mid || S \vec{P} ] \],

at each elimination step, including the initial step (note the last item is the absolute value of the potential pivot divided by its row scaling). Compute the relative residual of the approximate solution as the ratio of infinity-norms.

Use 4-Digit Exam Precision: Round to 4 significant digits only when you record an intermediate or final answer in your exam booklet; and continue calculations with these rounded, recorded numbers.

Computational Geometry

12. 

(a) What is meant by “sparsity” of points in a $k$-dimensional space? Explain how sparsity is used in nearest-neighbor problems.

(b) Sparsity can be defined over a $k$-dimensional radius-$d$ ball or over a $k$-dimensional radius-$d$ hypercube. Show that either definition is almost equivalent to the other,
involving only a minor parametric change. What is the change, if any, in converting a sparse ball to a sparse hypercube? What is the change, if any, in going in the other direction?

**Error-Correcting Codes and Cryptography**

13. Do there exist binary codes with the following parameters: [12, 6, 5], [5, 4, 2]? If no, show why not. If yes, give either a generator matrix or a parity-check matrix.

14. Suppose that a positive integer \( z < 400 \) is encrypted as a vector \((1, 1, 8)\), where the coordinates represent the remainders after dividing \( z \) by 3, 8, and 11, respectively. Is \( z \) uniquely determined? If so, then determine \( z \).
Problem 2 Solution (Turan)

a) \[ T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 2 = \]
\[ = 2 \left( 2 \cdot T\left(\frac{n}{4}\right) + 2 \right) + 2 = 4 \cdot T\left(\frac{n}{4}\right) + 4 + 2 = \]
\[ = 4 \left( 2 \cdot T\left(\frac{n}{8}\right) + 2 \right) + 4 + 2 = 8 \cdot T\left(\frac{n}{8}\right) + 8 + 4 + 2 = \]
\[ = \ldots = 2^i \cdot T\left(\frac{n}{2^i}\right) + 2^i + 2^{i-1} + \ldots + 2 = \]
\[ = 2^i \cdot T\left(\frac{n}{2^i}\right) + 2^{i+1} - 2 \]

For \( n = 2^k \), choose \( i = k - 1 \), then
\[ T(n) = 2^{k-1} \cdot T\left(\frac{n}{2^{k-1}}\right) + 2^k - 2 = \]
\[ = 2^{k-1} + 2^k - 2 = \frac{3}{2} n - 2 \]

b) \[
\begin{array}{c}
\square \quad \square \\
\hline
\frac{n}{2} \quad \frac{n}{2} \\
\hline
\end{array}
\]

MAX = maximum of MAX\(_1\) and MAX\(_2\)
MIN = minimum of MIN\(_1\) and MIN\(_2\)

recurrence for # of comparisons:
\[ T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 2, \quad T(2) = 1 \]

by part a) \( T(n) = \frac{3}{2} n - 2 \)
(1) For each machine, the sole machine-dependent necessity is to create an interpreter, which is far less work than writing a compiler from scratch for each machine.

(2) For programs which are to run on many machines, such as java applets, the code disseminated can universally be simply the intermediate code sequence ("bytecode" in java). The interpreter resident on the user computer then interprets this code.

(3) Intermediate code generation allows "safe" dialects of the full language, such as javascript, and allows safety of the intermediate code to be verified by permitting the interpreter to reject or not recognize lies of the intermediate code that are unsafe.

b) The main disadvantage is execution time, which is typically 5-20 times slower than well-compiled code, at least for "fast" language such as C++ or pascal. However, for short programs, a fast processor can render the slower execution time practically invisible since other operations on websites typically take much longer than the interpretive execution of an applet program, and this effect becomes more pronounced as processors become faster.
1. Find the number of \( n \)-letter words made from the letters A,B,C,D,E in which the total number of A’s and B’s is even (possibly zero) and the total number of C’s and D’s and E’s is also even.

Solution: The exponential generating function for the number of words made from A’s and B’s only is

\[
\sum_{n \geq 0, \text{ n even}} \frac{2^n x^n}{n!} = \frac{e^{2x} + e^{-2x}}{2}.
\]

Similarly, the exponential generating function for the number of words made from C’s and D’s and E’s only is

\[
\sum_{n \geq 0, \text{ n even}} \frac{3^n x^n}{n!} = \frac{e^{3x} + e^{-3x}}{2}.
\]

Therefore the total exponential generating function is

\[
\frac{1}{4}(e^{2x} + e^{-2x})(e^{3x} + e^{-3x}) = \frac{1}{4}(e^{5x} + e^{-5x} + e^{x} + e^{-x}),
\]

and the answer is the coefficient of \( e^{n}/n! \) in this function, namely

\[
\frac{1}{4} (5^n + (-5)^n + 1 + (-1)^n).
\]
2. Using generating functions, solve the recurrence

\[ a_n = 2a_{n-2} + n + 1, \quad n \geq 2 \]

\[ (a_0, a_1) = (0, 0). \]

Here is a useful identity that could save you time

\[ \frac{3x^2 - 2x^3}{(1-x)^2(1-2x^2)} = -4 \frac{1}{1-x} + \frac{-1}{(1-x)^2} + \frac{6x + 5}{1-2x^2}. \]

Solution: Let

\[ A(x) = \sum_{n \geq 0} a_n x^n. \]

Multiply the recurrence by \( x^n \) and sum over \( n \geq 2 \):

\[ \sum_{n \geq 2} a_n x^n = 2 \sum_{n \geq 2} a_{n-2} x^n + \sum_{n \geq 2} (n+1)x^n, \]

so

\[ A(x) = 2x^2A(x) + \sum_{n \geq 2} (n+1)x^n. \]

We have

\[ \sum_{n \geq 2} (n+1)x^n = \frac{d}{dx} \sum_{n \geq 2} x^{n+1} = \frac{d}{dx} \frac{x^3}{1-x} = \frac{3x^2 - 2x^3}{(1-x)^2}. \]

From this we obtain the generating function

\[ A(x) = \frac{3x^2 - 2x^3}{(1-x)^2(1-2x^2)} = -4 \frac{1}{1-x} + \frac{-1}{(1-x)^2} + \frac{6x + 5}{1-2x^2}, \]

so

\[ A(x) = -4 \sum_{m \geq 0} x^m - \sum_{m \geq 0} (m+1)x^m + (6x + 5) \sum_{m \geq 0} 2^m x^{2m}, \]

and by extracting the coefficient of \( x^n \), we obtain

\[ a_n = \begin{cases} 
-4 - (n+1) + 5 \cdot 2^{n/2}, & n \text{ even} \\
-4 - (n+1) + 6 \cdot 2^{(n-1)/2}, & n \text{ odd}.
\end{cases} \]
\[
\text{missis} \\
(\text{Lera)}
\]
Theory of Computation questions Master’s Full 2003

1. Given a language $L$, let $\text{Suff}(L)$ denote the set of all suffixes of strings in $L$, i.e.,

$$\text{Suff}(L) = \{ u : \text{there is a } u \text{ such that } uv \in L \}.$$  

a) Now let $L = \{ a^n b^m : n \geq 0 \}$. Determine $\text{Suff}(L)$.

b) Show that if $L$ is regular, then $\text{Suff}(L)$ is also regular.

2. Describe an algorithm that decides, given any context-free grammar $G = (V, \Sigma, R, S)$ and any $w \in \Sigma^*$, whether $w \in L(G)$ or not. Explain why the algorithm is correct.

SOLUTIONS:

1 a). $\text{Suff}(L) = \{ a^n b^m : 0 \leq n \leq m; m, n \in \mathbb{N} \}$.

b) If $L$ is regular, then there is a NFA $M$ such that $L = L(M)$, where $M = (Q, \Sigma, \delta, q_0, F)$. Let $M'$ be the NFA $(Q, \Sigma, \delta', q_0, F)$, where for $q \in Q$, $\sigma \in \Sigma \cup \{ \varepsilon \}$, $\delta'(q, \sigma) = \delta(q, \sigma)$ except when $q = q_0 \sigma = \varepsilon$:

$$\delta'(q_0, \varepsilon) = \delta(q_0, \varepsilon) \cup \{ q \in Q : q \text{ is on a path from } q_0 \text{ to a state } f \in F \}.$$  

Then for any string $v \in \Sigma^*$ we have $v \in L(M')$ if and only if there is a string $u \in \Sigma^*$ such that $uv \in L(M)$, i.e., if and only if $v \in \text{Suff}(L)$. Thus $\text{Suff}(L)$ is regular.


2. If $w = \varepsilon$, then $w \in L(G)$ iff $S \rightarrow \varepsilon$ is a rule of $G'$.

3. If $w \neq \varepsilon$, list all derivations in $G'$ with $2|w| - 1$ steps. If any of these derivations generate $w$, then $w \in L(G)$; if none of these derivations generate $w$, then $w \notin L(G)$.

Explanation: Every CFG grammar can be converted into an equivalent one in CNF, in a finite number of steps. In a grammar in CNF, a derivation of a string of terminals of length $n > 0$ takes precisely $2n - 1$ steps. Therefore, if $|w| = n > 0$, and $w$ is not among the strings derived in $2n - 1$ steps, then $w \notin L(G)$. The number of derivations of length $2n - 1$ is finite; they can be listed by a simple algorithm. Finally, if $w = \varepsilon$, then $w \in L(G)$ iff $S \rightarrow \varepsilon$ is a rule of $G'$.
Questions for MCS 320: Introduction to Symbolic Computation
Masters's exam Fall'03 CS option

1. Explain the difference between symbolic and automatic differentiation.
Illustrate with an example the difference between the two and give the two Maple commands you need.

Answer to the Question

1. Symbolic differentiation is the differentiation of formulas and is done by the command diff.
Automatic differentiation is the differentiation of procedures and is done by the command D.

For example,
diff(x^4,x$3$) gives the third derivative of the formula $x^4$
D[1]$\overrightarrow{\Delta}$(x \rightarrow $x^4$) gives the procedure which evaluates the third derivative of $x^4$
1. Newton’s Method Problem Answer:
Let \( f(x) = \exp(x/4) - 5.187/x \), so that \( f'(x) = \exp(x/4)/4 + 5.187/x^2 \). Since \( f(2) = (4r) - 0.9448 \) is greater in magnitude than \( f(3) = (4r) + 0.3880 \), \( x_1 = 3.0 \) is the better start for Newton’s method, \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( k_{/0} )</th>
<th>( x_k )</th>
<th>( f_k )</th>
<th>( f'_{k} )</th>
<th>( x_{k+1} )</th>
<th>( \Delta x_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>3.000</td>
<td>3.880e-1</td>
<td>1.106</td>
<td>2.649</td>
<td>-3.509e-1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>2.649</td>
<td>-1.888e-2</td>
<td>1.224</td>
<td>2.664</td>
<td>1.543e-2</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>2.664</td>
<td>-5.157e-5</td>
<td>1.217</td>
<td>2.665</td>
<td>4.237e-5</td>
</tr>
</tbody>
</table>

satisfying the joint stopping criteria with tolerance \( 5.e - 4 \) in \( x \) and \( y \).

2. FGE/VPP/VRS answer:

\[
\begin{bmatrix}
A & \|\bar{b}\| & \|\bar{x}\| & \|\bar{p}\|
\end{bmatrix}
\begin{bmatrix}
0.5148 & 7.149 & -2.426 & 49.39 & 1 & 7.149 & 0.7201e-1

7.035 & 3.535 & 3.535 & 69.83 & 2 & 7.035 & 1.000

-6.059 & -10.78 & 6.846 & 42.38 & 3 & 10.78 & 0.5621

(0.7318e-1)_m & 6.904 & -2.685 & 44.26 & 2 & 6.904 & 1.000

(7.035)_p & 3.535 & 3.535 & 69.83 & 1 & --- & --- & ---

(-0.8613)_m & -7.892 & 9.891 & 102.5 & 3 & 9.891 & 0.7979

(0.7318e-1)_m & 6.904 & -2.685 & 44.26 & 2 & --- & --- & ---

(7.035)_p & 3.535 & 3.535 & 69.83 & 1 & --- & --- & ---

(-0.8613)_m & -1.143 & 6.822 & 153.1 & 3 & --- & --- & ---
\end{bmatrix}
\]

Back Substitution:
\( x_3 = (4r)153.1/6.822 = (4r)22.44; \)
\( x_2 = (4r)(44.28 - (-2.685) * x_3)/6.904 = (4r)15.14; \)
\( x_1 = (4r)(69.83 - 3.535 * x_3 - 3.535 * x_2)/7.035 = (4r) - 8.566; \)

Accuracy:
\( f = \|\bar{b}\| - A * \bar{x} = (4r)[-3.505e - 2; 1.990 - 3; 6.357e - 2]; \)
\( relres = ||\bar{f}||/||\bar{b}|| + \infty = (4r)6.357e - 2/69.83 = (4r)9.104e - 4. \)
(a) Sparsity is expressed as a pair \((c, r)\) where, in dimensional \(k\)-space, no \(k\)-ball (resp. \(k\)-hypercube) of radius \(r\) may contain more than \(c\) points.

(b) Perforce, \(k\)-hypercube \((c, r)\) sparsity is also \((c, r)\) \(k\)-ball sparsity, since a \(k\)-ball fits inside a \(k\)-hypercube of the same radius. In the other direction, there is an expansion factor \(e(k)\) [an exponential function of \(k\)] expressing how many nonintersecting \(k\)-balls can be simultaneously packed so as to intersect a \(k\)-cube. Therefore \((c, r)\) \(k\)-ball sparsity translates to \((ce(k), r)\) sparsity in a \(k\)-hypercube.
Coding

Do there exist binary codes with the following parameters: \([12, 6, 5^-\), \([5, 4, 2^-\). If no, show why not. If yes, give either a generator matrix or a parity check matrix.

Solution: \([5, 4, 2^-\) exists, \(H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \)

\([12, 6, 5^-\) does not exist. If it existed it would have the following # of cosets of the following weights:

<table>
<thead>
<tr>
<th>wt</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>((\frac{12}{2}) = 6)</td>
</tr>
</tbody>
</table>

Since \(d = 5\), as such a code has only \(2^6 = 64\)
Cryptography

Users of RSA know that if both \( n \) and \( \phi(n) \) are known, then \( n \) can be factored. If \( n = 5 \cdot 183 \) and \( \phi(n) = 5040 \), factor \( n \).

Solution: \( \phi(q) = 5 \cdot 183 \), \((p-1)(q-1) = pq - q - p + 1 = 5040 \).

Hence \( p + q + 1 = 143 \), \( p + q = 142 \) so that \( p(142 - p) = 5 \cdot 183 \).

\[
p^2 - 144p + 5 \cdot 183 = 0
\]

\[
p = \frac{144 \pm \sqrt{20736 - 20732}}{2} = \frac{144 \pm 2}{2} = 72 \pm 1
\]

\[p + q = n = 71 \times 73\]