MASTERS EXAMINATION IN MATHEMATICS

PURE MATH OPTION, Fall 2015

Full points can be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth 20 points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS. When you have finished, insert all your answer sheets into the envelope provided, then seal it.

Any student whose answers need clarification may be required to submit to an oral examination.

Algebra

A1. Classify groups of order 55 up to isomorphism.

Solution. The 11-Sylow subgroup is $\mathbb{Z}/11\mathbb{Z}$; the 5-Sylow subgroup is $\mathbb{Z}/5\mathbb{Z}$. By Sylow’s theorem, the 11-Sylow subgroup is normal. Hence, the group is a semi-direct product of its 5 and 11-Sylow subgroups. Since $\text{Aut}(\mathbb{Z}/11\mathbb{Z}) = \mathbb{Z}/10\mathbb{Z}$ has a unique subgroup of order 5, there are up to isomorphism exactly two groups of order 55: the abelian group $\mathbb{Z}/55\mathbb{Z}$ and the group with presentation $\langle x, y | x^5 = y^{11} = 1, xyx^{-1} = y^3 \rangle$.

A2.

(1) Let $G$ be a group. Show that the center

$Z(G) := \{ g \in G \mid gh = hg \text{ for all } h \in G \}$

is a normal subgroup of $G$.

(2) Let $p$ be a prime number. Show that the center of a group of order $p^3$ has order $p$ or $p^2$.

Solution. (i) If $g, f \in Z(G)$ and $h \in G$, then $ghf = gfh = fgh$ and $g^{-1}f = g^{-1}fg^{-1} = g^{-1}fg^{-1} = f^{-1}g$, so also $g^{-1}, gh \in Z(G)$. Similarly, if $g \in Z(G)$ and $f \in G$, then $hgh^{-1} \in G \in Z(G)$. So $Z(G)$ is a normal subgroup.

(ii) Since $G$ has order the power of a prime, $G$ has non-trivial centre and thus $|Z(G)|$ is either $p$, $p^2$ or $p^3$. Now, if $|Z(G)| = p^2$, then $G/Z(G)$ is cyclic of order $p$ and thus there is an element $a \in G$ so that every element $g \in G$ can be written on the form $g = a^kx$ for some $x \in Z(G)$ and $k < p$. But then $G$ is clearly abelian and actually $Z(G) = G$.

A3. Calculate the Galois group of the splitting field of the polynomial $x^4 - 2$ over the field of rational numbers $\mathbb{Q}$. 

1
Solution. The roots of $x^4 = 2$ are $\pm 2^{1/4}$ and $\pm i2^{1/4}$. The splitting field is $\mathbb{Q}(\pm 2^{1/4}, i)$. The degree of the splitting field is 8, so the Galois group has order 8 and it is a transitive subgroup of $S_4$. Consequently, it has to be isomorphic to the dihedral group with 8 elements. Explicitly, it is generated by complex conjugation and $2^{1/4}$ maps to $i2^{1/4}$.

Complex Analysis

C1. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^4 + 1} \, dx$$

Solution. This is the real part of $\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^4 + 1} \, dx$. Since the integrand of the imaginary part is odd, the real part is the whole integral. We apply the residue theorem. There are two poles to consider, which are the roots of $x^4 + 1 = 0$ in the upper half plane. These are $z_1 = e^{\pi i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ and $z_2 = e^{\frac{3\pi}{4}i} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$. Thus the integral will be equal to

$$2\pi i \left( \text{Res} \left. \frac{e^{2iz}}{z^4 + 1} \right|_{z = z_1} + \text{Res} \left. \frac{e^{2iz}}{z^4 + 1} \right|_{z = z_2} \right)$$

The first residue is

$$\left. \frac{e^{2iz}}{(z^4 + 1)'} \right|_{z = z_1} = \frac{e^{2iz_1}}{4z_1^3}$$

$$= \frac{e^{2iz_1}}{4z_1^3}$$

$$= -\frac{1}{4} z_1 e^{2iz_1}$$

$$= -\frac{1}{4} \sqrt{2} e^{-\sqrt{2}} [\cos \sqrt{2} - \sin \sqrt{2} + i(\cos \sqrt{2} + \sin \sqrt{2})]$$

A similar calculation reveals that the second residue is

$$-\frac{1}{4\sqrt{2}} e^{-\sqrt{2}} [(-\cos \sqrt{2} + \sin \sqrt{2} + i(\cos \sqrt{2} + \sin \sqrt{2})]$$

Adding these together and multiplying by $2\pi i$ results in a final answer of

$$\frac{\pi}{\sqrt{2}} e^{-\sqrt{2}} (\cos \sqrt{2} + \sin \sqrt{2})$$

C2. Find the number of zeros of the polynomial

$$f(z) = 3z^9 + 8z^6 + z^5 + 2z^3 + 1$$

in the annulus $\{ z : 1 < |z| < 2 \}$.

Solution.
On $|z| = 2$, $|3z^9| = 3 \times 2^9 > 8 \times 2^6 + 2 \times 2^3 + 1 \geq |8z^6 + z^5 + 2z^3 + 1|$. Thus by Rouche’s Theorem, $3z^9$ and $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ have the same number of zeroes inside $|z| = 2$, namely 9.

On $|z| = 1$, $|8z^6| = 8 > 3 + 1 + 2 + 1 = |3z^9 + z^5 + 2z^3 + 1|$, so inside $|z| = 1$, $8z^6$ and $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ have the same number of zeroes inside $|z| = 1$, namely 6.

Hence on the annulus $1 < |z| < 2$, the polynomial $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ has $9 - 6 = 3$ zeroes (counting multiplicities).

C3. Prove the Fundamental Theorem of Algebra, which states that every polynomial with complex coefficients has a root.

Solution. Suppose $f(z)$ is a polynomial and $f(z) \neq 0$ for all $z \in \mathbb{C}$. Write $f(z) = a_nz^n + \ldots + a_1z + a_0$. Then there is an $R > 0$ such that if $|z| > R$ then

$$|f(z)| = |a_nz^n| \left| 1 + \frac{a_{n-1}}{a_n} \frac{1}{z} + \ldots + \frac{a_1}{a_n} \frac{1}{z^{n-1}} + \frac{a_0}{a_n} \frac{1}{z^n} \right|$$

$$> \frac{1}{2}|a_nz^n|$$

But then by Rouche’s Theorem, $f(z)$ and $a_0z^n$ have the same number of zeroes on $|z| < R$, namely $n$. Thus we have a contradiction.

Number Theory

N1.

a) Find all rational numbers $t$ such that $3t^3 + 10t^2 - 3t$ is an integer.

b) Show that for integers $a, b, c$ we have

$$\gcd(a, \text{lcm}(b, c)) = \text{lcm}(\gcd(a, b), \gcd(a, c))$$

Solution.

a). Write $t = a/b$ with $a$ and $b$ relatively prime integers with $b$ positive. Then

$$3t^3 + 10t^2 - 3t = \frac{3a^3 + 10a^2b - 3ab^2}{b^3}$$

If this is an integer, $b^3$ divides the numerator. So $b$ divides the numerator. Hence $b$ also divides $(3a^3 + 10a^2b - 3ab^2) - (10a^2b - 3ab^2) = 3a^3$. Since $a$ and $b$ are relatively prime $b = 1$ or 3. In the latter case,

$$3t^3 + 10t^2 - 3t = \frac{3a^3 + 30a^2 - 27a}{27}$$

$$= \frac{a^3 + 10a^2}{9} - a$$
Thus we need to know when $9$ divides $a^2(a + 10)$. This occurs when $a$ is a multiple of $3$ and when $a = -1 \pmod{9}$. Since $a$ and $b$ are relatively prime and $b = 3$, only the latter case may occur.

Thus either $b = 1$, or $b = 3$ and $a = -1 \pmod{9}$.

b) For a given prime $p$, let $p^k$ be the highest power of $p$ dividing $a$, $p^l$ be the highest power of $p$ dividing $b$, and $p^m$ be the highest power of $p$ dividing $c$. Then the highest power of $p$ dividing $\gcd(a, \text{lcm}(b, c))$ is $\min(k, \max(l, m))$, while the highest power of $p$ dividing $\text{lcm}(\gcd(a, b), \gcd(a, c))$ is $\max(\min(k, l), \min(k, m))$. In the event that $l \geq m$, these are both $\min(k, l)$, while in the event that $l \leq m$, these are both $\min(k, m)$. Hence for each prime $p$, the highest power of $p$ dividing $\gcd(a, \text{lcm}(b, c))$ and $\text{lcm}(\gcd(a, b), \gcd(a, c))$ are the same. Hence $\gcd(a, \text{lcm}(b, c)) = \text{lcm}(\gcd(a, b), \gcd(a, c))$.

N2.

Show that the $n$th element of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$ (each integer $m$ is repeated $m$ times) is $\left\lfloor \sqrt{2n + \frac{1}{2}} \right\rfloor$. Here $\lfloor x \rfloor$ is the integer part of $x$.

**Solution.** When $1 + 2 + \ldots + (x - 1) = x(x - 1)/2 < n \leq x(x + 1)/2 = 1 + 2 + \ldots + x$, the sequence takes $n$ to $x$. On the other hand, $\left\lfloor \sqrt{2n + \frac{1}{2}} \right\rfloor = x$ is equivalent to

$$x \leq \sqrt{2n + \frac{1}{2}} < x + 1$$

Subtracting $1/2$ and then squaring this is the same as

$$x^2 - x + 1/4 \leq 2n < x^2 + x + 1/4$$

Equivalently,

$$\frac{x^2 - x}{2} + 1/8 \leq n < \frac{x^2 + x}{2} + \frac{1}{8}$$

Since $(x^2 - x)/2$ and $(x^2 + x)/2$ are integers, this is equivalent to

$$\frac{x^2 - x}{2} < n \leq \frac{x^2 + x}{2}$$

N3. Let $d(n)$ be the number of positive divisors of a non-zero integer $n$.

a) Show that

$$\sum_{k=1}^{n} d(k) = \sum_{k=1}^{n} \lfloor n/k \rfloor.$$ 

b) Conclude that

$$\sum_{k=1}^{n} d(k) = n \log n + f(n)$$

with $|f(n)| \leq n$. 

4
Solution.

There are \([n/k]\) multiples of \(k\) between 1 and \(n\). Thus, the divisor \(k\) contributes \([n/k]\) to the sum \(\sum_{k=1}^{n} d(k)\). Adding over all \(k\) gives part a).

As for b), since \(n/k - 1 < [n/k] \leq n/k\), one has

\[
\sum_{k=1}^{n} \left( \frac{n}{k} - 1 \right) < \sum_{k=1}^{n} d(k) \leq \sum_{k=1}^{n} \frac{n}{k}
\]

If \(H_n\) denotes the \(n\)th harmonic number, this is the same as

\[
n H_n - n < \sum_{k=1}^{n} d(k) < n H_n
\]

By the integral test for example, one has

\[
\log n < H_n \leq \log n + 1
\]

So the above becomes

\[
n \log n - n < \sum_{k=1}^{n} d(k) \leq n \log n + n
\]

This gives part b).

Real Analysis

In each of the **True or False** problems: If true, give a proof of the statement. If false, give a counterexample or a proof that the statement is false.

R1.

(a) Given a sequence \(\{a_n\}\) of real numbers, give the definition of the statement: “\(\lim_{n \to \infty} a_n\) exists.”

(b) **True or False**: \(\lim_{n \to \infty} \frac{n(n!)}{(n+1)!} = 1\).

Solution.

a) The statement “\(\lim_{n \to \infty} a_n\) exists.” is equivalent to the following sentence. ”There exists a real number \(L\) such that for every \(\epsilon > 0\) there exists a natural number \(N\) such that if \(n\) is greater than \(N\), then \(|a_n - L| < \epsilon\).”

b) Note that \(\frac{n(n!)}{(n+1)!} = \frac{n}{n+1}\). Since \(\lim_{n \to \infty} \frac{n}{n+1} = 1\), the result is true.

R2.

(a) Give the definition of convergence for a series \(S = \sum_{n=1}^{\infty} b_n\) where \(S\) is a real number and \(b_n\) is a real number for each natural number \(n\).
(b) Give an example of a series \( \sum_{n=1}^{\infty} b_n \) that converges, but \( \sum_{n=1}^{\infty} |b_n| \) diverges.

**Solution.**

(a) Let \( a_n = \sum_{k=1}^{n} b_k \). Then \( \sum_{k=1}^{\infty} b_k \) is said to be convergent if the series \( a_n \) is a convergent sequence of real numbers. We use the definition of convergence given in problem 1. If \( S = \lim_{n \to \infty} a_n \), then we write \( S = \sum_{k=1}^{\infty} b_k \).

(b) Let \( b_n = (-1)^n/n \).

**R3.**

(a) Suppose that \( A \) is an open subset of the real line.

**True or False:** For no point \( x \in A \) is there an \( \epsilon > 0 \) so that the open interval \((x - \epsilon, x + \epsilon)\) is a subset of \( A \).

(b) **True or False:** There exist subsets of the real numbers that are both open and closed.

(c) **True or False:** A continuous real-valued function, defined on a closed bounded interval is bounded on that interval. (Recall that a function is said to be bounded if its range is a bounded subset of the real line.)

(d) **True or False:** Every open cover of the open interval \((0, 1)\) has a finite subcover.

(e) **True or False:** Given \( \epsilon > 0 \), if \( S \) is a countable subset of the real line, then \( S \) has an open cover by open intervals such that the sum of the lengths of these intervals can be made less than \( \epsilon \).

**Solution.** Part a) is false, although it would have been true if the second word "no" had been replaced by "every". Part b) is true, as the empty set and the real line are each both open and closed. Part c) is true, as this is the Bolzano-Weierstrass Theorem. Part d) is false, although it would have been true if the open interval were replaced by a closed interval. Part e) is true. Let \( S = \{s_1, s_2, \ldots\} \) be a countable enumeration of \( S \). Place and interval of length \( \epsilon/2^{n+1} \) around \( s_n \). Then the sum of the lengths of these intervals is equal to \( \epsilon(1/4 + 1/8 + 1/16 + \ldots) = \epsilon/2 < \epsilon \).

**Logic**

**L1.** Suppose that \( \Gamma \) is a consistent set of first order formulas in a countable language. Show that there is a consistent set of formulas \( \Delta \supseteq \Gamma \), such that for every formula \( \phi \), either \( \phi \in \Delta \) or \( \neg \phi \in \Delta \).

**Solution:**

Let \( \{\phi_n \mid n \in \mathbb{N}\} \) enumerate all formulas in the language. Define \( \Delta_n \) for \( n \in \mathbb{N} \) inductively as follows. \( \Delta_0 = \Gamma \). \( \Delta_{n+1} = \Delta_n \cup \{\phi_n\} \), if this is consistent, and otherwise set \( \Delta_{n+1} = \).
\( \Delta_n \cup \{ \neg \phi_n \} \). Set \( \Delta = \bigcup_n \Delta_n \). Clearly \( \Delta \supseteq \Gamma \), and is such that for every formula \( \phi \), either \( \phi \in \Delta \) or \( \neg \phi \in \Delta \). It remains to show that \( \Delta \) is consistent.

By compactness, it is enough to show that every \( \Delta_n \) is consistent. Do this by induction on \( n \). \( \Delta_0 = \Gamma \), which is consistent. Suppose \( \Delta_n \) is consistent, we have to show it for \( \Delta_{n+1} \). If \( \Delta_{n+1} = \Delta_n \cup \{ \phi_n \} \), this follows by definition. Otherwise, \( \Delta_{n+1} = \Delta_n \cup \{ \neg \phi_n \} \) and \( \Delta_{n+1} = \Delta_n \cup \{ \phi_n \} \) is not consistent.

Then for some \( \{ \alpha_1, ..., \alpha_k \} \subseteq \Delta_n \), we have that \( \{ \alpha_1, ..., \alpha_k, \phi \} \) is inconsistent. Suppose for contradiction, \( \Delta_{n+1} \) is not consistent. By compactness, let \( \{ \beta_1, ..., \alpha_m \} \subseteq \Delta_n \), be such that \( \{ \beta_1, ..., \beta_m, \neg \phi \} \) is inconsistent. Then:

- \( \{ \alpha_1, ..., \alpha_k \} \models \neg \phi \),
- \( \{ \beta_1, ..., \beta_m \} \models \phi \)

So, \( \Delta_n \) is not consistent. Contradiction with the inductive hypothesis. Suppose that \( \Gamma \) is a consistent set of first order formulas in a countable language. Show that there is a consistent set of formulas \( \Delta \supseteq \Gamma \), such that for every formula \( \phi \), either \( \phi \in \Delta \) or \( \neg \phi \in \Delta \).

L2.

(a) State the Soundness, Completeness and Compactness theorems.

(b) Prove the Compactness theorem, assuming Soundness and Completeness.

Solution.

(a)

(1) Soundness: if \( \Gamma \vdash \phi \), then \( \Gamma \models \phi \).

(2) Completeness: if \( \Gamma \models \phi \), then \( \Gamma \vdash \phi \).

(3) Compactness: if \( \Gamma \) is finitely satisfiable, then \( \Gamma \) is satisfiable.

(b) Suppose that \( \Gamma \) is finitely satisfiable. Suppose for contradiction, that \( \Gamma \) is not satisfiable. Then, vacuously, \( \Gamma \models \alpha \land \neg \alpha \). So, by Completeness, \( \Gamma \vdash \alpha \land \neg \alpha \). But since deductions are finite, there must be some finite subset \( \Delta \subseteq \Gamma \), such that \( \Delta \vdash \alpha \land \neg \alpha \). Then \( \Delta \) is inconsistent, and so by Soundness it is not satisfiable. Contradiction with our assumptions.

L3.

Let \( \lambda := \{ F \} \), where \( F \) is a unary function symbol.

(a) Write down an \( \lambda \)-sentence \( \sigma \) so that, for any \( \lambda \)-structure \( (M, f) \), we have \( (M, f) \models \sigma \) if and only if \( f : M \to M \) is a bijection with no fixed points.

(b) Suppose that \( X \) is an infinite set. Prove that there is a bijection \( f : X \to X \) with no fixed points.

Solution.

(a) \( \sigma := (\forall x)(\exists y)(y \neq x \land f(x) = y \land (\forall z)(f(x) = z \rightarrow y = z)) \land (\forall y)(\exists x)(f(x) = y \land (\forall z)(f(z) = y \rightarrow x = z)) \).
Let \( f = (\exists x_1)(\exists x_2)\ldots(\exists x_k)\Lambda_{i<j}x_i \neq x_j \).

Let \( \Sigma = \{\sigma\} \cup \{\lambda_k \mid k > 1\} \). Clearly any finite subset is satisfiable, and therefore so is \( \Sigma \).

Then by Lowenheim-Skolem, there is a model \( \mathfrak{A} \models \sigma \) with \( |\mathfrak{A}| = X \). Fix a bijection \( h : X \rightarrow |\mathfrak{A}| \), and define \( \phi : X \rightarrow X \) by \( \phi(i) = j \) iff
\[
\mathfrak{A} \models f(x) = y[h(i), h(j)].
\]

Then \( \phi \) is as desired.

**Topology**

**T1.** Let \( (X, d) \) be a compact metric space, and let \( f : X \rightarrow \mathbb{R} \) be a continuous function. Prove that \( f \) is uniformly continuous (start by defining the latter notion).

**Solution.** A function \( f : X \rightarrow \mathbb{R} \) on a metric space is uniformly continuous if given \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that whenever \( d(x, y) < \delta \) one has \( |f(x) - f(y)| < \varepsilon \).

Let \( (X, d) \) be a compact metric space, and \( f : X \rightarrow \mathbb{R} \) is continuous. Assume towards contradiction, that is not uniformly continuous. Thus there exists \( \varepsilon_0 > 0 \) so that for every \( \delta > 0 \) one would be able to find a pair of points \( x, y \in X \) with \( d(x, y) < \delta \) and \( |f(x) - f(y)| \geq \varepsilon_0 \).

In particular, for each \( n \) there is a pair \( x_n, y_n \in X \) with \( d(x_n, y_n) < 1/n \) and \( |f(x_n) - f(y_n)| \geq \varepsilon_0 \). Since \( X \) is compact, there is a subsequence \( n_1 < n_2 < \ldots \) and \( z \in X \), so that \( x_{n_i} \rightarrow z \). As \( f \) was assumed to be continuous, it is continuous at \( z \). Hence there exists \( \eta > 0 \) so that for any \( w \in X \) with \( d(z, w) < \eta \) one has \( |f(z) - f(w)| < \varepsilon_1 = \varepsilon_0/2 \). For \( i \) large enough, we have \( d(x_{n_i}, z) < \eta/2 \) and \( 1/n_i < \eta/2 \). Therefore, not only \( x_{n_i} \) is \( \eta/2 \)-close to \( z \), but also \( y_{n_i} \) is \( \eta \)-close to \( z \), because
\[
d(y_{n_i}, z) \leq d(y_{n_i}, x_n) + d(x_n, z) < \frac{1}{n_i} + d(x_{n_i}, z) < \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\]

It follows that \( |f(x_{n_i}) - f(z)| \) and \( |f(y_{n_i}) - f(z)| \) are less than \( \varepsilon_1 \), producing a contradiction
\[
\varepsilon_0 \leq |f(x_{n_i}) - f(y_{n_i})| \leq |f(x_{n_i}) - f(z)| + |f(z) - f(y_{n_i})| < \varepsilon_1 + \varepsilon_1 = \varepsilon_0
\]
to the assumption that \( f \) is not uniformly continuous.

**T2.** Suppose \( X \) is a Hausdorff space which has no isolated points. Prove that there is a subset \( A \subset X \) which is neither open nor closed.

**Solution.** In a Hausdorff space every singleton \( \{x\} \) is a closed set. Yet, if a point \( x \in X \) is not isolated, then \( W \setminus \{x\} \) is not closed for any open set \( W \) containing \( x \).

Let \( x \neq y \) be two distinct points in \( X \), and let \( x \in U, y \in V \) be their disjoint open neighborhoods. Consider the set \( A = \{x\} \cup (V \setminus \{y\}) \). Then \( A \) is not open, because \( A \cap U = \{x\} \) is not open; and \( A \) is not closed, because \( A \cap U^c = V \setminus \{y\} \) is not closed.
**T3.** A map \( f : X \to Y \) is *locally injective* if every point \( x \in X \) has an open neighborhood on which \( f \) is injective. Suppose \( f \) is locally injective and \( X \) is compact. Prove that there is some \( N \) so that every point of \( Y \) has at most \( N \) pre-images in \( X \).

**Solution.** By the assumption, for every \( x \in X \) there is an open set \( U_x \subset X \) so that \( f : U_x \to Y \) is one-to-one. By compactness, the open cover \( \{U_x \mid x \in X\} \) of \( X \) contains a finite subcover:

\[
X = U_{x_1} \cup \cdots \cup U_{x_n}.
\]

We now claim that every point \( y \in Y \) has no more than \( n \)-preimages. Indeed, assume towards contradiction that \( z_1, \ldots, z_{n+1} \) are distinct points in \( X \) all mapping under \( f \) to the same \( y \in Y \). Then at least one pair \( z_i \neq z_j \) lie in the same \( U_{x_k} \) set (pigeon-hole principle). But this contradicts injectivity of \( f \) on \( U_{x_j} \).