1 Classify groups of order 155.

2 Show that $C \otimes_R C$ and $C \otimes_C C$ are non isomorphic $R$-modules.

3 Show that the rings $F_{11}/(x^2 + 1)$ and $F_{11}/(y^2 + 2y + 2)$ are fields. Are these rings isomorphic?
(C1) Find Taylor or Laurent series for

\[ f(z) = \frac{1}{1 - z^2} \]

in the indicated domains

(a) \(|z| < 1\)  \hspace{1cm} (b) \(|z| > 1\)  \hspace{1cm} (c) \(0 < |z - 1| < 2\).

(C2) Use residues to evaluate the real integral

\[ \int_{-\infty}^{\infty} \frac{1}{x^4 + 3x^2 + 2} \, dx. \]

(C3) Let \(f(z)\) be analytic in \(\mathbb{C}\), except for a simple pole at \(z = 1/2\), where it has residue = 2. Also, \(f(0) = 1\). Evaluate the contour integral

\[ \frac{1}{2\pi i} \int_{C} \frac{f(z)}{z} \, dz, \]

where \(C\) is the unit circle taken counterclockwise.
1. For each of the following sentences determine which are logically valid and which are satisfiable. Justify your answer either by exhibiting a world (model) in which the sentence is true (to show satisfiable) or a world in which the sentence is false (to show not logically valid) or an argument to show the sentence is logically valid.

(a) \( (\exists x A(x) \land \exists x B(x)) \rightarrow \exists y (A(y) \land B(y)) \).
(b) \( \exists x (A(x) \rightarrow B(x)) \rightarrow (\exists x A(x) \rightarrow \exists x B(x)) \).
(c) \( \exists x \forall y C(x, y) \rightarrow \forall y \exists x C(x, y) \)

2. (a) State the compactness theorem for first order logic.

(b) Write a sentence \( \phi_3 \) which is true only if a world with more than three elements.

(c) Prove that if a sentence \( \phi \) of first order predicate logic has arbitrarily large finite models then it has an infinite model.

3. A formula of propositional logic is in negation normal form if every negation sign is applied directly to an atomic formula (propositional variable).

(a) Put the formula \( \neg(p \land (\neg q \lor r)) \) in negation normal form.
(b) What rules would you use to show that every propositional formula (with the propositional connectives \( \neg, \land, \lor \)) can be put in negation normal form.
Number Theory Master’s Exam Problems  
Spring 2004

1. If \( p = 4q + 1 \) and \( q = 3r + 1 \) are prime, show that 3 is a primitive root of \( p \).

2. Find (with proof) all positive integers \( n \) such that \( \phi(n) = 6 \). Here \( \phi \) is the Euler \( \phi \) or totient function.

3. (a) Compute the least positive residue modulo 47 of \( 2^{200} \).
    (b) Does 11 divide 674,310,976,375? Explain your answer.
1) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $\epsilon > 0$. Prove that there is a continuous function $g : [a, b] \to \mathbb{R}$ such that $g(x) \geq f(x)$ for all $x \in [a, b]$ and $\int_a^b g - \int_a^b f < \epsilon$.

2) Suppose that $f : [1, +\infty) \to \mathbb{R}$ is integrable on $[1, r]$ for all $r \geq 1$, $f$ is decreasing, and $f(x) > 0$ for all $x \geq 1$. Prove that $\int_1^{\infty} f$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges.

3) Let $g : [0, 1] \to \mathbb{R}$ be twice differentiable with $g''(x) > 0$ for all $x \in [0, 1]$. Assume that $g(0) > 0$ and $g(1) = 1$. Prove that there is $0 < c < 1$ with $g(c) = c$ if and only if $g'(1) > 1$. 

Real Analysis MS Problems
Masters Exam Questions—Topology, Spring 2004

1. Suppose that \( X \) is a Hausdorff space. Prove that the diagonal
\[
\Delta_X = \{(x, x) : x \in X\}
\]
is a closed subset of \( X \times X \) if \( X \times X \) is given the product topology.

2. Prove that the product of two connected topological spaces is always connected.

3. Suppose that \( f : X \to Y \) is a continuous map between topological spaces. Prove that for every set \( A \subset X \) we have \( f(\overline{A}) \subset \overline{f(A)} \).