1 Classify groups of order 155.
   Solution First notice that $155 = 31 \times 5$. Let $G$ be a group of order 155. By the Sylow theorem the number $n_{31}$ of subgroups of order 31 has form $31k+1$ and divides 5 i.e. $n_{13} = 1$. This implies that the Sylow subgroup $H_{31}$ is normal. Moreover $G$ is a semidirect product of $H_{31}$ and a Sylow subgroup $H_5$ of order 5. Hence $G$ defines the homomorphism $\phi : H_5 \to \text{Aut}H_{31}$. The group of automorphisms of the cyclic group of order 31 is cyclic and has the order equal to 30. If $\phi$ is trivial then $G$ is abelian and hence is a direct product of cyclic groups of order 31 and 5. In this case (i.e. when $\phi$ is trivial) $G$ is cyclic. Remaining 4 choices for $\phi$ yield isomorphic groups. Each isomorphism can be obtained by replacing the generator of $H_5$. Therefore there are, up to isomorphism, only two groups of order 155: one is cyclic and another is a semidirect product given by generators and relations as follows: $\{x, y, | x^{31} = 1, y^5 = 1, yxy^{-1} = x^2 \}$ (since 2 has order 5 in the multiplicative group $(\mathbb{Z}/31\mathbb{Z})^*$).

2 Show that $C \otimes_R C$ and $C \otimes_C C$ are non isomorphic $R$-modules.
   Solution The first tensor product is a tensor product over $R$ of two $R$-vector spaces having dimension equal to 2. Hence it has the dimension over $R$ equal to 4. On the other hand, the second tensor product is the tensor product of two one-dimensional $C$-vector spaces and hence has the dimension equal to one over $C$. Hence over $R$ it has dimension equal two 2. Hence two tensor products have different dimensions as $R$-vector spaces.

3 Show that the rings $F_{11}/(x^2 + 1)$ and $F_{11}/(y^2 + 2y + 2)$ are fields. Are these rings isomorphic?
   Solution Both polynomials $x^2 + 1$ and $y^2 + 2y + 2$ are irreducible over $F_{11}$ since reducible quadratic polynomial must have a linear factor and hence a root (neither of those polynomials has a root in $F_{11}$). This implies that both these rings are fields. Moreover, the degree each of these fields over $F_{11}$ is equal to 2 i.e. they both are fields with $11^2$ elements and since there is only one, up to isomorphism, field extension of given degree of a finite field these two rings are isomorphic.
(a) \[ f(z) = \sum_{n=0}^{\infty} z^{2n}, \quad |z| < 1 \]

(b) \[ f(z) = -\frac{1}{z^2} \left( 1 - \frac{1}{z^2} \right)^{-1} = -\sum_{m=0}^{\infty} \frac{1}{z^{2m+2}}, \quad |z| > 1 \]

(c) \[ f(z) = \frac{\frac{1}{2}}{1-z} + \frac{\frac{1}{2}}{1+z} \]

let \( z-1 = t \)

\[ f(z) = -\frac{1}{2t} + \frac{1}{2} \frac{1}{2+t} = -\frac{1}{2t} + \frac{1}{4} \sum_{l=0}^{\infty} (-1)^l \left( \frac{t}{2} \right)^l \]

(Laurent series converges for \( 0 < |z-1| < 2 \))
\[
\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 + 2}
\]

\[
f(z) = \frac{1}{z^4 + 3z^2 + 2} = \frac{1}{(z^2 + 1)(z^2 + 2)}
\]

has simple poles at \( z = \pm i, \pm \sqrt{2}i \)

Let \( R > 1 \)

Let \( C = C_R + [-R, R] \), by residue theorem

\[
\int_C f(z) \, dz = 2\pi i \left[ \text{res}(i) + \text{res}(\sqrt{2}i) \right]
\]

\[
= \int_R^{C_R} f(x) \, dx + \int_{C_R} f(z) \, dz
\]

\[
| \int_{C_R} f(z) \, dz | \leq \pi R \frac{1}{R^2 - 1} \xrightarrow{R \to \infty} 0
\]

\[
\text{res} (i) = \frac{1}{i^2 + 1} \cdot \frac{1}{2i} = \frac{1}{2i}
\]

\[
\text{res} (\sqrt{2}i) = \frac{1}{(\sqrt{2}i)^2 + 1} \cdot \frac{1}{2\sqrt{2}i} = -\frac{1}{2\sqrt{2}i}
\]

Thus

\[
\int_{-\infty}^{\infty} f(x) \, dx = \pi \left( 1 - \frac{1}{\sqrt{2}} \right)
\]
f(z) has a simple pole at \( z = \frac{1}{2} \) with residue = 2

thus, inside \( |z| = 1 \), \( \frac{f(z)}{z} \) has 2 simple poles, at \( z = 0 \) & \( z = \frac{1}{2} \)

by residue theorem

\[ \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} \, dz = \text{res}(0) + \text{res}(\frac{1}{2}) \]

\[ \text{res}(0) = \lim_{z \to 0} \left[ z \frac{f(z)}{z} \right] = f(0) = 1 \]

\[ \text{res}(\frac{1}{2}) = \lim_{z \to \frac{1}{2}} \left[ (2z - 1) \frac{f(z)}{2z} \right] \]

\[ = \frac{1}{2} \left[ \text{residue of } f(z) \text{ at } z = \frac{1}{2} \right] \]

\[ = 4 \]

thus \( \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} \, dz = 1 + 4 = 5 \)
1. For each of the following sentences determine which are logically valid and which are satisfiable. Justify your answer either by exhibiting a world (model) in which the sentence is true (to show satisfiable) or a world in which the sentence is false (to show not logically valid) or an argument to show the sentence is logically valid.

(a) \((\exists x A(x) \land \exists x B(x)) \to \exists y (A(y) \land B(y))\).  
(b) \(\exists x (A(x) \to B(x)) \to (\exists x A(x) \to \exists x B(x))\).  
(c) \(\exists x \forall y C(x, y) \to \forall y \exists x C(x, y)\)

SOLUTION:
(a) Satisfiable: Consider a world with two objects 0 and 1 in which \(A(0)\) and \(A(1)\) are both true. Not logically valid: Consider a world consisting only of two objects 0 and 1 in which \(A(0)\) and \(B(1)\) are true and \(A(1)\) and \(B(0)\) are false.

(b) Satisfiable: Consider a world with an object 0 in which \(A(0)\) and \(B(0)\) are both true. Not logically valid: Consider a world consisting only of two objects 0 and 1 in which \(A(0)\) is true and \(A(1), B(0)\) and \(B(1)\) are false.

(c) If \(\exists x \forall y C(x, y)\) is true in some world then there is an object \(a\) in this world for which \(C(a, y)\) is true for every \(y\). But then \(\forall y \exists x C(x, y)\) is true in this world.

2. (a) State the compactness theorem for first order logic.
(b) Write a sentence \(\phi_3\) which is true only if a world with more than three elements.
(c) Prove that if a sentence \(\phi\) of first order predicate logic has arbitrarily large finite models then it has an infinite model.

SOLUTION:
(a) If every finite subset of an infinite set of first order sentences \(\Phi\) is satisfiable then the entire set has a model.
(b) \(\phi_3\) is \(\exists x (\exists y)(\exists z) x \neq y \land x \neq z \land y \neq z\).
(c) Let \( \Phi \) be the set of all sentences \( \phi_n \), where \( \phi_n \) is true only in worlds with more than \( n \) elements. By hypothesis \( \bigwedge_{i<N} \phi_i \) is satisfiable for each \( N \). By the compactness theorem \( \Phi \) is satisfiable.

3. A formula of propositional logic is in negation normal form if every negation sign is applied directly to an atomic formula (propositional variable).

(a) Put the formula \( \neg(p \land (\neg q \lor r)) \) in negation normal form.

(b) What rules would you use to show that every propositional formula (with the propositional connectives \( \neg, \land, \lor \)) can be put in negation normal form.

SOLUTION:

(a) \( \neg p \lor (q \land \neg r) \).

(b) Replace any subformula of the form \( \neg(A \land B) \) by \( \neg A \lor \neg B \). Replace any subformula of the form \( \neg(A \lor B) \) by \( \neg A \land \neg B \). Replace any subformula of the form \( \neg\neg A \) by \( A \). Repeat as long as the formulas is not in negation normal form.
1. If $p = 4q + 1$ and $q = 3r + 1$ are prime, show that 3 is a primitive root of $p$.

We must show that the order of 3 is $p-1 = 4q$. For this, it suffices to verify that $3^{(p-1)/2} \not\equiv 1 \pmod{p}$ and $3^{(p-1)/4} \not\equiv 1 \pmod{p}$. (It’s clear that the order of 3 isn’t 1, because $p$ isn’t 2. The first statement shows that the order of 3 divides $p-1$ but not $(p-1)/2$, so is divisible by 4. The second shows that the order is not divisible by 4, so must be divisible by $q$.) Now if $3^{(p-1)/4} = 3^4 \equiv 1 \pmod{p}$ then $p$ divides 80. Therefore either $p = 2$ (which can’t be) or $p = 5$. Fortunately, 3 is a primitive root mod 5. If $3^{(p-1)/2} \equiv 1 \pmod{p}$, then by Euler’s criterion 3 is a primitive root mod $p$. However, $p$ is visibly congruent to 1 mod 4, so quadratic reciprocity then says that $p$ is 1 mod 3. This is impossible under the given conditions on $p$. Therefore 3 is a primitive root.

2. Find (with proof) all positive integers $n$ such that $\phi(n) = 6$. Here $\phi$ is the Euler $\phi$ or totient function.

Use the formula for $\phi(n)$ in terms of the factorization of $n$. If $n$ is divisible by a prime $p$ greater than or equal to 11, then $\phi(n)$ is divisible by a number at least 10. So all prime factors of $n$ must be chosen from 2, 3, 5, 7. The numbers 7 and 14 both work. However, if $n$ is a higher multiple of 7 then $\phi(n)$ will be a proper multiple of 6. So any other solutions must have only divisors 2, 3, 5. If $n$ is divisible by 5 or by 8, then $\phi(n)$ is divisible by 4, and if $n$ is divisible by 27 then $\phi(n)$ is divisible by 9. On the other hand, unless $n$ is divisible by 9, $\phi(n)$ won’t be divisible by 3. The only remaining numbers are 9, 18, and 36, or which 9 and 18 work and 36 doesn’t. So the solutions are 7, 9, 14, 18.

3. (a) Compute the least positive residue modulo 47 of $2^{200}$.
Use Euler's Theorem (really Fermat's Little Theorem). 
\[ 200 = 4 \cdot 46 + 16. \text{ So } 2^{200} \equiv 2^{16} \pmod{47}. \text{ Now } 2^{10} = 1024 \text{ and } 1024 \equiv 37 \pmod{47}. \text{ Also } 2^6 = 64 \text{ and } 64 \cdot 37 = 2368 \text{ and } 2368 \equiv 18 \pmod{47}. \]

(b) Does 11 divide 674,310,976,375? Explain your answer.

Use the alternating sum of the digits test, a consequence of place value and the fact that \[10 \equiv -1 \pmod{11}.\] We get

\[ 6 - 7 + 4 - 3 + 1 - 0 + 9 - 7 + 6 - 3 + 7 - 5 = 8 \]

which isn't zero, so the answer is NO.
1) Suppose $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable and $\epsilon > 0$. Prove that there is a continuous function $g : [a, b] \rightarrow \mathbb{R}$ such that $g(x) \geq f(x)$ for all $x \in [a, b]$ and $\int_{a}^{b} g - \int_{a}^{b} f < \epsilon$.

We can find a partition $P = \{ x_{0}, \ldots, x_{n} \}$ such that $U(f, P) - \int_{a}^{b} f < \epsilon / 2$. For $i = 1, \ldots, n$ let

$$M_{i} = \sup \{ f(x) : x \in [x_{i-1}, x_{i}] \}.$$ 

Consider the step function $h : [a, b] \rightarrow \mathbb{R}$ where

$$h(x) = \begin{cases} M_{i} & \text{if } x \in [x_{i-1}, x_{i}) \\ M_{n} & \text{if } x = b \end{cases}.$$ 

Then $h(x) \geq f(x)$ for all $x \in [a, b]$ and

$$\int_{a}^{b} h = \sum_{i=1}^{n} M_{i}(x_{i} - x_{i-1}) = U(f, P).$$

We next modify $h$ slightly to build a continuous piecewise linear function $g$ such that $g(x) \geq h(x)$ for all $x \in [a, b]$ and $\int_{a}^{b} g - \int_{a}^{b} h < \epsilon / 2$.

If $M_{i-1} < M_{i}$, we will choose $c_{i} < x_{i}$ and define $g$ on $[c_{i}, x_{i}]$ by

$$g(x) = \frac{M_{i} - M_{i-1}}{x_{i} - c_{i}}(x - c_{i}) + M_{i-1}.$$ 

We choose $c_{i}$ so that the triangle with points $(c_{i}, M_{i-1}), (x_{i}, M_{i-1}), (x_{i}, M_{i})$ has area less than $\epsilon / (2n)$. If $M_{i} < M_{i-1}$ we choose $c_{i} > x_{i}$ and proceed in a similar manner. We will have $g(x) \geq h(x) \geq f(x)$ for all $x \in [a, b]$ and

\[
\int_{a}^{b} g - \int_{a}^{b} f \leq \left( \int_{a}^{b} g - \int_{a}^{b} h \right) + \left( \int_{a}^{b} h - \int_{a}^{b} f \right) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
2) Suppose that $f : [1, +\infty) \to \mathbb{R}$ is integrable on $[1, r]$ for all $r \geq 1$, $f$ is decreasing, and $f(x) > 0$ for all $x \geq 1$. Prove that $\int_1^\infty f$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges.

Let $s_n = \sum_{i=1}^n f(i)$. The sequence $(s_n)$ is increasing. Thus it converges if and only if it is bounded. Similarly, the function $F(x) = \int_1^x f$ is increasing. Thus $\int_1^\infty f$ converges if and only if $F$ is bounded.

Consider the partition $P = \{1, \ldots, n\}$. Since $f$ is decreasing,

$$s_{n-1} = U(f, P) \geq \int_1^x f \geq L(f, P) = s_n - f(1).$$

If $(s_n)$ converges to $L$, then $F(x) \leq L$ for all $x$ and $\int_1^\infty f$ converges. On the other hand, if $\int_1^\infty f$ converges to $M$, then the $(s_n)$ is bounded by $M + f(1)$, and, hence, converges.

3) Let $g : [0, 1] \to \mathbb{R}$ be twice differentiable with $g''(x) > 0$ for all $x \in [0, 1]$. Assume that $g(0) > 0$ and $g(1) = 1$. Prove that there is $0 < c < 1$ with $g(c) = c$ if and only if $g'(1) > 1$.

$(\Rightarrow)$ Suppose $g(c) = c$. By the Mean Value Theorem, there is $d \in (c, 1)$ such that

$$g'(d) = \frac{g(1) - g(c)}{1-c} = \frac{1-c}{1-c} = 1.$$ 

Since $g$ is twice-differentiable and $g'' > 0$, $g'$ is increasing. Thus $g'(1) > g'(d) = 1$.

$(\Leftarrow)$ Let $h(x) = g(x) - x$. If $g'(1) > 1$, then $h'(1) > 0$. Since $h'$ is differentiable, it is continuous. Thus there is $0 < a < 1$ such that $h' > 0$ on $(a, 1)$. Thus the function is increasing on this interval and there is $b \in (0, 1)$ with $h(b) < 0$. Since $h(0) > 0$ and $h$ is continuous, there is $c \in (0, 1)$ with $h(c) = 0$. 

1. Let $\Delta_X$ be the diagonal in $X \times X$ and let $(x, y) \not\in \Delta_X$, so $x \neq y$. Since $X$ is Hausdorff, there are open sets $U$ and $V$ with $U \cap V = \emptyset$ and $x \not\in U$, $y \not\in V$. In the product topology on $X \times X$, $U \times V$ is a neighborhood of $(x, y)$ which does not meet $\Delta_X$. Therefore the complement of $\Delta_X$ is open, so $\Delta_X$ is closed in $X \times X$.

2. Let $X$ and $Y$ be connected topological spaces. If $X \times Y$ were not connected it could be written as the union of two disjoint nonempty sets, $W_1$ and $W_2$. Let $(x_1, y_1) \in W_1$ and $(x_2, y_2) \in W_2$. In the subspace topology,

$$X \times y_1 \subset X \times Y$$

is homeomorphic to $X$. Since $X$ is connected and $(x_1, y_1) \in W_1$, we also have $(x_2, y_1) \in W_1$. Similarly

$$x_2 \times Y \subset X \times Y$$

is homeomorphic to $Y$. Since $Y$ is connected and $(x_2, y_1) \in W_1$, we have $(x_2, y_2) \in W_1$, a contradiction. Therefore $X \times Y$ is connected.

3. Suppose $f : X \rightarrow Y$ is continuous and $A$ is any subset of $X$. Let $f^{-1}$ be the map taking subsets of $Y$ to subsets of $X$. By definition of $f^{-1}$, we have

$$f(A) \subset f(A) \quad \text{iff} \quad \bar{A} \subset f^{-1}(f(A)).$$

Again by the definition of $f^{-1}$, $A \subset f^{-1}(f(A)) \subset f^{-1}(f(A))$. Now $f$ is continuous, so $f^{-1}(f(A))$ is closed. It follows that $\bar{A} \subset f^{-1}(f(A))$. 

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