Full points can be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth the same number of points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. **DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS.** When you have finished, insert all your answer sheets into the envelope provided, then seal and print your name on it.

Any student whose answers need clarification may be required to submit to an oral examination.
Algebra

A1. Let $G$ be a group of order $231 = 11 \cdot 7 \cdot 3$ and let $P, Q,$ and $R$ be a Sylow 11-subgroup, Sylow 7-subgroup, and a Sylow 3-subgroup of $G$ respectively.

(a) Show that at least two of $P, Q,$ and $R$ are normal subgroups of $G$.
(b) Show that $PQ$ and $PR$ are subgroups of $G$ which are abelian.
(c) Show that $P$ is in the center of $G$.

Solution:

A.1 (a) Several things by the Sylow Theorems. A Sylow $p$-subgroup of $G$ is a normal subgroup of $G$ if and only if $G$ has exactly one Sylow $p$-subgroup of $G$. The number of Sylow $p$-subgroups of $G$ is $pn + 1$ for some non-negative integer $n$ and this number divides $|G| = 231 = 11 \cdot 7 \cdot 3$.

Case 1: $p = 11$. $11n + 1$ is among $1, 3, 7, 21$. Therefore $n = 0$. Hence $P$ is a normal subgroup of $G$.

Case 2: $p = 7$. $7n + 1$ is among $1, 3, 11, 33$. Therefore $n = 0$. Hence $Q$ is a normal subgroup of $G$.

(b) Since $P$ is a normal subgroup of $G$ it follows that $PH$ is a subgroup of $G$ for all subgroups $H$ of $G$. Thus $PQ$ and $PR$ are subgroups of $G$.

Now $Q$ is a normal subgroup of $PQ$ since it is a normal subgroup of $G$. Since the number of Sylow 3-subgroups of $PR$ is $3n + 1$ and divides $|PR| = |P||R|/|P \cap R| = 11 \cdot 3/1 = 11 \cdot 3$, it follows $n = 0$. Thus $R$ is a normal subgroup of $PR$.

Now $P \cap Q = (e) = P \cap R$ by Lagrange’s Theorem. (This justifies the assertion $|P \cap R| = 1$ in the preceding paragraph.) If $H, K$ are normal, $HK$ is the direct product of $H$ and $K$. Since $P, Q,$ and $R$ are cyclic (hence abelian) as they have prime order, it follows that $PQ$ and $PR$ are commutative.

(c) Consider the set $PQR$. By an argument similar to one in part (b) it follows that $|PQR| = |G|$. Therefore $PQR = G$. Since $PQ$ and $PR$ are commutative, every element of $P$ commutes with every element of $P, Q,$ and $R$. Thus $P$ is in the center of $G$.

A2. Let $Q$ denote the field of rational numbers.

(a) State the Division Algorithm for $Q[x]$.
(b) Show that $Q[x]$ is a principal ideal domain.
SOLUTION:

A.2 (a) Let \( f(x), g(x) \in \mathbb{Q}[x] \), where \( g(x) \neq 0 \). Then there is a unique pair \( q(x), r(x) \in \mathbb{Q}[x] \) which satisfies (1) \( f(x) = q(x)g(x) + r(x) \) and (2) either \( r(x) = 0 \) or \( \text{Deg } r(x) < \text{Deg } g(x) \).

(b) Let \( I \) be an ideal of \( \mathbb{Q}[x] \). Since \( I = (0) \) is principal we may assume \( I \neq (0) \). Since \( I \neq (0) \) there is a non-zero polynomial in \( I \). Let \( f(x) \in I \) be a non-zero polynomial in \( I \) of least degree. We will show that \( I = (f(x)) \).

Since \( f(x) \in I \) it follows that \( (f(x)) \subseteq I \). Suppose that \( g(x) \in I \). By the Division Algorithm \( g(x) = q(x)f(x) + r(x) \) for some \( q(x), r(x) \in \mathbb{Q}[x] \) where \( r(x) = 0 \) or \( \text{Deg } r(x) < \text{Deg } f(x) \). Since \( r(x) = g(x) - q(x)f(x) \in I \) necessarily \( r(x) = 0 \) since \( f(x) \) has least degree of all non-zero polynomials in \( I \). Therefore \( g(x) = q(x)f(x) \in (f(x)) \).

We have shown \( I \subseteq (f(x)) \). Therefore \( I = (f(x)) \).

A3. Let \( F \) be a field and \( f(x) \in F[x] \).

(a) Describe, in terms of \( f(x) \), what it means for \( F[x]/(f(x)) \) to be a field.

(b) Find an \( f(x) \in F_3[x] \) such that \( F[x]/(f(x)) \) is a field of 27 elements. (You need not list the elements of the field, but explain why there are 27 elements.)

SOLUTION:

(a) \( F[x]/(f(x)) \) is a field if and only if \( (f(x)) \) is maximal ideal of \( F[x] \) if and only if \( f(x) \in F[x] \) is irreducible. Hence \( f(x) \) is irreducible.

(b) We need an irreducible \( f(x) \in F_3[x] \) of degree 3. In this case

\[
F[x]/(f(x)) = \{a + bx + cx^2 + (f(x)) \mid a, b, c \in F_3\}
\]

has 27 elements. Reason: essentially by the Division Algorithm.

Let \( f(x) = x^3 + x^2 - 1 \) for example. Then \( f(0) = -1, f(1) = 1 \) and \( f(-1) = -1 \). Since the degree of \( f(x) \) is 2 or 3 and \( f(x) \) has no roots in \( F = F_3 \) it follows that \( f(x) \) is irreducible.

Complex Analysis

C1. Use contour integration to evaluate

\[
\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta}.
\]

Solution: We write

\[
\sin \theta = \frac{1}{2i}(z - \frac{1}{z}),
\]

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where \( z = e^{i\theta}, \ dz = ie^{i\theta}d\theta \). Then

\[
I = \int_0^{2\pi} \frac{d\theta}{5 + 4\sin \theta} = \int_{|z|=1} \frac{dz}{2z^2 + 5iz - 2} = \int_{|z|=1} \frac{dz}{(z + 2i)(2z + i)}.
\]

The integrand has a single pole inside the circle \(|z| = 1\), at \( z = -i/2 \). The pole has order 1 and the residue is

\[
\frac{1}{2 \cdot 3i/2} = \frac{1}{3i}.
\]

Hence by the residue theorem,

\[
I = \frac{2\pi i}{3i} = \frac{2\pi}{3}.
\]

C2. Let \( C \) be a simple closed contour, described counterclockwise, and let \( z_0 \) be a point not lying on \( C \). Compute

\[
\int_C \frac{z(z+1)}{(z-z_0)^3} \, dz
\]

in the following cases:

(a) \( C \) does not enclose \( z_0 \);
(b) \( C \) encloses \( z_0 \) and \( z_0 \) is not equal to 0 or \(-1\);
(c) \( C \) encloses \( z_0 \) and \( z_0 = -1 \).

Solution:

(a) In this case the integrand has no poles in the region enclosed by \( C \), so the integral is 0.

(b) In this case there is a pole of order 3 at \( z = z_0 \), and the residue is

\[
\frac{1}{2} \frac{d^2}{dz^2} (z^2 + z) \bigg|_{z=z_0} = 1.
\]

Since \( z_0 \) is inside the contour, we conclude that

\[
\int_C \frac{z(z+1)}{(z-z_0)^3} \, dz = 2\pi i.
\]

(c) In this case the integral may be rewritten as

\[
\int_C \frac{z}{(z+1)^2} \, dz.
\]

The integrand has a pole of order 2 at \(-1\), and the residue is

\[
\frac{d}{dz} (z) \bigg|_{z=-1} = 1.
\]

We again get the value \( 2\pi i \) for the integral.
C3. Determine the roots of the equations

(a) \( \cos z = \cosh 3 \) and

(b) \( z^3 = 1 + i \).

Solution:

(a) If \( \cos z = \cosh 3 \) then \( \cos x \cosh y = \cosh 3 \) and \( \sin x \sinh y = 0 \). Hence \( x = 2\pi n \) for some integer \( n \) and \( y = 3 \). The roots are therefore

\[ z = 2\pi n + 3i, \quad n = 0, \pm 1, \pm 2, \ldots \]

(b) We rewrite the equation as \( z^3 = \sqrt{2}e^{\pi i/4} \). This gives the roots

\[ z = 2^{1/6}e^{\pi i/12}, \quad z = 2^{1/6}e^{3\pi i/4}, \quad z = 2^{1/6}e^{17\pi i/12}. \]

Logic

L1. Any set of connectives for propositional logic with the capability to express all truth tables is said to be adequate. For example the standard set of connectives for propositional logic \( \{\land, \lor, \neg\} \) is adequate. Thus, a set \( C \) of connectives is adequate if every one of the connectives in the standard set of connectives can be expressed using only connectives in \( C \).

(a) Show that \( \{\lor, \neg\} \) is an adequate set of connectives.

(b) Is \( \{\rightarrow, \neg\} \) an adequate set of connectives? Justify your answer.

(c) Is \( \{\rightarrow, \land\} \) an adequate set of connectives? Justify your answer.

Solution:

(a) \( x \land y = \neg(\neg x \lor \neg y) \)

(b) Yes. \( x \lor y = (\neg x) \rightarrow y \). Now use part (a).

(c) No. \( T \rightarrow T = T \) and \( T \land T = T \). So there is no way to express \( \neg \) using only \( \rightarrow \) and \( \land \) since \( \neg T = F \).

L2. Given a first-order language \( \mathcal{L} \), let \( F \) be a sentence, \( S \) a set of sentences, and \( S \) a structure (or model) for this language.

(a) What does it mean to say that \( F \) is satisfiable?

(b) State the Compactness Theorem for first-order logic.

(c) Let \( \mathcal{L} = \{r\} \) be the language of graphs. A vertex \( v \) in a graph \( S \) is called isolated if there is no vertex \( w \) for which \( (v, w) \in r \) or \( (w, v) \in r \), i.e., no edge of \( S \) contains \( v \).
Write a first-order sentence $G_3$ such that for a graph $S$, we have $S \models G_3$ if and only if $S$ has at least 3 isolated vertices.

(d) Let $F$ be sentence for $\mathcal{L}$ such that for every positive integer $n$ there exists a graph $S_n$ that has at least $n$ isolated vertices and for which $S_n \models F$. Prove that there exists an graph $S$ such that $S$ has an infinite number of isolated vertices and $S \models F$.

Solution:

(a) $F$ is satisfiable if $F$ has a model. That is, there is a structure $S$ for which $F$ is true in $S$.

(b) The compactness theorem states that if every finite set of sentences in $S$ has a model, then $S$ has a model.

(c) Let $G_3$ be the sentence

$$\exists x \exists y \exists z (\neg(x \approx y) \land \neg(x \approx z) \land (x \approx y) \land \forall w (\neg(r(x, w) \land \neg r(w, x) \land \neg r(y, w) \land \neg r(w, y) \land \neg r(z, w) \land \neg r(w, z))).$$

(d) For each positive integer $n$ let $G_n$ be the sentence that states that at least $n$ vertices in a graph are isolated. Let $S$ be $\{F, G_1, G_2, G_3, \ldots\}$. Let $S_0$ be any finite subset of $S$. Let $m = \max \{ n | G_n \in S_0 \}$. Then $S_m$ is a model of $S_0$. So by the compactness theorem $S$ has a model $S$. Clearly $F$ is satisfiable in $S$ and $S$ has infinitely many isolated points.

L3. Let $\mathcal{L} = \{ f, c \}$ where $f$ is a binary function symbol and $c$ is a constant symbol. For each of the following sentences determine whether or not the sentence is valid. Justify your answers.

1. $\forall x \exists y (f(x, y) = c) \rightarrow \exists y \forall x (f(x, y) = c)$.
2. $\forall w (f(w, c) = f(c, c)) \rightarrow \exists x \forall y \forall z (f(y, x) = f(z, x))$.

Solution: (1.) The sentence is not valid. Consider the reals and interpret $f(x, y)$ as $x + y$ and interpret $c$ as 0. Then $\forall x \exists y (x + y = 0)$ is true but $\exists y \forall x (x + y = 0)$ is false, so the sentence is false for this model.

(2.) This sentence is valid. In any structure, interpret $x$ as $c$. Then $f(y, c) = f(z, c) = f(c, c)$.

Number Theory

N1. Consider the Diophantine equation:

$$253x + 111y = 2$$
a) Find a solution to this equation, or show that no solution exists.

**Solution**
The euclidean algorithm gives:

\[
\begin{align*}
253 &= 2 \times 111 + 31 \\
111 &= 3 \times 31 + 18 \\
31 &= 18 + 13 \\
18 &= 13 + 5 \\
13 &= 2 \times 5 + 3 \\
5 &= 2 \times 3 - 1 \\
\end{align*}
\]

From which:

\[
\begin{align*}
1 &= 2 \times 3 - 5 \\
   &= 2(13 - 2 \times 5) - 5 = 2 \times 13 - 5 \times 5 \\
   &= 7 \times 13 - 5 \times 18 \\
   &= 7 \times 31 - 12 \times 18 \\
   &= 43 \times 31 - 12 \times 111 \\
   &= 43 \times 253 - 98 \times 111 \\
\end{align*}
\]

So \(\gcd(253, 111) = 1\) and the equation has a solution:

\[
(x, y) = (86, -196)
\]

b) Describe all solutions to the equation.

**Solution**
Since \(\gcd(253, 111) = 1\), the general solution is

\[
(x, y) = (86 + 111n, -196 - 253n)
\]

N2. a) Prove Wilson’s theorem:
If \(p\) is an odd prime, then \((p - 1)! \equiv -1 \pmod{p}\)

**Solution**
If \(0 < x < p\), and \(xa \equiv xb \pmod{p}\), we have \(p|a - b\), since \(p\) does not divide \(x\), and so multiplication by \(x\) is injective on congruence classes \(\pmod{p}\). But since any injective map from a finite set to itself is also bijective, there is a unique \(y\), \(0 < y < p\) such that \(xy \equiv 1 \pmod{p}\). Since \(p|(x^2 - 1) \iff p|(x - 1)\) or \(p|(x + 1)\), the only solutions of \(x^2 \equiv 1 \pmod{p}\) with \(0 < x < p\) are \(p = 1, p = -1\), and since \(p\) is odd these are distinct.

Hence, by grouping the numbers \(x\), \(1 < x < p - 1\) into pairs \((x, y)\) such that \(xy \equiv 1 \pmod{p}\), we find that \(\prod_{1 < x < p - 1} x \equiv 1 \pmod{p}\), and hence \((p - 1)! \equiv -1 \pmod{p}\).
b) Show that

\[ 12!24! \equiv -1 \pmod{37} \]

Solution

\[ -1 \equiv 36! = 24!(25)(26)\ldots(36) \pmod{37} \]
\[ = 24!(37 - 12)(37 - 11)\ldots(37 - 1) \]
\[ \equiv 24!12!(-1)^{12} = 24!12! \pmod{37} \]

c) Show that if \( p = 4k + 1 \) is a prime congruent to 1 modulo 4, then \(-1\) is a square modulo \( p \).

Solution

If \( p = 4k + 1 \), then \( \frac{p-1}{2} \) is even, and

\[ -1 \equiv (p - 1)! = \left( \frac{p-1}{2} \right)! \left( \frac{p-1}{2} + 1 \right) \left( \frac{p-1}{2} + 2 \right) \ldots (p - 1) \pmod{p} \]
\[ \equiv \left( \frac{p-1}{2} \right)! ((p - \frac{p-1}{2})(p - (\frac{p-1}{2} - 1)) \ldots (p - 1) \]
\[ \equiv \left( \frac{p-1}{2} \right)! (-1)^{\frac{p-1}{2}} = \left( \frac{p-1}{2} \right)!^2 \pmod{p} \]

N3. Show that 2 is a primitive root mod 19, and find all primitive roots mod 38.

Solution

Since \( \phi(19) = 19 - 1 = 18 \), it is enough to show that \( 2^n \) is not congruent to 1 for any proper divisor \( n \) of 18. Since the maximal proper divisors of 18 are 9 and 6, we must compute \( 2^6 \) and \( 2^9 \) modulo 19.

\[ 2^4 = 16 \equiv -3 \pmod{19} \]
\[ 2^6 = -3 \times 4 = 12 \equiv 7 \pmod{19} \]
and

\[ 2^9 \equiv -3 \times 2^3 = 7 \times 8 = 56 \equiv -1 \pmod{19} \]

Let’s start by finding the other primitive roots mod 19. They are of the form \( 2^k \) for \( \gcd(k, 18) = 1 \), i.e. \( k = 1, 5, 7, 11, 13, 17 \).

\[ 2^5 = 32 \equiv 13 \pmod{19} \]
\[ 2^7 \equiv 13 \times 4 = 52 \equiv 14 \pmod{19} \]
\[ 2^{11} = 2 \times (2^5)^2 \equiv 2 \times (-6)^2 = 2 \times 36 \equiv 2 \times (-2) = -4 \equiv 15 \pmod{19} \]
\[ 2^{13} \equiv 15 \times 4 = 60 \equiv 3 \pmod{19} \]
\[ 2^{17} = 2^{13} \times 2^4 = 3 \times 16 = 48 \equiv 10 \pmod{19}. \]

So the complete set of primitive roots mod 19 are \( \{2, 13, 14, 15, 3, 10\} \).

To find the primitive roots mod 38, notice that \( \phi(38) = \phi(2)\phi(19) = 1 \times 18 = 18 \), and so reducing mod 19 gives a bijection from primitive roots mod 38 onto primitive roots mod 19. Since \( \gcd(n, 38) = 1 \) if and only if \( n \) is odd and prime to 19, we need only look for the odd representative mod 38 of the primitive roots mod 19:

\[ \{21, 13, 33, 15, 3, 29\} \]

**Real Analysis**

**R1.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function which is injective (same as one-to-one). Assume \( f(a) < f(b) \). Prove

(i) If \( a < c < b \) then \( f(a) < f(c) < f(b) \).

(ii) Deduce that \( f \) is strictly increasing on \([a, b]\).

**Solution:** (i) Claim \( f(a) < f(c) \). Otherwise, since \( f \) is injective, \( f(a) > f(c) \), so there exists \( v \) with \( f(c) < v < f(a) < f(b) \). By the intermediate value theorem there are \( x_1, x_2 \) with \( a < x_1 < c < x_2 < b \) and \( f(x_1) = v = f(x_2) \). This contradicts the injectivity of \( f \).

Claim \( f(c) < f(b) \). Otherwise \( f(c) > f(b) \), so there exists \( v \) with \( f(a) < f(b) < v < f(c) \). Again by the intermediate value theorem there are \( x_1, x_2 \) with \( a < x_1 < c < x_2 < b \) and \( f(x_1) = v = f(x_2) \) and again this contradicts the injectivity of \( f \).

Thus \( f(a) < f(c) < f(b) \).

(ii) If \( a < c < d < b \), apply part (i) to \( f \) on the interval \([c, b]\) to conclude that \( f(c) < f(d) < f(b) \). Thus \( f(c) < f(d) \) and \( f \) is strictly increasing.

**R2.** Let \( f : [0, \infty) \to \mathbb{R} \) be a differentiable function, and assume that the limit

\[ A = \lim_{x \to \infty} f'(x) \]

exists. Assume \( f \) is bounded: for some \( b \), \( |f(x)| < b \) for \( 0 \leq x \). Prove that \( A = 0 \).

**Solution:** If \( A \neq 0 \), assume \( A > 0 \) replacing \( f \) by \( -f \) if necessary. Then there is an \( x_0 \) such that \( \xi \geq x_0 \implies f'(\xi) > A/2 \). By the mean value theorem, \( f(x) - f(x_0) = f'(\xi)(x - x_0) \) for some \( \xi \in (x_0, x) \). Hence \( b > f(x) > f(x_0) + (A/2)(x - x_0) \), a contradiction for \( x > x_0 + (2/A)(b - f(x_0)) \).

**R3.** Consider the series \( f(x) = \sum_{k=1}^{\infty} \frac{e^{-kx}}{k} \).
(i) For what values of \( x \in \mathbb{R} \) does it converge?

(ii) Compute \( f'(x) \) for \( x > 0 \). [Justify your method].

**Solution:** (i) The ratio of the absolute value of two successive terms is \( \frac{k}{k+1}e^{-x} \) which tends to \( e^{-x} \) as \( k \to \infty \). Hence the series converges absolutely for \( x > 0 \) and diverges for \( x < 0 \). For \( x = 0 \) the series is the harmonic series which diverges.

(ii) The term-wise derivative is \( \sum_{k=1}^{\infty} -e^{-kx} \) which converges for \( x > 0 \) and converges uniformly on \([a, \infty)\) for \( a > 0 \) by the Weierstrass M-test: \( x \geq a \implies e^{-ka} \geq e^{-kx} \). Let \( g(x) = \sum_{k=1}^{\infty} -e^{-kx} \). Since this series converges uniformly on \([a, \infty)\), we have

\[
\int_{a}^{x} g(t) \, dt = \sum_{k=1}^{\infty} \int_{a}^{x} -e^{-kt} \, dt = \sum_{k=1}^{\infty} \left( \frac{1}{k} e^{-kx} - \frac{1}{k} e^{-ka} \right) = f(x) - f(a).
\]

By the fundamental theorem of calculus \( f'(x) = g(x) \).

Since \( g(x) \) is defined by a geometric series, we know \( g(x) = -e^{-x}/(1 - e^{-x}) \). In fact, the series for \( \log(1 + x) \) leads to \( f(x) = -\log(1 - e^{-x}) \).

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**Topology**

**T1.** Show that the product of two connected nonempty topological spaces is connected.

**Solution:** Let \( X \) and \( Y \) be nonempty connected spaces. For any \( x \in X \), the map \( f : Y \to X \times Y \) given by \( f(y) = (x, y) \) is continuous, since its coordinate functions are continuous. Thus its image \( \{x_0\} \times Y \) is a connected subspace of \( X \times Y \). Similarly the subspace \( X \times \{y\} \) is connected. Now fix a point \( x_0 \in X \). For each \( y \in Y \), define \( C_y = \{x_0\} \times Y \cup X \times \{y\} \). The set \( C_y \) is a union of two connected sets, each of which contains \( (x_0, y) \). Thus \( C_y \) is connected. Observe that \( X \times Y = \bigcup_{y \in Y} C_y \). We have shown that \( X \times Y \) is a union of connected sets, each of which contains the non-empty subset \( \{x_0\} \times Y \). Thus \( X \times Y \) is connected.

**T2.** Let \( T \) be a topological space and let \( \Delta(T) = \{(x, x) \mid x \in T\} \) be the diagonal in \( T \times T \). Show that \( T \) is Hausdorff if and only if \( \Delta(T) \) is a closed subset of \( T \times T \).

**Solution:** \((\implies)\) Suppose \( T \) is Hausdorff. We will show that \( T \times T - \Delta(T) \) is open in \( T \times T \). Suppose \( (x, y) \in T \times T - \Delta(T) \). Then \( x \neq y \). Since \( T \) is Hausdorff, there are open sets \( U \) and \( V \) in \( T \) such that \( x \in U \), \( y \in V \), and \( U \cap V = \emptyset \). The subset \( U \times V \) is a basic open set in \( T \times T \). Moreover, \((t, t) \not\in U \times V \) for any \( t \in T \), since otherwise we would have \( t \in U \cap V \). It follows that \( U \times V \subset T \times T - \Delta(T) \). For each point \( (x, y) \) of \( T \times T - \Delta(T) \) we have found an open set \( U \times V \) such that \( (x, y) \in U \times V \subset T \times T - \Delta(T) \). Thus \( T \times T - \Delta(T) \) is open.

\((\impliedby)\) Now, suppose that \( \Delta(T) \) is closed in \( T \times T \). Then \( T \times T - \Delta(T) \) is open. Thus, for any \( (x, y) \in T \times T - \Delta(T) \), there exists a basic open set \( U \times V \) such that
The points \((x, y) \in U \times V \subset T \times T - \Delta(T)\). By definition of the basis for the topology on \(T \times T\), the sets \(U\) and \(V\) are open in \(T\). Observe that \(U \cap V = \emptyset\), since if \(t \in U \cap V\), then \((t, t) \in U \times V\), which implies \(U \times V \cap \Delta(T) \neq \emptyset\), a contradiction. Thus, \(x \in U\), and \(y \in V\) and \(U \cap V = \emptyset\). Since \(x\) and \(y\) were arbitrary points, this shows that \(T\) is Hausdorff.

**T3.** Recall that the Cantor middle thirds set \(C\) is a subspace of the interval \([0, 1]\) that can be described as

\[
C = \bigcap_{n \in \mathbb{N}} A_n,
\]

where \(A_1 = [0, \frac{1}{3}] \cup \left[\frac{2}{3}, 1\right]\) and

\[
A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right).
\]

(Thus each set \(A_n\) is a union of \(2^n\) disjoint closed intervals \(J_{i,n}\) of length \(\frac{1}{3^n}\), for \(i = 1, \ldots, 2^n\).) Let \(\mathcal{I}\) be the collection of all subsets of \(C\) of the form \(C \cap J_{i,n}\).

1. Show that each set in \(\mathcal{I}\) is both open and closed (clopen) in the space \(C\).
2. Show that \(\mathcal{I}\) is a basis for the topology of \(C\).
3. Show that any clopen subset of \(C\) is a finite union of sets in \(\mathcal{I}\).

**Solution:**

1. Let \(J_{i,n}\) be an interval of \(A_n\). Then \(J_{i,n}\) is closed, so \(J_{i,n} \cap C\) is closed. Since \(A_n = \bigcup_{i=1}^{2^n} J_{i,n}\) is a disjoint union of closed intervals, we also have that \(A_n - J_{i,n}\) is closed, so \(C - (J_{i,n} \cap C) = C \cap (A_n - J_{i,n})\) is closed, and thus \(J_{i,n} \cap C\) is open in \(C\).

2. Let \(x \in C\). Then for any \(\epsilon > 0\), there exists \(n\) such that \(3^{-n} < \epsilon\). Then there exists \(i\) such that \(x \in J_{i,n}\), since \(C \subset A_n\). Since the length of \(J_{i,n}\) is equal to \(3^{-n}\), we have \(x \in J_{i,n} \cap C \subset (x - \epsilon, x + \epsilon)\). Thus, the sets \(\mathcal{I}\) form a basis for the topology on \(C\), since the balls \((x - \epsilon, x + \epsilon) \cap C\) are in \(C\), \(\epsilon > 0\) form a basis for the topology on \(C\).

3. Let \(U \subset C\) be a clopen subset. Since \(C\) is a closed and bounded subspace of \(\mathbb{R}\), it is compact. Therefore the closed subset \(U\) of \(C\) is compact. Since \(U\) is open in \(C\), and the sets \(J_{i,n} \cap C\) form a basis for the topology of \(C\), we have that for every \(x \in U\) there exists \(J_{i,n}\) such that \(x \in J_{i,n} \cap C \subset U\). Thus, \(U = \bigcup_{J \in \mathcal{I}, J \subset U} J\). But since \(U\) is also compact, there are only finitely many \(J \in \mathcal{I}\), \(J \subset U\) needed to cover \(U\). Thus we see that \(U\) is a finite union of sets in \(\mathcal{I}\). In particular, there exists \(n\) and \(K \subseteq \{1, \ldots, 2^n\}\) such that \(U = \bigcup_{i \in K} J_{i,n} \cap C\).