Algebra

A1. Assume $H$ and $K$ are subgroups of a group $G$. Show that $HK$ is a subgroup of $G$ if and only if $HK = KH$.

Solution: Assume first that $HK = KH$ and let $a, b \in HK$. We prove $ab^{-1} \in HK$ so $HK$ is a subgroup. Let $a = h_1k_1$ and $b = h_2k_2$, with $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Thus $b^{-1} = k_2^{-1}h_2^{-1}$ and $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1}$. Let $k_3 = k_1k_2^{-1} \in K$ and $h_3 = h_2^{-1} \in H$, so $ab^{-1} = h_1k_3h_3$. Since $HK = KH$, $k_3h_3 = h_4k_4$ with $h_4 \in H$ and $k_4 \in K$. Thus $ab^{-1} = h_1h_4k_4 \in HK$ as desired.

Conversely, assume $HK$ is a subgroup of $G$. Since $K \leq HK$ and $H \leq HK$, by closure $KH \subseteq HK$. To show the reverse containment, let $hk \in HK$. Since $HK$ is a subgroup, write $hk = a^{-1}$ for some $a \in HK$. If $a = h_1k_1$, then $hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$. Thus $KH = HK$.

A2. Let $\mathcal{K} = \{k_1, \ldots, k_m\}$ be a conjugacy class in the finite group $G$. Prove that the element $K = k_1 + \ldots + k_m$ is in the center of the group ring $RG$.

Solution: Given $g \in G$ and $k \in \mathcal{K}$, then $gkg^{-1} = k'$ with $k' \in \mathcal{K}$, so $g\mathcal{K}g^{-1} \subseteq \mathcal{K}$. So action by conjugation is an automorphism and gives a permutation of the elements in $\mathcal{K}$. Thus, in $RG$, $gKg^{-1} = K$ and $K$ is in the center of $RG$.

A3. Show that if $p$ is a prime congruent to 3 modulo 4, then $p$ is prime in the ring of Gaussian integers $\mathbb{Z}[i]$.

Solution: If $p = ab$ then one has for the norms $N(a)N(b) = p^2$. If the norm of one of the factors is 1 then this factor is a unit. Hence $N(a) = N(b) = p$. But for $\alpha = a + bi$ one has $N(\alpha) = a^2 + b^2$ and hence $N(\alpha)$ cannot be congruent to 3 modulo 4.
Complex Analysis

C1. Compute the integral
\[ \int_0^{2\pi} \frac{\cos \theta}{5 + 4 \cos \theta} \, d\theta. \]

**Solution:** To compute the integral, we recall that
\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \]
and so, by plugging this into the integral and remembering that \( z(\theta) = e^{i\theta}, 0 \leq \theta \leq 2\pi \) is a parametrization of the unit circle \( C \), positively oriented, we get:

\[
\int_0^{2\pi} \frac{\cos \theta}{5 + 4 \cos \theta} \, d\theta = \int_0^{2\pi} \frac{e^{i\theta} + e^{-i\theta}}{10 + 4(e^{i\theta} + e^{-i\theta})} \, d\theta = \int_0^{2\pi} \frac{1 + e^{-2i\theta}}{4e^{i\theta} + 10 + 4e^{-i\theta}} \, e^{i\theta} \, d\theta \\
= \frac{1}{i} \int_C \frac{1 + \frac{1}{z^2}}{4z^3 + 10z^2 + 4z} \, dz = \frac{1}{i} \int_C \frac{z^2 + 1}{4z^3 + 10z^2 + 4z} \, dz
\]

The last integral can easily be computed using the Residue Theorem. Indeed, notice that the denominator can be factored as
\[ 4z^3 + 10z^2 + 4z = 2z(2z^2 + 5z + 2) = 2z(z + 2)(2z + 1) \]
and so the function we are integrating has two simple poles, at \( z = 0 \) and \( z = -\frac{1}{2} \), inside the unit circle, with residues

\[
\text{Res} \left( \frac{z^2 + 1}{4z^3 + 10z^2 + 4z}; z = 0 \right) = \frac{1}{4}
\]
and
\[
\text{Res} \left( \frac{z^2 + 1}{4z^3 + 10z^2 + 4z}; z = -\frac{1}{2} \right) = -\frac{5}{12}.
\]

So, by the Residue Theorem,
\[
\int_C \frac{z^2 + 1}{4z^3 + 10z^2 + 4z} \, dz = 2\pi i \left( \frac{1}{4} - \frac{5}{12} \right) = -\frac{\pi i}{3}.
\]

Putting it all together we get
\[
\int_0^{2\pi} \frac{\cos \theta}{5 + 4 \cos \theta} \, d\theta = -\frac{\pi}{3}.
\]
C2. Let $f$ be an entire function such that

$$|f(z)| \leq 1 + |z|^{3/2} \quad \text{for all } z \in \mathbb{C}.$$ 

Prove that there exist $a_0, a_1 \in \mathbb{C}$ such that

$$f(z) = a_0 + a_1 z.$$ 

Solution: Since $f$ is an entire function, we know that it has a power series expansion around 0 with infinite radius of convergence,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

and, from the Cauchy integral formula,

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_{C_R} \frac{f(w)}{w^{n+1}} \, dw,$$

where $C_R$ is the circle of radius $R > 0$, positively oriented.

Since $|f(w)| \leq 1 + R^{3/2}$ on $C_R$ and $C_R$ has length $2\pi R$, we find for each $n \geq 0$ and each $R > 0$,

$$|f^{(n)}(0)| = \frac{n!}{2\pi} \left| \oint_{C_R} \frac{f(w)}{w^{n+1}} \, dw \right| \leq \frac{n!}{2\pi} \cdot \frac{1 + R^{3/2}}{R^{n+1}} \cdot 2\pi R.$$ 

But for $n \geq 2$ the right-hand side

$$\frac{n!(1 + R^{3/2})}{R^n} \to 0 \quad \text{as} \quad R \to \infty,$$

and so

$$f^{(n)}(0) = 0 \quad \text{for all } n \geq 2.$$ 

Putting this into the powers series for $f$ we conclude that

$$f(z) = f(0) + f'(0)z = a_0 + a_1 z, \quad \text{for } a_0 = f(0), \ a_1 = f'(0).$$

C3. Let $f(z) = \sin z$. Find

$$\max \{|f(z)| : z \in K\},$$

where

$$K = \{x + iy : 0 \leq x, y \leq 2\pi\}.$$
Solution: Since
\[ f(z) = \sin z = \frac{e^{iz} - e^{-iz}}{2i} \]
is analytic on the close square \( K \), we know from the Maximum Modulus Principle that \( |f| \) will achieve its maximum on the boundary of \( K \),

\[ \partial K = \{ x + iy : x = 0 \text{ or } x = 2\pi \text{ or } y = 0 \text{ or } y = 2\pi \} . \]

On the two vertical edges, \( x = 0 \) and \( x = 2\pi \), we have
\[ |f(iy)| = |f(2\pi + iy)| = \frac{|e^{-y} - e^{y}|}{2} = \frac{e^{y} - e^{-y}}{2} . \]

Since
\[ \frac{d}{dy} \left( \frac{e^{y} - e^{-y}}{2} \right) = \frac{e^{y} + e^{-y}}{2} > 0 , \]
we see that the maximum of \( f \) is achieved at \( 2\pi i \) and \( 2\pi + 2\pi i \), and it equals
\[ |f(2\pi i)| = |f(2\pi + 2\pi i)| = \frac{e^{2\pi} - e^{-2\pi}}{2} . \]

On the horizontal edge, \( y = 0 \), we get
\[ |f(x)| = |\sin x| \leq 1 \]
and it achieves its maximum value, 1, at \( x = \frac{\pi}{2} \) and \( x = \frac{3\pi}{2} \).

Finally, on the other horizontal edge, \( y = 2\pi \),
\[ |f(x + 2\pi i)| = |\sin(x + 2\pi i)| = \frac{1}{2} |\cos x(e^{-2\pi} - e^{2\pi}) + i \sin x(e^{-2\pi} + e^{2\pi})| \]
\[ = \frac{1}{2} \sqrt{\cos^2 x(e^{-2\pi} - e^{2\pi})^2 + \sin^2 x(e^{-2\pi} + e^{2\pi})^2} \]
\[ = \frac{1}{2} \sqrt{e^{4\pi} + e^{-4\pi} + 2(\sin^2 x - \cos^2 x)} . \]

This function clearly achieves its maximum when \( \sin^2 x = 1 \) and \( \cos^2 x = 0 \), in other words when \( x = \frac{\pi}{2} \) or \( x = \frac{3\pi}{2} \), in which cases
\[ \max_{y=2\pi} |f(z)| = \frac{1}{2} \sqrt{e^{4\pi} + e^{-4\pi} + 2} = \frac{e^{2\pi} + e^{-2\pi}}{2} . \]

Summing up, we see that
\[ \max_{y=0} |f(z)| = 1 < \max_{x=0, x=2\pi} |f(z)| = \frac{e^{2\pi} - e^{-2\pi}}{2} < \max_{y=2\pi} |f(z)| = \frac{e^{2\pi} + e^{-2\pi}}{2} , \]
and therefore
\[ \max_{z \in K} |\sin z| = \frac{e^{2\pi} + e^{-2\pi}}{2} , \]
achieved at \( z = \frac{\pi}{2} + 2\pi i \) and \( z = \frac{3\pi}{2} + 2\pi i \).
_logic

Convention: If \( \mathcal{M} \) is a structure, we denote the universe of \( \mathcal{M} \) by \(|\mathcal{M}|\).

L1. Let \( L \) be a first-order language with equality, a unary function symbol \( f \), and a binary relation symbol \( < \).

(a) Write down a sentence \( \theta \) asserting that \( < \) is a linear ordering (that is, \( < \) is irreflexive, transitive, and any two elements are comparable).

(b) Is the set of \( L \)-sentences \( S := \{ \forall x (x < y \rightarrow f(x) < f(y)), \exists x (f(f(f(x)))) = x \land f(x) \neq x \} \) satisfiable? If yes, give an example of a structure that satisfies it; if not, explain why not.

(c) Is the set \( S \cup \{ \theta \} \) satisfiable? (\( \theta \) is the sentence in part (a).)

Solution: (a) Let \( \theta \) be the sentence

\[
\forall x \; \neg x < x \\
& \forall x \; \forall y \; \forall z \; ((x < y \land y < z) \rightarrow x < z) \\
& \forall x \; \forall y \; (x < y \lor x = y \lor y < x)
\]

(b) Yes. For example, if \( \mathcal{M} \) is the structure with exactly two distinct elements \( x, y \) satisfying \( f^\mathcal{M}(x) = y, \; f^\mathcal{M}(y) = x \) and \( x <^\mathcal{M} x <^\mathcal{M} y <^\mathcal{M} y <^\mathcal{M} x \), then \( \mathcal{M} \models S \).

(c) No. For if \( \mathcal{M} \models S \cup \{ \theta \} \), then \( <^\mathcal{M} \) is a strict linear ordering of \(|\mathcal{M}|\). So suppose \( x \in |\mathcal{M}| \) satisfies \( f(f(f(x)))) = x \) and \( f(x) \neq x \), say \( x < f(x) \). Then also \( f(x) < f(f(x)), \; f(f(x)) < f(f(f(x))) \) and \( f(f(f(x))) < f(f(f(f(x)))) \), whereby \( x < f(f(f(x)))) \) and hence also \( x \neq f(f(f(x)))) \), contradicting the assumption on \( x \). A similar argument applies if \( f(x) < x \).

L2. Suppose \( L \) is a first order language with equality, \( T \) an \( L \)-theory and \( \phi(x) \) an \( L \)-formula with exactly one free variable \( x \). Suppose that for every finite number \( C \) there is an infinite model \( \mathcal{M} \) of \( T \) such that

\[
\{ x \in |\mathcal{M}| : \mathcal{M} \models \phi(x) \}
\]

is finite, but of cardinality \( \geq C \).

Show that \( T \) has a model \( \mathcal{N} \) such that

\[
A = \{ x \in |\mathcal{N}| : \mathcal{N} \models \phi(x) \}
\]

and its complement \(|\mathcal{N}| \setminus A \) are infinite subsets of \(|\mathcal{N}|\).
**Solution:** Let \((c_n)_{n \in \mathbb{N}}\) and \((d_n)_{n \in \mathbb{N}}\) be disjoint sets of new constant symbols and let \(S\) be the theory having axioms
\[
\{\phi(c_0), \phi(c_1), \phi(c_2), \ldots, \neg\phi(d_0), \neg\phi(d_1), \ldots, c_i \neq c_j, d_i \neq d_j\}_{i \neq j}.
\]

By assumption on \(T\), \(T \cup S\) is finitely satisfiable and hence has a model \(\mathcal{N}\). Moreover,
\[
\{c^\mathcal{N}_i\}_{i \in \mathbb{N}} \subseteq A = \{x \in |\mathcal{N}| : \mathcal{N} \models \phi(x)\}
\]
while
\[
\{d^\mathcal{N}_i\}_{i \in \mathbb{N}} \subseteq |\mathcal{N}| \setminus A.
\]
So \(A\) is both infinite and coinfinite.

**L3.** A first order formula is called **existential** if it is of the form
\[
\exists x_1 \exists x_2 \ldots \exists x_n \psi,
\]
where \(\psi\) is a formula without quantifiers. Similarly, a formula of the form
\[
\forall x_1 \forall x_2 \ldots \forall x_n \psi,
\]
where \(\psi\) is a formula without quantifiers, is called **universal**.

Two formulae \(\phi(x_1, \ldots x_n)\) and \(\psi(x_1, \ldots x_n)\) are **equivalent modulo** \(T\) if every model of \(T\) satisfies
\[
\forall x_1 \ldots \forall x_n (\psi \leftrightarrow \phi).
\]

For a given theory \(T\), show that if every formula is equivalent modulo \(T\) to a universal formula, then every formula is equivalent modulo \(T\) to an existential formula.

**Solution:** Suppose \(L\) is the language of \(T\). Let \(\phi(x_1, \ldots x_n)\) be any \(L\)-formula and find a quantifier free \(L\)-formula \(\psi(x_1, \ldots x_n, y_1, \ldots y_m)\) such that
\[
T \models \forall x_1 \ldots \forall x_n (\neg\phi(x_1, \ldots x_n) \leftrightarrow \forall y_1 \ldots \forall y_m \psi(x_1, \ldots x_n, y_1, \ldots y_m)).
\]

Then also
\[
T \models \forall x_1 \ldots \forall x_n (\phi(x_1, \ldots x_n) \leftrightarrow \exists y_1 \ldots \exists y_m \neg\psi(x_1, \ldots x_n, y_1, \ldots y_m)).
\]
So \(\phi\) is equivalent modulo \(T\) to an existential formula.
Number Theory

N1. Find all integers \( n \) such that \( \phi(n) = 4 \).

**Solution:** The Euler function \( \phi \) is the number of integers \( k \) relatively prime to \( n \) with \( 1 \leq k \leq n \). For \( p \) prime and \( e \geq 1 \), \( \phi(p^e) = p^e - 1 \). For \( a \) and \( b \) relatively prime, \( \phi(ab) = \phi(a)\phi(b) \). For \( n > 1 \), \( \phi(n) < n \).

If \( \phi(n) = 4 \) then \( n > 4 \). If \( p \) divides \( n \), then \( p \leq 5 \). Hence \( n = 2^a3^b5^c \) where \( a \leq 3 \), \( b \leq 1 \), and \( c \leq 1 \). Further if \( c = 1 \) then \( b = 0 \). The possibilities are:

\[
\begin{array}{cccc}
a & b & c & n \\
0 & 0 & 1 & 5 \\
1 & 0 & 1 & 10 \\
2 & 0 & 1 & 12 \\
3 & 0 & 0 & 8 \\
\end{array}
\]

N2. Find all integral \( x \) such that

\[
\begin{align*}
3x & \equiv 2 \pmod{7} \\
x & \equiv 5 \pmod{11}
\end{align*}
\]

**Solution:** Since \( 3 \cdot 5 \equiv 1 \pmod{7} \), if \( 3x \equiv 2 \pmod{7} \), then

\[
x \equiv 15x \equiv 10 \equiv 3 \pmod{7}.
\]

Hence \( x = 7k + 3 \) for some integer \( k \). Then we have:

\[
\begin{align*}
7k + 3 & \equiv 5 \pmod{11}, \\
7k & \equiv 2 \pmod{11} \\
-k & \equiv 21k \equiv 6 \pmod{11} \\
k & \equiv 5 \pmod{11} \\
k & = 5 + 11\ell \\
x & = 7(5 + 11\ell) + 3 = 38 + 77\ell \text{ for any } \ell \in \mathbb{Z}.
\end{align*}
\]

N3. (a) Show that if \( p \) is a prime congruent to 3 mod 4, then

\[
\left( \left( \frac{p-1}{2} \right)! \right)^2 \equiv 1 \pmod{p}.
\]

(b) Show that \( 30!6! \) is congruent to \(-1 \) mod 37.
Solution: (a) If \( p \) is prime, then Wilson’s theorem states
\[
(p - 1)! \equiv -1 \pmod{p}.
\]
For \( p = 4k + 3 \),
\[
(p - 1)! = 1 \cdot 3 \cdot 3 \cdots (2k + 1)(p - 2k - 1)(p - 2k - 2) \cdots (p - 1)
\equiv ((2k + 1)!)^2(-1)^{2k+1} \equiv -((2k + 1)!)^2 \pmod{p}.
\]
Hence \(
\left(\frac{p - 1}{2}\right)! \equiv 1 \pmod{p}.
\)
(b) Again by Wilson’s theorem with \( p = 37 \) (and noting that 6 is even)
\[
30! \cdot 6! = 30!(37 - 6)(37 - 5) \cdots (37 - 1)
\equiv 37! \pmod{37}
\equiv -1 \pmod{37}.
\]

Real Analysis

R1. Suppose \( g(x) \) is a function on \( \mathbb{R} \) such that \( |g(x) - g(y)| \leq |x - y|^\frac{1}{2} \) for all \( x \) and \( y \). Let \( g_n(x) = g(x + \frac{1}{n}) \) for all positive integers \( n \). Show that \( g_n(x) \) converges uniformly to \( g(x) \) on \( \mathbb{R} \) as \( n \to \infty \).

Solution: \( |g_n(x) - g(x)| = |g(x + \frac{1}{n}) - g(x)| \leq |(x + \frac{1}{n}) - x|^\frac{1}{2} = n^{-\frac{1}{2}} \). Thus for any \( \epsilon > 0 \), if \( n \) satisfies \( n^{-\frac{1}{2}} < \epsilon \) (i.e. if \( n > \frac{1}{\epsilon^2} \)) then \( |g_n(x) - g(x)| < \epsilon \), uniformly in \( x \).

R2. Define the sequence \( \{a_n\}_{n=0}^\infty \) of numbers as follows. Let \( a_0 = 2 \). For \( n > 0 \), define \( a_n \) by
\[
a_n = \frac{1}{2} + \frac{a_{n-1}}{2}.
\]

a) Show that \( 1 < a_n < a_{n-1} \) for all \( n \).
b) Show that the sequence \( \{a_n\}_{n=0}^\infty \) is convergent and find its limit.

Solution: (a) By induction assume \( a_{n-1} > 1 \). Then \( a_n \) is the sum of \( \frac{1}{2} \) and a number greater than \( \frac{1}{2} \) and thus is greater than 1. The inequality \( a_n < a_{n-1} \) can be rewritten as \( \frac{1}{2} < \frac{a_{n-1}}{2} \) or \( a_{n-1} > 1 \) which again holds by the induction hypothesis.

(b) By completeness of the reals, a decreasing sequence of numbers bounded below has a limit \( L \). Taking limits in both sides of the recursion \( a_n = \frac{1}{2} + \frac{a_{n-1}}{2} \) as \( n \) goes to infinity gives \( L = \frac{1}{2} + \frac{L}{2} \) which is solved by \( L = 1 \).
R3. For which real values of $x$ does the series $\sum_{n=2}^{\infty} \frac{(2x-1)^n}{n \log n}$ converge? (Distinguish between absolute and conditional convergence; state convergence criteria that you are using.)

Solution: There are four cases: (1) $x < 0$ or $x > 1$, (2) $0 < x < 1$, (3) $x = 0$, (4) $x = 1$.

In case (1) the ratio test for the absolute values $a_n = |2x - 1|^n / n \log n$ gives

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = |2x - 1| \cdot \lim_{n \to \infty} \frac{(n+1) \cdot \log(n+1)}{n \cdot \log n} = |2x - 1| > 1.$$ 

So $a_n \not\to 0$ and the series diverges.

In case (2) the same ratio test gives absolute convergence.

In case (3) the series is $\sum (-1)^n / n \log n$; it converges conditionally by Leibnitz’s rule because $1/n \log n \to 0$ monotonically.

In case (4) the series is $\sum 1/n \log n = +\infty$. This can be deduced from Cauchy criterion comparing the series to $\sum 2^k a_{2k} = \text{const} \cdot \sum 1/k$, or by the integral criterion with change of variables $u = e^x$.

So the series converges for $x \in [0, 1)$ with conditional convergence at $x = 0$.

Topology

T1. Let $X$ and $Y$ be topological spaces and $f : X \to Y$ a function. Prove that $f$ is continuous if and only if for every subset $A \subset X$ we have $f(A) \subset f(A)$.

Solution: Assume $f$ is continuous, and let $A$ be any subset of $X$. Since $f(A)$ is closed in $Y$ and $f$ is continuous, $f^{-1}(f(A))$ is closed in $X$. Further, $A \subset f^{-1}(f(A)) \subset f^{-1}(f(A))$. Thus $f^{-1}(f(A))$ is a closed set containing $A$ and therefore $A \subset f^{-1}(f(A))$. Applying $f$ gives $f(A) \subset f(A)$ as claimed.

Assume for every subset $A \subset X$ we have $f(A) \subset f(A)$. Let $U$ be an open subset of $Y$ and $x$ a point with $f(x) \in U$. To show $f$ continuous we need to see that $f^{-1}(U)$ contains a neighborhood of $x$. Equivalently, we need to show that $x \notin f^{-1}(U^c)$. Let $A = f^{-1}(U^c)$. By assumption $f(A) \subset f(A)$. Since $f(A) = f(f^{-1}(U^c) \subset U^c$ and $U^c$ is closed, $f(A) \subset U^c$, and so $f(A) \subset U^c$. Since $f(x) \in U$ this proves $x \notin A$ as needed.

T2. Let $X$ and $Y$ be topological spaces with $X$ and $Y$ compact and $Y$ Hausdorff. Prove that a function $f : X \to Y$ is continuous if and only if the subset

$$Gr_f = \{(x, f(x)) : x \in X\} \subset X \times Y$$

is closed when $X \times Y$ is given the product topology.
Solution: Assume $f$ is continuous. Let $(x, y)$ be a point of $\overline{Gr_f}$. We need to prove $y = f(x)$. If not, then because $Y$ is Hausdorff there are disjoint open sets $U$ and $V$ in $Y$ containing $y$ and $f(x)$ respectively. By the continuity of $f$, $V' = f^{-1}(V) \subset X$ is open in $X$. Therefore $V' \times U$ is an open subset of $X \times Y$ containing $(x, y)$. By construction $f(V')$ and $U$ are disjoint, so $V' \times U$ is disjoint from $Gr_f$, contradicting the fact that $(x, y) \in \overline{Gr_f}$.

Assume $Gr_f$ is closed. We show $f$ is continuous by showing $f^{-1}(C)$ is closed for all closed $C \subset Y$. Because $Y$ is compact, $C$ is compact and therefore $X \times C$ is compact in $X \times Y$. Since $Gr_f$ is closed, the subset $Z = X \times C \cap Gr_f$ is also compact. The projection $X \times Y$ to $X$ is continuous, so the projection of $Z$ to $X$ is compact and hence closed. This projection is precisely $f^{-1}(C)$, so $f$ is continuous.

T3. Let $\tau$ be the collection of subsets of $\mathbb{R}$ given by

$$\tau := \{ A \subset \mathbb{R} : \text{ Either } A = \emptyset \text{ or } \mathbb{R} \setminus A \text{ is compact in the standard topology} \}.$$ 

(a) Is $\mathbb{R}$ with the topology $\tau$ connected?

(b) Is $\mathbb{R}$ with the topology $\tau$ Hausdorff?

(c) Is $\mathbb{R}$ with the topology $\tau$ compact?

Solution: By the definition of $\tau$, any non-empty $\tau$-open set contains the semi-infinite intervals $(a, +\infty)$ and $(-\infty, -a)$ for some $a \geq 0$. In particular, no two non-empty $\tau$-open sets are disjoint. Thus:

(a) $(\mathbb{R}, \tau)$ is connected.

(b) $(\mathbb{R}, \tau)$ is not Hausdorff.

(c) $(\mathbb{R}, \tau)$ is compact: If $U_\alpha$ is any open cover, let $U_0$ be one of the sets which is non-empty. Since, by definition, the complement $U^c_0$ is compact in the usual topology, and since all $\tau$-open sets are also open in the standard topology, there is a finite subset of the $U_\alpha$ that cover $U^c_0$. Together with $U_0$ this gives a finite subcover of $\mathbb{R}$. 

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