MASTERS EXAMINATION IN MATHEMATICS
PURE MATH OPTION, FALL 2011

Algebra

A1. Suppose that $G$ is a group, with normal subgroups $A$ and $B$ so that $G/A$ and $G/B$ are abelian. Prove that $G/(A \cap B)$ is abelian.

Solution. A group $G/K$ is abelian if and only if the commutator subgroup $[G,G]$ is contained in $K$.

If $G/A$ and $G/B$ are abelian then $[G,G] \leq A$ and $[G,G] \leq B$ so $[G,G] \leq A \cap B$. Therefore, $G/(A \cap B)$ is abelian.

A2. Prove that if the order of $G$ is 132, then $G$ is not simple.

Solution. Let $n_p$ denote the number of $p$-Sylow subgroups. Since $n_{11}$ is 1 mod 11 and divides 12, we know $n_{11}$ is either 1 or 12. If it is 1, then it is normal and we are done. So instead, assume $n_{11}$ is 12. Then there must be 120 elements of order 11 in $G$, leaving only 12 more elements available.

Since $n_3$ is 1 mod 3 and divides 44, we know that $n_3$ is 1, 4 or 22. If it is 1, then again it is normal and we would be done. If it is 22, that would require 44 elements of order 3 which is not possible. If $n_3$ is 4 that only requires 8 elements of order 3. This leaves 4 elements available which must form the unique Sylow 2-subgroup, which is then normal.

A3. Prove that the polynomial $x^2 - \sqrt{2}$ is irreducible over $\mathbb{Z}[\sqrt{2}]$.

Solution. Suppose not. Since $x^2 - \sqrt{2}$ is monic, if it factorizes as a product of two non-units, these non-units must each be linear.

Then there are $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[\sqrt{2}]$ so that

$$x^2 - \sqrt{2} = (\alpha x + \beta)(\gamma x + \delta).$$

Since $\alpha \gamma = 1$ we get

$$x^2 - \sqrt{2} = (x + \beta \gamma)(x + \delta \alpha).$$

Let $\beta \gamma = u$ and $\delta \alpha = v$.

Then we have $uv = -\sqrt{2}$ and $u + v = 0$, which means that $u^2 = \sqrt{2}$.

However, if $u = a + b\sqrt{2}$ then $u^2 = (a^2 + 2b^2) + (2ab)\sqrt{2}$. Since $a, b \in \mathbb{Z}$ it is clear that we cannot have $u^2 = \sqrt{2}$.

Complex Analysis

1
C1. Does the integral
\[ \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 2x + 2} \, dx \]
converge? If so, what is its value?

**Solution.** The integral converges since
\[ \left| \frac{\cos(x)}{x^2 + 2x + 2} \right| \leq \frac{1}{x^2 + 2x + 2} \]
and
\[ \int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} \, dx \]
converges. To compute its value we note that
\[ \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 2x + 2} \, dx = \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 2x + 2} \, dx \]
\[ = \lim_{R \to \infty} \text{Re} \int_{-R}^{R} \frac{e^{ix}}{x^2 + 2x + 2} \, dx \]
\[ = \text{Re} \left. 2\pi i \text{Res}_{z=-1+i} \left[ \frac{e^{iz}}{z^2 + 2z + 2} \right] \right|_{z=-1+i} \]
\[ - \lim_{R \to \infty} \text{Re} \int_{C_R} \frac{e^{iz}}{z^2 + 2z + 2} \, dz, \]
where \( C_R \) is the upper half of the circle \( |z| = R \) from \( z = R \) to \( z = -R \). Note that
\[ \text{Res}_{z=-1+i} \left[ \frac{e^{iz}}{z^2 + 2z + 2} \right] = \frac{e^{-1-i}}{2i} \]
and for \( R \) large enough
\[ \left| \int_{C_R} \frac{e^{iz}}{z^2 + 2z + 2} \, dz \right| \leq \pi R \cdot \frac{1}{R^2 - 2R - 2} \to 0, \quad \text{as} \quad R \to \infty. \]
Therefore
\[ \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + 2x + 2} \, dx = \text{Re} \pi e^{-2-i} = \pi e^{-2} \cos 1. \]

C2. Find all singular points of the function
\[ h(z) = e^{\frac{1}{z}} \frac{\sin^2(z)}{(z-1)z^2} \]
and classify them as removable, poles, or essential singularities. Then find the residue of 
$h(z)$ at each pole.

**Solution.** \(z = 3\) is an essential singularity, \(z = 0\) is a removable singular point, \(z = 1\) is a simple pole.

\[
\text{Res}_{z=1} h(z) = e^{-1/2} \sin^2 1.
\]

**C3.** Let \(D = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 < 1 \text{ and } (x - \frac{1}{2})^2 + y^2 > \frac{1}{4}\}\), let \(\bar{D}\) be the closure of \(D\), and let \(D' = \bar{D} \setminus \{(0, 0)\}\).

Find a function \(\psi(x, y)\) on \(D'\) that is harmonic on \(D\) and continuous on \(D'\) such that \(\psi(x, y) = 30\) when \((x - 1)^2 + y^2 = 1\) and \(\psi(x, y) = 10\) when \((x - \frac{1}{2})^2 + y^2 = \frac{1}{4}\).

**Hint:** use a linear fractional transformation with a pole \(z = 0\).

**Solution.** The transformation \(w = T(z) = 1/z\) maps the domain into a channel with boundaries \(\text{Re } w = 1/2\) and \(\text{Re } w = 1\). Clearly \(f(w) = 50 - 40 \text{ Re } w\) is harmonic. Then

\[
\phi(x, y) = f(T(z)) = 50 - 40 \text{ Re } \frac{1}{z} = 50 - \frac{40x}{x^2 + y^2}
\]

is a harmonic function satisfying given boundary conditions.

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**Number Theory**

**N1.** Show that for any integer \(n\) the expression

\[
\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}
\]

is always an integer.

**Solution 1.** \(\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} = \frac{3n^5 + 5n^3 + 7n}{15}\), so the goal is to show that 15 divides \(3n^5 + 5n^3 + 7n\) for all \(n\). Equivalently, one must show that 3 and 5 divide \(3n^5 + 5n^3 + 7n\) for all \(n\).

Modulo 3, \(3n^5 + 5n^3 + 7n = 5n^3 + 7n = -n^3 + n = -n(n - 1)(n + 1)\). Since \(n - 1\), \(n\), and \(n + 1\) are consecutive integers, 3 must divide one of them. Thus 3 always divides \(3n^5 + 5n^3 + 7n\).

Similarly, modulo 5, \(3n^5 + 5n^3 + 7n = -2n^5 + 2n = 2n(1 - n^4) = 2n(1 - n)(1 + n)(1 + n^2) = -2n(n - 1)(n + 1)(n^2 + 1)\). If \(n = 0, 1\), or \(-1\) (mod 5) then 5 divides \(n, n - 1, \text{ or } n + 1\).
respectively. If \( n = 2 \) or \( 3 \) (mod 5), then \( n^2 + 1 = 0 \) (mod 5). Thus in all cases 5 divides \( 3n^5 + 5n^3 + 7n \) and we are done.

**Solution 2.** By Fermat’s little theorem, \( n^5 \equiv n \) (mod 5), so that \( 3n^5 + 5n^3 + 7n \equiv 3n + 7n \equiv 10n \equiv 0 \) (mod 5), while similarly \( n^3 \equiv n \) (mod 3), so that \( 3n^5 + 5n^3 + 7n \equiv 5n^3 + 7n \equiv 5n + 7n \equiv 12n \equiv 0 \) (mod 3).

**N2.** Find all integers \( x \) such that

\[
\begin{align*}
x &\equiv 1 \text{ mod } 2; \\
x &\equiv 1 \text{ mod } 3; \\
x &\equiv 1 \text{ mod } 5; \\
x &\equiv 1 \text{ mod } 7.
\end{align*}
\]

**Solution.** This is an immediate consequence of the Chinese remainder theorem. One can also do it directly: \( x = 1 \) (mod \( p \)) implies that \( p \) divides \( x - 1 \). So in the situation of the problem, 2, 3, 5, 7 all divide \( x - 1 \). Since these numbers are all prime, their product 210 divides \( x - 1 \), which in turn means that \( x = 1 \) (mod 210). Conversely, if \( x = 1 \) (mod 210) then 210 divides \( x - 1 \), so that 2, 3, 5 and 7 all divide \( x - 1 \) and thus \( x = 1 \) (mod \( p \)) for \( p = 2, 3, 5, 7 \). Hence the \( x \) satisfying the conditions of the problem are exactly those \( x \) congruent to 1 modulo 210; that is, the \( x \) of the form 210\( n + 1 \) for some integer \( n \).

**N3.** Let \( p \geq 7 \) be a prime. Let \( a \) be such that \( a \not\equiv 1 \) mod \( p \) and \( a^3 \equiv 1 \) mod \( p \).

1. Show that \( a^2 + a + 1 \equiv 0 \) mod \( p \).

2. Show that for \( 1 \leq k \leq 5 \) we have \((a + 1)^k \not\equiv 1 \) mod \( p \), and finally show that \((a + 1)^6 \equiv 1 \) mod \( p \).

**Solution.** Since \( a^3 \equiv 1 \) (mod \( p \)), one has that \( p \) divides \( a^3 - 1 = (a - 1)(a^2 + a + 1) \). Since \( a \not\equiv 1 \) (mod \( p \)), \( p \) does not divide \( a - 1 \), so it must divide \( a^2 + a + 1 \), giving part 1).

For part 2, note that since \( a^2 + a + 1 = 0 \) (mod \( p \)), \( a + 1 = -a^2 \) (mod \( p \)). So \((a + 1)^6 = (-a^2)^6 = (a^3)^4 = 1 \) (mod \( p \)). So as an element of \( \mathbb{Z}_p^* \), the order of \( a + 1 \) divides 6. To see it has order exactly 6, thereby giving part b), note that \((a + 1)^2 = (-a^2)^2 = a^4 = a \not\equiv 1 \) (mod \( p \)), while \((a + 1)^3 = (-a^2)^3 = -a^6 = -(a^3)^2 = -1 \) mod \( p \).

**Real Analysis**

**R1.**
1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function with continuous derivative \( f'(x) \). Let \( a, b \in \mathbb{R} \) with \( a < b \). Prove that there exists \( M \) such that
\[
|f(x) - f(y)| \leq M|x - y|
\]
for all \( x, y \in [a, b] \).

2. Let \( M > 0 \) and \( a < b \). Find a function \( f : \mathbb{R} \to \mathbb{R} \) satisfying (1) for all \( x, y \in [a, b] \) but which is not differentiable everywhere on \((a, b)\).

**Solution.**

(a) Let \( x, y \in [a, b] \). We may assume without loss of generality that \( x < y \). By the Mean Value Theorem, there exists \( z \in (x, y) \) such that
\[
f(x) - f(y) = f'(z) \cdot (x - y).
\]
Since \( f' \) is continuous and \([a, b]\) is a closed interval, \( f' \) is bounded on \([a, b]\) and thus there exists \( M \geq 0 \) such that
\[
- M \leq f'(t) \leq M \text{ for all } t \in [a, b].
\]
It therefore follows that
\[
|f(x) - f(y)| = |f'(z)| \cdot |x - y| \leq M|x - y|.
\]

(b) Choose \( z \in (a, b) \) arbitrary and define \( f(x) = M|x - z| \). It then follows that
\[
|f(x) - f(y)| \leq M||x - z| - |y - z|| \leq M|x - y|
\]
for all \( x, y \in [a, b] \). We now show that \( f \) is not differentiable at \( z \). On the one hand, if \( y \in (z, b) \) then
\[
\frac{f(y) - f(z)}{y - z} = M
\]
and hence
\[
\lim_{y \to z^+} \frac{f(y) - f(z)}{y - z} = M.
\]
On the other hand, if \( y \in (a, z) \) then
\[
\frac{f(y) - f(z)}{y - z} = - M
\]
and hence
\[
\lim_{y \to z^-} \frac{f(y) - f(z)}{y - z} = -M.
\]
This shows that the right and left limits are not equal and therefore the limit does not exist.

R2. Let \( f \) and \( g \) be continuous functions on \([a, b]\) with \( g(x) \geq 0 \) for all \( x \in [a, b] \). Prove that there exists \( x \in [a, b] \) such that
\[
\int_a^b f(t)g(t)dt = f(x) \int_a^b g(t)dt.
\]
Solution. Since \( f \) is continuous, it attains its minimum \( m \) and its maximum \( M \) on \([a, b]\).

Since \( g \) is non-negative, it follows that
\[
mg(t) \leq f(t)g(t) \leq Mg(t)
\]
for all \( t \in [a, b] \) and hence
\[
mL \leq \int_a^b f(t)g(t)dt \leq ML,
\]
where \( L := \int_a^b g(t)dt \). Now, note that the function \( F : [a, b] \to \mathbb{R} \) given by \( F(x) := f(x) \cdot L \) is continuous on \([a, b]\) and attains its minimum \( mL \) and its maximum \( ML \) on \([a, b]\). By the Intermediate Value Theorem, there thus exists \( x \in [a, b] \) such that \( F(x) = \int_a^b f(t)g(t)dt \).

R3. Let \( f_n : [0, 1) \to \mathbb{R} \) be the function defined by
\[
f_n(x) := \sum_{k=1}^n \frac{x^k}{1+x^k}.
\]

1. Prove that \( f_n \) converges to a function \( f : [0, 1) \to \mathbb{R} \).

2. Prove that for every \( 0 < a < 1 \) the convergence is uniform on \([0, a]\).

3. Prove that \( f \) is differentiable on \((0, 1)\).

Solution. (a) Let \( x \in (0, 1) \). By the ratio test the series \( \sum_{k=1}^\infty \frac{x^k}{1+x^k} \) converges absolutely, thus \( f_n(x) \) converges to the limit \( f(x) := \sum_{k=1}^\infty \frac{x^k}{1+x^k} \).

(b) Let \( a \in (0, 1) \). Clearly, we have
\[
0 \leq \frac{x^k}{1+x^k} \leq a^k
\]
for every \( x \in [0, a] \) and every \( k \geq 1 \). Since \( \sum_{k=1}^\infty a^k \) converges it follows from the Weierstrass M-test that the sequence \( f_n \) of functions converges uniformly on \([0, a]\).

(c) Note that
\[
\left. \frac{d}{dx} \right|_x \left( \frac{x^k}{1+x^k} \right) = \frac{kx^{k-1}}{(1+x^k)^2}
\]
for all \( x \in (0, 1) \) and hence
\[
f'_n(x) = \sum_{k=1}^n \frac{kx^{k-1}}{(1+x^k)^2}
\]
for all \( x \in (0, 1) \). One shows as in (a) and (b) that the sequence \( f'_n \) of functions converges to a function \( g : (0, 1) \to \mathbb{R} \), uniformly on \((0, a)\) for every \( a \in (0, 1) \). Since every \( f'_n \) is
continuous it follows that \( g \) is continuous. It follows from this that \( f \) is differentiable on \((0, 1)\) with derivative \( g(x) \) at \( x \) for every \( x \in (0, 1) \).

**Logic**

**L1.** Prove or disprove the following claim.

**Claim:** Suppose \( L \) is a propositional language and \( F, G \) are two \( L \)-formulas with no common propositional variables. Then whenever
\[
| F \rightarrow G,
\]
either \( \neg F \) or \( G \) is a tautology.

**Solution.** The claim is true. To see this, suppose towards a contradiction that neither \( \neg F \) nor \( G \) is a tautology and let \( v_F \) and \( v_G \) be \( L \)-valuations such that \( v_F^*(\neg F) = 0 \) and \( v_G^*(G) = 0 \), where \( v^* \) is the canonical extension of \( v \) to the set of all \( L \)-formulas. Now let \( v \) be any valuation that agrees with \( v_F \) on all propositional variables occurring in \( F \) and agrees with \( v_G \) on all propositional variables occurring in \( G \). This is possible, since \( F \) and \( G \) have no common variables. Then \( v^*(\neg F) = 0 \), while \( v^*(G) = 0 \), whence \( v^*(F \rightarrow G) = 0 \), contradiction our assumption.

**L2.** Let \( L = \{ f \} \), where \( f \) is a unary function symbol. Find an \( L \)-sentence \( \phi \) such that

(a) for any \( n \geq 1 \), \( \phi \) has a model of size \( 2^n - 1 \), and

(b) any finite \( L \)-structure satisfying \( \phi \) will have odd cardinality.

**Solution.** We let \( \phi \) say that \( f \) is an involution with exactly one fixed point, i.e., \( \phi \) is the sentence
\[
\forall x \, ffx = x \land \exists x \, (ffx = x \land \forall y \, (fy = y \rightarrow x = y)).
\]

**L3.** Decide whether the following argument is valid by either providing a proof in a proof system of your choice or by providing a counter-example.

**Solution.** Let \( \mathcal{M} \) consist of elements \( a, b \) such that
\[
P^Ma, \quad P^Mb, \quad R^Maaa, \quad R^Mbbb, \quad R^Mabb, \quad R^Mbba.
\]

Then \( \mathcal{M} \) satisfies the premiss, but not the conclusion and thus the argument is invalid.

\[
\forall x \, (P(x) \rightarrow \forall z \, (P(z) \rightarrow \exists y \, R(x, y, z))) \quad \nabla x \, (P(x) \rightarrow \exists y \, \forall z \, (P(z) \rightarrow R(x, y, z)))
\]
**Topology**

**T1.** Let $X$ be an infinite set. The **Zariski topology** on $X$ is defined by the collection of subsets

$$\mathcal{T} = \{ U = X - A \mid A \subset X, \text{ A is finite } \} \cup \{ \emptyset \}$$

a) Show that $\mathcal{T}$ satisfies the axioms of a topology.

b) What is the closure of the set $A = \{1/n \mid n = 1, 2, \ldots \}$ in the Zariski topology on $\mathbb{R}$?

**Solution.**

a) The set $A = \emptyset$ is finite, so $X = X - \emptyset$ is open.

Let $\{U_\alpha \mid \alpha \in A\}$ be an arbitrary collection of open sets for $\mathcal{T}$. Then for each $\alpha$ there is a finite set $A_\alpha \subset X$ with $U = X - A_\alpha$. Then by DeMorgan’s Laws,

$$\bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} (X - A_\alpha) = X - \bigcap_{\alpha \in A} A_\alpha = X - A$$

where $A = \bigcap_{\alpha \in A} A_\alpha$ is the intersection of finite sets, so is finite. Thus, $X - A$ is open in $\mathcal{T}$.

Let $\{U_1, \ldots, U_n\}$ be a finite collection of open sets for $\mathcal{T}$. Then for each $i$ there is a finite set $A_i \subset X$ with $U = X - A_i$. Then by DeMorgan’s Laws,

$$\bigcap_{i=1}^n U_i = \bigcap_{i=1}^n (X - A_i) = X - \bigcup_{i=1}^n A_i = X - A$$

where $A = A_1 \cup \cdots \cup A_n$ is a finite union of finite sets, so finite. Thus, $X - A$ is open in $\mathcal{T}$.

b) The closure $\overline{A}$ of $A$ is the intersection of all closed subsets in $\mathcal{T}$ containing $A$. The closed subsets of $\mathcal{T}$ are the complements of the open subsets in $\mathcal{T}$, so either a closed subset is a finite subset of $X$, or all of $X$. No finite subset contains the infinite set $A$, thus $X$ is the only closed subset containing $A$. Thus, $\overline{A} = X$. □

**T2.** Let $X$ be a topological space and $A, B \subset X$. Define $\partial A = \overline{A} - int(A)$, where $int(A) \subset A$ is the largest open subset of $X$ contained in $A$. Show that if $B$ is connected, and $B \cap A \neq \emptyset$, and $B \setminus A \neq \emptyset$, then $B \cap \partial A \neq \emptyset$.

**Solution.** Suppose that $B \cap \partial A = \emptyset$. Then $\overline{A} = A \cup \partial A = int(A) \cup \partial A$ is a closed subset of $X$, so

$$U = B - \overline{A} = B \cap (X - (int(A) \cup \partial A)) = B - int(A)$$
is an open, non-empty, proper subset of $B$. But $\text{int}(A)$ is an open subset of $X$, so $B - \text{int}(A)$ is also closed. Thus, the set $V = \text{int}(A) \cap B$ is open in $B$ and disjoint from $B$.

We assume that $B \cap A \neq \emptyset$ and $B \cap \partial A = \emptyset$, hence $V \neq \emptyset$. Thus, the connected set $B = U \cup V$ where $U$ and $V$ are open, disjoint non-empty subsets, which is a contradiction. □

**T3.** Let $X$ be a Hausdorff topological space and $A \subset X$ a subset. Suppose there exists a continuous map $f : X \to A$ (for the relative topology on $A$) with $f(x) = x$ for every $x \in A$. Prove that $A$ is closed in $X$.

**Solution.** If $A = X$ we are done, as $X$ is always closed in itself. Thus, assume there exists $x \in X - A$ and we must show there exists an open subset $W \subset X - A$ with $x \in W$.

Set $y = f(x) \in A$. As $x \notin A$ and $y = f(x) \in A$ we have $x \neq y$. By the Hausdorff assumption, there exists disjoint open sets $U, V \subset X$ with $x \in U$ and $y \in V$.

As $V \cap A$ is open in $A$ and $f$ is continuous, $V' = f^{-1}(V \cap A)$ is open in $X$ and $f(x) = y$ implies that $x \in V'$.

Set $W = U \cap V'$, an open set in $X$, then $x \in W$.

Suppose that $z \in W$ then $f(z) \in V \cap A$ and $z \in U$. As $U \cap V = \emptyset$, this implies that $f(z) \neq z$, hence $z \notin A$. Thus, $W \subset X - A$ is an open neighborhood of $x$ disjoint from $A$. This shows that $X - A$ is open, hence $A$ is closed. □