Full points can be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth 20 points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. **DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS.** When you have finished, insert all your answer sheets into the envelope provided, then seal and print your name on it.

Any student whose answers need clarification may be required to submit to an oral examination.
A1. Let $p$ be a prime other than 3, and $k \geq 1$ an integer. Prove that a group of order $3p^k$ is not simple.

**Solution.** Suppose that $G$ is a group of order $3p^k$. The Sylow $p$-subgroups of $G$ have order $p^k$, and there are either 1 or 3 of them. If there is a unique Sylow $p$-subgroup, then $G$ cannot be simple. The group $G$ acts faithfully on the set of its Sylow $p$-subgroups, by conjugation. If there are 3 Sylow $p$-subgroups of $G$, then we get a nontrivial homomorphism from $G$ to $S_3$, the kernel of which is a proper subgroup of $G$ of index at most 6.

In either case, $G$ cannot be simple.

A2.

1. Let $G$ be a group and $D = \{(g, g) \mid g \in G\} \leq G \times G$. Prove that $D$ is normal in $G \times G$ if and only if $G$ is abelian.

2. If $G$ is abelian, prove that $(G \times G)/D \cong G$.

**Solution.** If $G$ is abelian then $G \times G$ is also abelian. Every subgroup of an abelian group is normal, so in particular $D$ is a normal subgroup of $G \times G$.

On the other hand, suppose that $G$ is nonabelian. Then there are $g, h \in G$ so that $hg \neq gh$, so that $g^{-1}hg \neq h$. Then in $G \times G$ we have

$$(g, 1)^{-1}(h, h)(g, 1) = (g^{-1}hg, h) \notin D,$$

so $D$ is not normal. Define $\phi : G \times G \to G$ by $\phi(g, h) = gh^{-1}$.

Note that since $G$ is abelian we have

$$\phi(g_1, h_1)\phi(g_2, h_2) = g_1h_1^{-1}g_2h_2^{-1} = g_1g_2(h_1h_2)^{-1} = \phi((g_1, h_1)(g_2, h_2)).$$

So $\phi$ is a homomorphism. Now, $\phi(g, h) = 1$ if and only if $g = h$, so the kernel of $\phi$ is exactly $D$. Also, $g = \phi(g, 1)$, so $\phi$ is surjective. The First Isomorphism Theorem Implies the required statement.

A3. Show that $\mathbb{Z}[\sqrt{-7}]$ is not a unique factorization domain.

**Solution.** Let $R = \mathbb{Z}[\sqrt{-7}]$. Note that

$$8 = 2^3 = (1 - \sqrt{-7})(1 + \sqrt{-7}).$$
I claim that 2, \((1 - \sqrt{-7})\) and \((1 + \sqrt{-7})\) are all irreducible.

To see this, define \(N : \mathbb{R} \rightarrow \mathbb{Z}\) by
\[
N(a + b\sqrt{-7}) = a^2 + 7b^2.
\]

It is easy to see that
\[
N(\alpha \beta) = N(\alpha)N(\beta),
\]
and so, since \(N(1) = 1\) a unit \(u \in \mathbb{R}\) must have \(N(u) = 1\). Conversely, the only elements with \(N(u) = 1\) are \(u = \pm 1\).

Also, \(N(2) = 4\), \(N(1 - \sqrt{-7}) = N(1 + \sqrt{-7}) = 8\). If we have non-units \(x, y \in \mathbb{R}\) so that
\[
xy = 2
\]
we must have \(N(x) = N(y) = 2\). But it is easy to see that there are no integers \(a, b\) so that \(a^2 + 7b^2 = 2\). Similarly, if there are non-units \(x\) and \(y\) so that
\[
xy = 1 \pm \sqrt{-7}
\]
then one of \(x\) and \(y\) has norm 2 and the other norm 4. In any case, we see that these elements are irreducible.

Finally, it is obvious that 2 is not associate to \(1 \pm \sqrt{-7}\), again since the only units are \(\pm 1\).

Thus, we have written 8 as a product of irreducibles in two essentially different ways, so \(\mathbb{R}\) is not a UFD.

**Complex Analysis**

**C1.** Determine the number of complex roots (counting multiplicities) of the polynomial
\[
P(z) = z^5 + 7z^4 + z^3 - 2z^2 + z + 1
\]
which lie inside the unit circle.

**Solution.** We use Rouché’s Theorem. Write \(P(z) = f(z) + g(z)\) where \(f(z) = 7z^4 + 1\) and \(g(z) = z^5 + z^3 - 2z^2 + z\). For \(|z| = 1\) we have
\[
|g(z)| \leq |z^5| + |z^3| + 2|z^2| + |z| \leq 5
\]
and
\[
|f(z)| \geq 7|z^4| - 1 > 5.
\]
Thus we have \(|g(z)| < |f(z)|\) for all \(z\) lying on the unit circle, and Rouché’s theorem states that \(f(z)\) and \(P(z) = f(z) + g(z)\) have the same number of zeros inside the unit circle.

However, the roots of \(f(z)\) are exactly the fourth roots of \(-1/7\), which all have modulus \(\sqrt[4]{1/7} < 1\), and hence lie inside the unit circle. This shows that \(P\) has four roots inside the unit circle.

**C2.** Evaluate the improper integral \(\int_0^\infty \frac{dx}{(x^2 + 1)^2}\).
Thus we may compute the value of the improper integral as
\[
\frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \lim_{R \to \infty} \int_{S_R} \frac{dz}{(z^2 + 1)^2}.
\]
The basic estimate for contour integrals shows that
\[
\left| \int_{C_R} \frac{dz}{(z^2 + 1)^2} \right| \leq \frac{2\pi R}{R^4} \to 0 \text{ as } R \to \infty.
\]
Thus we may compute the value of the improper integral as
\[
\int_{0}^{\infty} \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \lim_{R \to \infty} \int_{S_R + C_R} \frac{dz}{(z^2 + 1)^2}.
\]
The function \( f(z) = 1/(z^2 + 1)^4 \) has two poles of order 2, at the points \( i \) and \(-i\). For \( R > 1 \), the closed contour \( S_R + C_R \) encloses only the pole at \( i \). Thus the integral in the above expression is independent of \( R \), for \( R > 1 \), and equals \( 2\pi i \) times the residue of \( f \) at \( i \). To evaluate this residue we write \( f(z) = g(z)/(z-i)^2 \) where \( g(z) = 1/(z+i)^2 \). Then \( \text{Res}_{z=i} f(z) = g(i) = 1/4i \). Thus
\[
\int_{0}^{\infty} \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} (2\pi i)(\frac{1}{4i}) = \frac{\pi}{4}.
\]

**Solution.** Set \( f(z) = 1/(z^2 + 1)^2 \), let \( S_R \) be the positively oriented segment from \(-R\) to \( R \) on the real line, and let \( C_R \) be the counter-clockwise semicircle in the upper half-plane which meets the real line at the points \(-R\) and \( R \).

The value of the improper integral is equal to
\[
\frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \lim_{R \to \infty} \int_{S_R} \frac{dz}{(z^2 + 1)^2}.
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**Solution.** Set \( f(z) = 1/(z^2 + 1)^2 \), let \( S_R \) be the positively oriented segment from \(-R\) to \( R \) on the real line, and let \( C_R \) be the counter-clockwise semicircle in the upper half-plane which meets the real line at the points \(-R\) and \( R \).

The value of the improper integral is equal to
\[
\frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \lim_{R \to \infty} \int_{S_R} \frac{dz}{(z^2 + 1)^2}.
\]

**Solution.** Set \( f(z) = 1/(z^2 + 1)^2 \), let \( S_R \) be the positively oriented segment from \(-R\) to \( R \) on the real line, and let \( C_R \) be the counter-clockwise semicircle in the upper half-plane which meets the real line at the points \(-R\) and \( R \).

The value of the improper integral is equal to
\[
\frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{(x^2 + 1)^2} = \frac{1}{2} \lim_{R \to \infty} \int_{S_R} \frac{dz}{(z^2 + 1)^2}.
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**Solution.** Set \( f(z) = 1/(z^2 + 1)^2 \), let \( S_R \) be the positively oriented segment from \(-R\) to \( R \) on the real line, and let \( C_R \) be the counter-clockwise semicircle in the upper half-plane which meets the real line at the points \(-R\) and \( R \).

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\]
Number Theory

N1. Prove that for any $n \in \mathbb{N} - \{0, 1\}$, we have $1 + \frac{1}{2} + \ldots + \frac{1}{n} \notin \mathbb{Z}$.

Solution. Note that

$$1 + \frac{1}{2} + \ldots + \frac{1}{n} = \frac{\prod_{j \neq i} j}{n!} \quad \text{(1)}$$

By Bertrand’s Postulate, there is a prime $p$ such that $\frac{n}{2} < p < n$. Thus $p$ divides the denominator $n!$ exactly once. It also divides every term of the numerator, except $\prod_{j \neq p} j$. As a result, $p$ does not divide the numerator. Hence the denominator of (1) does not divide the numerator of (1), and consequently $1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is not an integer.

N2. We recall Partial Summation: Let $(c_n)_{n \geq 1}$ be a sequence of complex numbers. Let $f : [1, \infty] \rightarrow \mathbb{C}$ be a function with continuous derivative. Then

$$\sum_{n \leq x} c_n f(n) = \left( \sum_{n \leq x} c_n \right) f(x) - \int_1^x \left( \sum_{n \leq t} c_n \right) f'(t) \, dt.$$ 

Use Partial Summation to show that, as $x \to \infty$,

$$\sum_{d \leq x} \frac{1}{d} = \log x + O(1).$$

Let $\tau(n)$ denote the number of divisors of $n$. Show that, as $x \to \infty$,

$$\sum_{n \leq x} \tau(n) = x \log x + O(x).$$

Solution.

1. We apply partial summation with $c_n = 1 \ \forall n \geq 1$ and $f(x) = \frac{1}{x}$. 


2. We have:

\[
\sum_{n \leq x} \tau(n) = \sum_{n \leq x} \sum_{d \mid n} 1 \\
= \sum_{d \leq x} \sum_{n \leq x \atop n \mid d} 1 \\
= \sum_{d \leq x} \sum_{m \leq \frac{x}{d}} 1 \\
= \sum_{d \leq x} \left\lfloor \frac{x}{d} \right\rfloor \\
= \sum_{d \leq x} \left( \frac{x}{d} - \left\{ \frac{x}{d} \right\} \right) \\
= x \sum_{d \leq x} \frac{1}{d} + O \left( \sum_{d \leq x} 1 \right) \\
= x \log x + O(x),
\]

by part (1).

N3. Let \( p \) be an odd prime.

1. Let \( d \in \mathbb{Z} \). Show that

\[
\# \{ [x] \in \mathbb{Z}/p\mathbb{Z} : x^2 \equiv d \pmod{p} \} = 1 + \left( \frac{d}{p} \right),
\]

where \([x]\) denotes the equivalence class of \( x \) modulo \( p \) and \( \left( \frac{\cdot}{p} \right) \) denotes the Legendre symbol modulo \( p \).

2. Let \( a, b \in \mathbb{Z} \). Show that

\[
\# \{ ([x], [y]) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} : y^2 \equiv x^3 + ax + b \pmod{p} \} = p + \sum_{[x] \in \mathbb{Z}/p\mathbb{Z}} \left( \frac{x^3 + ax + b}{p} \right).
\]

Solution.

1. We recall that \( \left( \frac{d}{p} \right) \) is 0 if \( p \mid d \), 1 if \( p \nmid d \) and there exists a solution to the equation \( x^2 \equiv d \pmod{p} \), and \(-1\) if \( p \nmid d \) and there exists no solution to the equation \( x^2 \equiv d \pmod{p} \).

We also observe that, if \( p \nmid d \) and there exists a solution to the equation \( x^2 \equiv d \pmod{p} \), then there exist exactly two solutions. Moreover, if \( p \mid d \), then there exists exactly one solution, \( 0 \pmod{p} \), to the equation \( x^2 \equiv d \pmod{p} \).
Thus
\[ 1 + \left( \frac{d}{p} \right) \]
equals the number of solutions to the equation \( x^2 \equiv d \pmod{p} \), which is what we wanted to prove.

2. This is an immediate application to (1).
Real Analysis

R1. Let $\alpha, C > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \mathbb{R}$.

1. Prove that $f$ is uniformly continuous on $\mathbb{R}$.

2. Prove that if $\alpha > 1$ then $f$ is constant, that is, $f(x) = f(0)$ for all $x \in \mathbb{R}$.

Solution. a) Let $\varepsilon > 0$ and set $\delta := (\varepsilon/C)^{\frac{1}{\alpha}}$. Then for any $x, y \in \mathbb{R}$ with $|x - y| \leq \delta$ we have

$$|f(x) - f(y)| \leq C|x - y|^\alpha \leq C\delta^\alpha = \varepsilon.$$

(b) Let $x > 0$ and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ so large that $\frac{Cx^\alpha}{n^\alpha} \leq \varepsilon$. Set $x_k := \frac{kx}{n}$ for $k = 0, 1, \ldots, n$ and note that $|x_{k-1} - x_k| = \frac{x}{n}$. We calculate

$$|f(0) - f(x)| \leq |f(x_0) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_{n-1}) - f(x_n)|$$

$$\leq C|x_0 - x_1|^\alpha + C|x_1 - x_2|^\alpha + \cdots + C|x_{n-1} - x_n|^\alpha$$

$$= Cn \frac{x^\alpha}{n^\alpha} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this shows that $f(x) = f(0)$. The case $x < 0$ is proved analogously.

R2. Let $f$ and $g$ be continuous functions on $[a, b]$ satisfying

$$\int_a^b f(t)dt = \int_a^b g(t)dt.$$

Prove that there exists $x \in [a, b]$ such that $f(x) = g(x)$.

Solution. If $f(x) \neq g(x)$ for all $x \in [a, b]$ then, since $f$ and $g$ are continuous, either $f(x) > g(x)$ for all $x \in [a, b]$ or $f(x) < g(x)$ for all $x \in [a, b]$. In the first case, we would have

$$\int_a^b f(t)dt \geq \int_a^b g(t)dt,$$

and in the second case we would have

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt.$$
both cases being impossible.

**R3.** Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the function defined by
\[
f_n(x) := \sum_{k=1}^{n} \frac{1}{2k^2 + k \cos(kx)}.
\]

1. Prove that \( f_n \) converges uniformly to a continuous function \( f : \mathbb{R} \to \mathbb{R} \).

2. Prove that \( f \) is differentiable.

**Solution.**

(a) Note first that
\[
0 < \frac{1}{2k^2 + k \cos(kx)} \leq \frac{1}{k^2}
\]
for all \( x \in \mathbb{R} \) and all \( k \geq 1 \). Since the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges it follows from the Weierstrass \( M \)-test that the sequence \( f_n \) converges uniformly on \( \mathbb{R} \) to a function \( f \). Since every \( f_n \) is continuous, so is \( f \).

(b) Note that
\[
\frac{d}{dx} \left( \frac{1}{2k^2 + k \cos(kx)} \right) = \frac{\sin(kx)}{(2k + k \cos(kx))^2}
\]
and that the absolute value of the term on the right hand side is bounded above by \( \frac{1}{k^2} \) for all \( x \in \mathbb{R} \) and all \( k \geq 1 \). It thus follows from the Weierstrass \( M \)-test that the sequence \( f'_n \) converges uniformly to a function \( g : \mathbb{R} \to \mathbb{R} \). Since every \( f'_n \) is continuous, so is \( g \). It follows from this that \( f \) is differentiable with derivative \( g(x) \) at \( x \).
L1. Let \( L \) be a propositional language. A set \( \Sigma \) of \( L \)-sentences is said to be \textit{maximally consistent} if (1) it is consistent and (2) there is no consistent set of \( L \)-sentences strictly containing \( \Sigma \).

(a) Suppose \( \Sigma \) is maximally consistent and \( P \in L \) is a propositional variable. Show that either \( P \in \Sigma \) or \( \neg P \in \Sigma \).

(b) Suppose \( \Sigma_1 \) and \( \Sigma_2 \) are two maximally consistent sets containing the same literals, i.e., propositional variables and negations of such. Show that \( \Sigma_1 = \Sigma_2 \).

L2. Show that for every \( n \geq 1 \) there is a first-order sentence \( \phi_n \) containing only unary predicates and no constant symbols, function symbols, nor equality “=”, such that every model of \( \phi_n \) has at least \( n \) elements.

L3. Let \( L = \{<\} \) be a first-order language, where \(<\) is a binary relation symbol used to represent a linear ordering.

(a) Show that the class of finite linearly ordered sets is \textit{not axiomatisable}, i.e., there is no set of \( L \)-sentences \( \Sigma \) such that an \( L \)-structure \( \mathcal{M} \) satisfies \( \Sigma \) if and only if \( \mathcal{M} \) is a finite set linearly ordered by \( <^\mathcal{M} \).

(b) Show that the class of infinite linearly ordered sets is axiomatisable in the language \( L \).

(c) Show that the set of infinite linearly ordered sets is \textit{not finitely axiomatisable}, i.e., cannot be axiomatised by a finite set of \( L \)-sentences.
T1. Let $X$ be a topological space, and $A \subset X$ a subset. Define the interior $\text{int}(A) \subset A$ to be the largest open subset of $X$ contained in $A$. Define the boundary of $A \subset X$ by the equation:

$$\partial A = \overline{A} \cap \overline{X - A}$$

a) Show that $\text{int}(A)$ and $\partial A$ are disjoint.
b) Show that $\overline{A} = \text{int}(A) \cup \partial A$
c) Show that $\partial A = \emptyset$ if and only if $A$ is both open and closed.

Solution.

a) The set $\text{int}(A)$ is open, so $X - \text{int}(A)$ is a closed set containing $X - A$, hence

$$\text{int}(A) \cap \overline{X - A} \subset \text{int}(A) \cap (X - \text{int}(A)) = \emptyset$$

b) $\overline{X - A}$ is closed, so its complement is open, hence

$$A - \overline{X - A} = A \cap (X - \overline{X - A}) \subset \text{int}(A)$$

Then

$$\overline{A} = \overline{A} \cap \{(\overline{X - A}) \cup (X - \overline{X - A})\} = \{\overline{A} \cap \overline{X - A}\} \cup (A - \overline{X - A}) \subset \partial A \cup \text{int}(A) \subset \overline{A}$$

c) Suppose that $\partial A = \emptyset$, then $\overline{A} = \text{int}(A)$ by b) so $A$ is open.

Also, $\partial A = \emptyset$ implies that $\overline{A} \cap (X - A) = \emptyset$ so

$$A = X - (X - A) = \overline{A} \cap (X - (X - A)) = \overline{A} - (\overline{A} - \overline{A} \cap (X - A)) = \overline{A}$$

Thus, $A = \overline{A}$ showing that $A$ is closed.

Conversely, assume that $A$ is open and closed. Then $A$ closed implies $\partial A \subset A$, while $A$ open implies that $X - A$ is closed, hence $(X - A) \subset X - A$. Thus,

$$\partial A = \overline{A} \cap \overline{X - A} \subset A \cap (X - A) = \emptyset$$

T2. Let $X$ be a Hausdorff topological space. Let $K \subset X$ be a compact subset, and let $x \in X$ with $x \notin K$. Prove that there exists disjoint open sets $U, V \subset X$ such that

$$x \in U \; ; \; K \subset V \; ; \; U \cap V = \emptyset$$
Solution. For each \( y \in K \) the Hausdorff hypotheses implies there exists open sets \( U_y, V_y \subset X \) with \( x \in U_y \), \( y \in V_y \) and \( U_y \cap V_y = \emptyset \). The collection \( \{ V_y \mid y \in K \} \) forms a covering of \( K \) by open sets, as for each \( y \in K \) we have \( y \in V_y \). As \( K \) is assumed compact, there exists a finite subcovering, say \( \{ V_{y_i} \mid i = 1, 2, \ldots, n \} \).

The finite intersection of open sets is open, so \( U = \bigcap_{i=1}^{n} U_{y_i} \) is an open set with \( x \in U \).

The arbitrary union of open sets is open, so \( V = \bigcup_{i=1}^{n} V_{y_i} \) is an open set with \( K \subset V \).

Note that each \( U_x \cap V_{x_i} = \emptyset \) by choice, so \( U \cap V = \emptyset \) by DeMorgan’s Laws. □

T3. Show that if \( U \) is an open, connected subspace of \( \mathbb{R}^2 \), then \( U \) is path connected.

Solution. If \( U = \emptyset \), then there is nothing to prove, so let \( x \in U \). Let \( W \subset U \) be the set of points in \( U \) which are connected to \( x \) by a path in \( U \). We claim that \( W = U \).

First, we claim that \( W \) is an open subset of \( U \). Let \( y \in W \), then there exists a path \( \sigma : [0, 1] \rightarrow W \) with \( \sigma(0) = x \) and \( \sigma(1) = y \). As \( U \) is open, there exists \( \epsilon > 0 \) such that the open \( \epsilon \)-ball \( B(y, \epsilon) \subset U \). Let \( z \in B(y, \epsilon) \) then there is a radial path \( \tau : [0, 1] \rightarrow B(y, \epsilon) \) with \( \tau(0) = y \) and \( \tau(1) = z \). The concatenation \( \tau * \sigma \) is then a path in \( U \) from \( x \) to \( z \). Thus, \( z \in W \). This shows that \( W \) is an open subset of \( U \).

Suppose that \( V = U - W \neq \emptyset \), then we claim that \( V \) is also open. Let \( \xi \in V \), then as \( U \) is open, there exists \( \delta > 0 \) such that \( B(\xi, \delta) \subset U \). If \( B(\xi, \delta) \cap W \neq \emptyset \), then as above, there exists a path in \( U \) from \( x \) to \( \xi \), which implies that \( \xi \in W \), contrary to assumption. Thus, \( B(\xi, \delta) \subset V \). As \( \xi \in V \) was arbitrary, \( V \) is open.

By definition, \( W \cap V = \emptyset \), so we have \( U = W \cup V \) is a decomposition of \( U \) into open disjoint subsets, contrary to the assumption that \( U \) is connected. We conclude \( U = W \) as claimed. □
1. Exercise

Let $L$ be a propositional language. A set $Σ$ of $L$-sentences is said to be maximally consistent if (1) it is consistent and (2) there is no consistent set of $L$-sentences strictly containing $Σ$.

(a) Suppose $Σ$ is maximally consistent and $P ∈ L$ is a propositional variable. Show that either $P ∈ Σ$ or $¬P ∈ Σ$.

(b) Suppose $Σ_1$ and $Σ_2$ are two maximally consistent sets containing the same literals, i.e., propositional variables and negations of such. Show that $Σ_1 = Σ_2$.

Solution

(a) Since $Σ$ is consistent, there is a valuation $v : L → \{T, F\}$ such that $v^*(S)$ for any formula $S ∈ Σ$, where $v^*$ is the canonical extension of $v$ to all formulas of $L$. I.e., $v$ satisfies $Σ$. Now, either $v(P) = T$ or $v(P) = F$ and so, in the latter case, $v^*(¬P) = T$. In the first case, one sees that $Σ ∪ \{P\}$ is consistent and, in the second case, $Σ ∪ \{¬P\}$ is consistent. Since $Σ$ is maximally consistent, it follows that either $P ∈ Σ$ or $¬P ∈ Σ$.

(b) Since $Σ_i$ is consistent, there is some truth valuation $v_i : L → \{T, F\}$ satisfying $Σ_i$. And for any $L$-formula $S$, if $v_i^*(S) = T$, then $Σ_i ∪ \{S\}$ is consistent and so $S ∈ Σ_i$. It follows that

$$Σ_i = \{S ∈ Form(L) \mid v_i^*(S) = T\}.$$

Now, if $Σ_1$ and $Σ_2$ contain the same literals, then, by (a), $v_1(P) = v_2(P)$ for any $P ∈ L$, i.e., $v_1^* = v_2^*$, and so $Σ_1 = Σ_2$.

2. Exercise

Show that for every $n ≥ 1$ there is a first-order sentence $φ_n$ containing only unary predicates and no constant symbols, function symbols, nor equality “=”, such that every model of $φ_n$ has at least $n$ elements.

Solution

Let $U_1, \ldots, U_n$ be distinct unary predicates and let $φ_n$ be the formula

$$∃x_1 ∃x_2 \ldots ∃x_n \bigwedge_{i=1}^n (U_i(x_i) & \bigwedge_{j \neq i} ¬U_j(x_i)).$$

Then any model of $φ_n$ contains elements $a_1, \ldots, a_n$ such that $a_i$ satisfies $U_j$ if and only if $i = j$. It follows that the $a_i$ are all distinct.
3. Exercise

Let $L = \langle \rangle$ be a first-order language, where $\langle \rangle$ is a binary relation symbol used to represent a linear ordering.

(a) Show that the class of finite linearly ordered sets is not axiomatisable, i.e., there is no set of $L$-sentences $\Sigma$ such that an $L$-structure $\mathcal{M}$ satisfies $\Sigma$ if and only if $\mathcal{M}$ is a finite set linearly ordered by $\langle \rangle$.

(b) Show that the class of infinite linearly ordered sets is axiomatisable in the language $L$.

(c) Show that the set of infinite linearly ordered sets is not finitely axiomatisable, i.e., cannot be axiomatised by a finite set of $L$-sentences.

Solution

(a) Suppose $\Sigma$ is a set of $L$-sentences such that if $\mathcal{M}$ is a finite set linearly ordered by $\langle \rangle$ then $\mathcal{M} \models \Sigma$. For every $n$ let $\phi_n$ be the $L$-sentence

$$\exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i < x_j$$

and note that if $\mathcal{M}$ is a finite set linearly ordered by $\langle \rangle$ with at least $n$ elements then $\mathcal{M} \models \phi_m$ for any $m \leq n$. Since we can build linear orderings of arbitrary large finite cardinality, it follows that for every $n$, there is a model of $\Sigma \cup \{\phi_1, \ldots, \phi_n\}$.

By the compactness theorem, we see that $\Sigma \cup \{\phi_n\}_{n \in \mathbb{N}}$ is consistent and thus has a model $\mathcal{M}$, which must necessarily be infinite. Thus, no theory $\Sigma$ can axiomatise the class of finite linear orderings.

(b) Let $\Sigma$ be the theory consisting of $\{\phi_n\}_{n \in \mathbb{N}}$ along with the axioms for linear orderings, i.e.,

$$\sigma_1 = \forall x \forall y (x = y \lor x < y \lor y < x),$$

$$\sigma_2 = \forall x \forall y \forall z (x < y \land y < z \rightarrow x < z),$$

$$\sigma_3 = \forall x \neg x < x.$$

Then $\Sigma$ axiomatises the class of infinite linear orderings.

(c) Suppose towards a contradiction that $\Delta = \{\psi_1, \ldots, \psi_k\}$ is an $L$-theory axiomatising the class of infinite linear orderings. Then $\psi = \bigwedge_{i=1}^k \psi_i$ is a single sentence axiomatising the class and hence an $L$-structure $\mathcal{M}$ is a model of $\neg \psi$ if and only if $\mathcal{M}$ is not an infinite linear ordering. It follows that $\mathcal{M}$ is a model of $\Sigma = \{\neg \psi, \sigma_1, \sigma_2, \sigma_3\}$ if and only if $\mathcal{M}$ is not an infinite linear ordering, but is a linear ordering, i.e., if and only if $\mathcal{M}$ is a finite linear ordering. Since, by (a), there is no such theory $\Sigma$, it follows that $\Delta$ cannot exist.