A1. Show that every group of order $294 = 49 \times 2 \times 3$ has a normal subgroup of order 147.

Solution. Let $G$ be a group of order 294 and $n_7$ be the number of Sylow 7-subgroups of $G$. By Sylow theorems $n_7 \equiv 1 \mod 7$ and $n_7 | 6$. Thus $n_7 = 1$. Let $H$ be the unique Sylow 7-subgroup of $G$. If $g \in G$ then $gHg^{-1}$ is a Sylow 7-subgroup. Thus $gHg^{-1} = H$ for all $g \in G$. Let $K$ be a Sylow 3-subgroup of $G$.

We have $H < G$, $K < G$, $|H| = 49$, $|K| = 3$ and $H \cap K = \{1\}$. Thus $HK$ is a subgroup of $G$ and $|HK| = 147$. Finally, since $[G : HK] = 2$, $HK$ is a normal subgroup of $G$.

A2. Show that $(\mathbb{Z}/7\mathbb{Z})[X]/(X^3 - 3)$ is isomorphic to $(\mathbb{Z}/7\mathbb{Z})[X]/(X^3 - 5)$.

Solution. Let $L = (\mathbb{Z}/7\mathbb{Z})[X]/(X^3 - 3)$ and $K = (\mathbb{Z}/7\mathbb{Z})[X]/(X^3 - 5)$. To show that $L \cong K$ we will use the fact that finite fields having the same number of elements are isomorphic.

Since $x^3 - 3$ and $x^3 - 5$ do not have roots in $\mathbb{Z}/7\mathbb{Z}$, $x^3 - 3$ and $x^3 - 5$ are irreducible in $(\mathbb{Z}/7\mathbb{Z})[X]$. Thus $L$ and $K$ are fields. Moreover, $|L| = 7^3 = |K|.$

A3. Exhibit a subgroup of the symmetric group $S_8$ isomorphic to the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Do this explicitly by giving the permutation corresponding to each of the eight elements of $Q$.

Solution. The action of $Q$ on itself by left multiplication defines an injective homomorphism $\varphi$ from $Q$ to $S_Q$. The bijection $f : Q \rightarrow \{1, 2, 3, 4, 5, 6, 7, 8\}$ with $f(1) = 1$, $f(-1) = 2$, $f(i) = 3$, $f(-i) = 4$, $f(j) = 5$, $f(-j) = 6$, $f(k) = 7$ and $f(-k) = 8$ defines an isomorphism $\psi : S_Q \rightarrow S_8$. Then $\psi \circ \varphi(Q)$ gives a required subgroup with

- $\psi \circ \varphi(1) = (1)$
- $\psi \circ \varphi(-1) = (12)(34)(56)(78)$
- $\psi \circ \varphi(i) = (1324)(5768)$
- $\psi \circ \varphi(-i) = (1423)(5867)$
- $\psi \circ \varphi(j) = (1526)(3847)$
- $\psi \circ \varphi(-j) = (1625)(3748)$
- $\psi \circ \varphi(k) = (1728)(3546)$
- $\psi \circ \varphi(-k) = (1827)(3645)$
Complex Analysis

C1. Find the Taylor or Laurent Series for

\[ f(z) = \frac{1}{z^2 - 4} \]

in the indicated domains

(a) \(0 < |z| < 2\), (b) \(|z| > 2\), (c) \(0 < |z - 2| < 4\)

Solution.

(a) For \(0 < |z| < 2\), we rewrite the function \(f(z)\) in terms of \(z/2\) and use that \(|z| < 2\) implies \(|z/2| < 1\) to form the power series expansion:

\[
\begin{align*}
    f(z) &= \frac{1}{z^2 - 4} = \frac{-1/4}{1 - (z/2)^2} = \frac{-1}{4} \cdot \left\{ 1 + (z/2)^2 + (z/2)^4 + (z/2)^6 + \cdots \right\} \\
    &= - \left\{ \frac{1}{2^2} + \frac{z^2}{2^4} + \frac{z^4}{2^6} + \frac{z^6}{2^8} + \cdots \right\} \\
    &= - \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{2n+2}}
\end{align*}
\]

(b) For \(|z| > 2\), we rewrite the function \(f(z)\) in terms of \(2/z\) and use that \(|z| > 2\) implies \(|2/z| < 1\) to form the power series expansion:

\[
\begin{align*}
    f(z) &= \frac{1}{z^2 - 4} = \frac{1}{z^2(1 - (2/z)^2)} \\
    &= \frac{1}{z^2} \cdot \left\{ 1 + (2/z)^2 + (2/z)^4 + (2/z)^6 + \cdots \right\} \\
    &= \frac{1}{z^2} + \frac{2^2}{z^4} + \frac{2^4}{z^6} + \frac{2^6}{z^8} + \cdots \\
    &= \sum_{n=0}^{\infty} \frac{2^{2n}}{z^{2n+2}}
\end{align*}
\]

(c) For \(0 < |z - 2| < 4\), we find the Laurent series for the function \(f(z)\) about \(z = 2\), and use that \(0 < |z - 2| < 4\) to expand it in a power series in terms of \((z - 2)\):

\[
\text{(c) For } 0 < |z - 2| < 4, \text{ we find the Laurent series for the function } f(z) \text{ about } z = 2, \text{ and use that } 0 < |z - 2| < 4 \text{ to expand it in a power series in terms of } (z - 2):}
\]
\[ f(z) = \frac{1}{z^2 - 4} = \frac{1}{(z + 2)(z - 2)} = \frac{1}{(z - 2)} \cdot \frac{1}{4 + (z - 2)} \]
\[ = \frac{1}{4(z - 2)} \cdot \left\{ 1 + \frac{(z - 2)}{4} + \frac{(z - 2)^2}{4^2} + \frac{(z - 2)^3}{4^3} + \ldots \right\} \]
\[ = \frac{(z - 2)^{-1}}{4} + \frac{1}{4^2} + \frac{(z - 2)^2}{4^3} + \frac{(z - 2)^3}{4^4} + \ldots \]
\[ = \sum_{n=-1}^{\infty} \frac{(z - 2)^n}{4^{n+2}} \]

C2. Use the Cauchy Residue Theorem to calculate the Fourier transform

\[ F(k) = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1 + x^2} \, dx \]

of the function \( f(x) = \frac{1}{1 + x^2} \), where \( k > 0 \) is real.

Solution. The function \( f(z) = \frac{1}{1 + z^2} \) has poles at \( z = \pm i \), so we can use the Cauchy Residue Theorem to evaluate the indefinite integral. Note that for \( z = x + iy \) where \( y \geq 0 \), the function \( e^{-ikz} \) is unbounded, so we must choose the contour in the half plane \( y \leq 0 \), so about the pole \( z = -i \). Also note that then the integral traverses the boundary curve \( y = 0 \) in a clockwise direction, which introduces a factor of \(-1\) in front of the residue. Thus, we obtain

\[ \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1 + x^2} \, dx = -2\pi i \cdot \text{Res}_{z=-i} \frac{e^{-ikz}}{z+i} \]
\[ = -2\pi i \cdot \frac{e^{-k}}{(-i)-i} \]
\[ = \pi e^{-k} \]

C3. Let \( f \) be a holomorphic function defined on all of \( \mathbb{C} \). Suppose that \( f \) satisfies the identity

\[ f(z) = f(z + 1) = f(z + i) \quad \text{for all } z \in \mathbb{C}. \]

Prove that \( f \) must be constant.

Solution. The identity \( f(z) = f(z + 1) \) implies the identity \( f(z) = f(z + n) \) for all integers \( n \) by induction, and likewise \( f(z) = f(z + i) \) implies \( f(z) = f(z + mi) \) for all integers \( m \).

Since \( |f(z)| \) is continuous, it is bounded on the closed disk \( \{ z \mid |z| \leq 2 \} \) by a constant \( M \).
Finally, observe that for any \( z \in \mathbb{C} \), write \( z = x + iy \), then there exists an integer \( n \) such that \( 0 \leq x - n < 1 \) and an integer \( m \) such that \( 0 \leq y - m < 1 \). Then \( (x - n)^2 + (y - m)^2 \leq 2 \) and so \( (x - n) + i(y - m) \in \{ z \mid |z| \leq 2 \} \).

Then \( |f(z)| = |f((x - n) + i(y - m))| \leq M \), so that \( |f(z)| \leq M \) for all \( z \in \mathbb{C} \). This implies that \( f(z) \) is a bounded holomorphic function on \( \mathbb{C} \), hence must be constant by Louiville’s Theorem.

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**PURE MATH OPTION, Spring 2012**

**Number Theory**

**N1.** Determine all primes \( p \) such that the equation

\[
x^2 + 3x + 6 \equiv 0 \pmod{p}
\]
is solvable modulo \( p \).

**N2.** Let \( \phi(n) = \# \{ 1 \leq k \leq n : (k, n) = 1 \} \) be the Euler function.

(1) Show that if \( n \) has at least three distinct prime factors, then \( \phi(n) \geq 8 \).
(2) Find \( n \) such that \( \phi(n) = 4 \).
(3) Find \( n \) such that \( \phi(n) = 6 \).
(4) Find \( n \) such that \( \phi(n) = \frac{n}{3} \).

**Solution.** Let

\[
n = \prod_{i=1}^{k} p_i^{\alpha_i}
\]
be the prime factorization of \( n \). Then

(1)

\[
\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1).
\]

1. By hypothesis, \( k \geq 3 \). Then, by (1),

\[
\phi(n) \geq (2 - 1)(3 - 1)(5 - 1) = 8.
\]

2. Using part 1 and the hypothesis \( \phi(n) = 4 \), we deduce that \( k \leq 2 \).

If \( n = p^a \) (i.e. \( k = 1 \)), then

\[
p^a - 1(p - 1) = 4,
\]

which implies \( n \in \{5, 8\} \).

If \( n = p_1^{\alpha_1}p_2^{\alpha_2} \) (i.e. \( k = 2 \)), then

\[
p_1^{\alpha_1 - 1}p_2^{\alpha_2 - 1}(p_1 - 1)(p_2 - 1) = 4,
\]

which implies \( n \in \{10, 12\} \).

3. Reasoning as in part 2, we obtain that \( n \in \{7, 9, 14, 18\} \).

4. Let us assume that \( k \geq 3 \). Then, using the hypothesis and (1), we deduce that

\[
3(p_1 - 1)(p_2 - 1) \ldots (p_k - 1) = p_1p_2 \ldots p_k.
\]
By the uniqueness of the prime factorization, we deduce that $p_1 = 3$, hence
\[2(p_2 - 1) \ldots (p_k - 1) = p_2 \ldots p_k.\]
Again by the uniqueness of the prime factorization, we deduce that $p_2 = 2$, hence
\[(p_3 - 1) \ldots (p_k - 1) = p_3 \ldots p_k.\]
Invoking one more time the uniqueness of the prime factorization, we are led to a contradiction.

The above shows that $k \leq 2$ and
\[n \in \{2^{\alpha_1}3^{\alpha_2} : \alpha_1, \alpha_2 \in \mathbb{N}\}.\]

**N3.**

1. Use Fermat’s Little Theorem to determine all the primes $p$ for which
   \[3p^2 + 11p^2 \equiv 0 \pmod{p^2}\]
2. Use Fermat’s Little Theorem to determine all the primes $p$ for which
   \[4^{2p^2} + 3^{2p^2} \equiv 0 \pmod{p^2}\]

**Solution.**

1. By Fermat’s Little Theorem,
   \[3p^2 = (3^p)^p \equiv 3^p \equiv 3 \pmod{p}\]
   and
   \[11p^2 \equiv 11 \pmod{p}.\]
   This gives
   \[0 \equiv 3p^2 + 11p^2 \equiv 3 + 11 = 14 \pmod{p},\]
   which implies $p \in \{2, 7\}$.
   Conversely, only $p = 7$ satisfies the hypothesis. Thus $p = 7$ is the only solution.
2. Similarly to part 1, we have
   \[4^{2p^2} + 3^{2p^2} \equiv 16 + 9 = 25 \pmod{p}.\]
   This gives $p = 5$. Conversely, $p = 5$ satisfies the hypothesis, so this is the only solution.

**Real Analysis**

**R1.** Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by
\[a_1 = 1, \quad a_{n+1} = \frac{3a_n - 1}{a_n}.\]
Prove that (i) the sequence is monotonic, (ii) bounded. (iii) Find the limit.

**Solution.** First, we examine what the limit should be. Solving $L = \frac{3L-1}{L}$, we get $L^2 - 3L + 1 = 0$ or $L = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$, two positive roots, one of which will be the limit. By induction on $n$, we
show that \( a_{n+1} > a_n \) and that \( a_n \) is always between the two roots, which will give (i) and (ii). One can verify it directly for \( n = 1 \) to start the induction. So we assume we know the result for \( n \) and wish to show it for \( n + 1 \). Note that
\[
a_n - a_{n+1} = a_n - \frac{3a_n - 1}{a_n} = \frac{a_n^2 - 3a_n + 1}{a_n}
\]
If \( a_n \) is between the two roots (so in particular \( a_n > 0 \)), then \( \frac{a_n^2 - 3a_n + 1}{a_n} < 0 \) and we see \( a_n < a_{n+1} \). Furthermore, we have
\[
a_{n+1}^2 - 3a_n + 1 = (3 - \frac{1}{a_n})^2 - 3(3 - \frac{1}{a_n}) + 1 = 9 - \frac{6}{a_n} + \frac{1}{a_n^2} - 9 + \frac{3}{a_n} + 1
\]
\[
= \frac{1}{a_n^2} - \frac{3}{a_n} + 1
\]
\[
= \frac{a_n^2 - 3a_n + 1}{a_n^2}
\]
This quantity is negative since by induction hypothesis \( a_n \) is between the two roots, so \( a_{n+1} \) is between these two roots as well and we have shown the induction step.

Lastly, to find the limit, one takes the limit in both sides of \( a_{n+1} = \frac{3a_n - 1}{a_n} \), getting \( L^2 - 3L + 1 = 0 \). We get the larger root \( \frac{3}{2} + \frac{\sqrt{5}}{2} \) since the sequence is increasing and stays between the two roots.

R2. Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be two bounded sequences. Prove that
\[
\limsup_{n \to \infty} (a_n - b_n) \leq \limsup_{n \to \infty} (a_n) - \liminf_{n \to \infty} (b_n).
\]
Give an example where the inequality is strict.

**Solution.** \( \limsup_{n \to \infty} (a_n - b_n) = \lim_{k \to \infty} \sup_{n \geq k} (a_n + (-b_n)) \). The supremum is subadditive, so this is at most \( \lim_{k \to \infty} (\sup_{n \geq k} a_n + \sup_{n \geq k} (-b_n)) \). By additivity of limits (note we use boundedness here), this is \( \lim_{k \to \infty} \sup_{n \geq k} a_n + \lim_{k \to \infty} \sup_{n \geq k} (-b_n) = \lim_{k \to \infty} \sup_{n \geq k} a_n + \lim_{k \to \infty} (-\inf_{n \geq k} b_n) = \limsup_{n \to \infty} a_n - \liminf_{n \to \infty} b_n \). The examples 1, 0, 1, 0, ... and 0, -1, 0, -1, 0, ... show the inequality may be strict.

R3. Show that the series
\[
\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}x\right)
\]
converges to a differentiable function on \(( -\infty, \infty )\) and that it can be differentiated term by term.

**Solution.** It suffices to show the result on any interval \(( -R, R )\). Using that \( |\sin(y)| \leq |y| \) for all \( y \), we have that \( |\sin(\frac{1}{n^2}x)| \leq \frac{1}{n^2}R \) on the interval \(( -R, R )\). The derivative of \( \sin(\frac{1}{n^2}x) \) is \( \frac{1}{n^2}\cos(\frac{1}{n^2}x) \), and since \( |\cos(y)| \leq 1 \) for all \( y \) we have \( \left| \frac{d}{dx} \sin(\frac{1}{n^2}x) \right| \leq \frac{1}{n^2} \) on \(( -R, R )\). Because \( \sum_n \frac{1}{n^2}R \) and \( \sum_n \frac{1}{n^2} \) are both finite, the sum of the terms converges uniformly on \(( -R, R )\), as does the sum of their derivatives. As a result, the series converges to a differentiable function on \(( -R, R )\) which may be differentiated term by term.
L1. A 2-coloured graph \((V, E, C_1, C_2)\) consists of a set \(V\) of vertices, a binary edge relation \(E\) on \(V\) that is irreflexive and symmetric, and a partition \(V = C_1 \cup C_2\) into two disjoint sets such that no two vertices in \(C_1\) are \(E\)-related and no two vertices in \(C_2\) are \(E\)-related.

(a) Let \(L = \{E, C_1, C_2\}\) be the language consisting of a binary relation symbol \(E\) and unary relation symbols \(C_1\) and \(C_2\). Write an \(L\)-sentence \(\sigma\) whose models are exactly the 2-coloured graphs.

(b) Let \(\phi(x, y)\) be the \(L\)-formula
\[Exy \lor \exists z (Exz \& Ez \& Ey) \lor \exists v \exists w (Exz \& Ez \& Ew \& Ewy)\]
and let \(\tau\) be the \(L\)-sentence
\[\forall x \forall y (x \neq y \rightarrow \phi(x, y)).\]
Construct a model of \(\{\sigma, \tau\}\) with exactly 5 elements. Clearly state the domain and the interpretations of all the relations in your model.

(c) Let \(\theta\) be the \(L\)-sentence
\[\forall x \exists y \exists z (y \neq z \& Exy \& Ez \& \forall v (v \neq y \& v \neq z \rightarrow \neg Exv)).\]
Decide if \(\{\sigma, \tau, \theta\}\) has a model with exactly 5 elements by either constructing such a model or arguing that it cannot exist.

Solution.
(a) \[\sigma = \forall x (C_1 x \lor C_2 x) \& \neg \exists x (C_1 x \& C_2 x) \& \forall x \forall y (Exy \rightarrow Ey)\]
\[\& \forall x \forall y (Exy \rightarrow ((C_1 x \& C_2 y) \lor (C_1 y \& C_2 x))).\]

(b) Let \(\mathcal{M}\) be the structure with underlying set \(\{0, 1, 2, 3, 4\}\). Let \(E^\mathcal{M} := \{(i, i+1) : 0 \leq i \leq 3\}\). Let \(C_1^\mathcal{M} := \{0, 2, 4\}\) and \(C_2^\mathcal{M} := \{1, 3\}\).

(c) Suppose, for contradiction, there is a model \(\mathcal{M}\) of the set \(\{\sigma, \tau, \theta\}\), where the underlying set of \(\mathcal{M}\) contains exactly 5 elements. Since \(\mathcal{M} \models \sigma\), \(\mathcal{M}\) is a 2-coloured graph. Since \(\mathcal{M} \models \tau\) and \(\mathcal{M}\) is size-5, \(\mathcal{M}\) is a connected graph. Because \(\mathcal{M} \models \theta\), all edges in \(\mathcal{M}\) have degree exactly two. A connected graph with all edges of degree 2 must be a cycle. In other words, we know that for some labelling of the points in \(\mathcal{M}\) as \(\{0, 1, 2, 3, 4\}\), \(E^\mathcal{M} = \{(i, (i+1) \mod 5) : 0 \leq i \leq 4\}\). Without loss of generality, 0 is in \(C_1^\mathcal{M}\). Then, by a short induction, the fact that \(\mathcal{M}\) is 2-coloured yields that 4 is also in \(C_1^\mathcal{M}\). However, \((0, 4)\) is an edge, contradicting that \(\mathcal{M}\) is a 2-coloured graph. Thus, no such model exists.

L2. Let \(L\) be a first order language. Prove or disprove each of the following two statements.

(a) Every \(L\)-formula is logically equivalent to an \(L\)-sentence.

(b) Every \(L\)-formula is satisfiable in a finite model.
(b) Every $L$-sentence is logically equivalent to an $L$-formula that is not a sentence.

Solution.

(a) False. Consider the $L$-formula $\psi := (x \equiv y)$. Suppose, for contradiction, that $\theta$ is an $L$-sentence that is logically equivalent to $\psi$. By completeness,
\[ \emptyset \vdash ((x \equiv y) \leftrightarrow \theta) \]
By Eqn. (2) and laws of deductions,
\[ \emptyset \vdash \forall x \forall y ((x \equiv y) \leftrightarrow \theta) \]
Now fix an arbitrary $L$-structure $\mathcal{M}$. By Eqn. (3),
\[ \mathcal{M} \models \forall x \forall y ((x \equiv y) \leftrightarrow \theta) \]
Choose arbitrary points $a, b$ in $\mathcal{M}$. We have two equations (we let “$a$” refer to a constant denoting $a$ when it appears in a formula):
\[ \mathcal{M} \models ((a \equiv b) \leftrightarrow \theta) \]
\[ \mathcal{M} \models ((a \equiv a) \leftrightarrow \theta) \]
Since $\theta$ has no free variables,
\[ \mathcal{M} \models ((a \equiv b) \leftrightarrow (a \equiv a)) \]
So we have shown that for any $L$-structure $\mathcal{M}$, for any points $a, b$ in $\mathcal{M}$, we must have that $a = b$, which is a contradiction.

(b) True. Let $\varphi$ be any $L$-sentence. Then the following is an $L$-formula which is not an $L$-sentence: $\psi := \varphi \& (x \equiv x)$. Any interpretation satisfying $\varphi$ must satisfy $\psi$. Any interpretation satisfying $\psi$ trivially satisfies $\varphi$. Thus $\varphi$ and $\psi$ are logically equivalent.

L3. Let $L = \{c, f, <\}$, where $c$ is a constant symbol, $f$ is a unary function symbol and $<$ is a binary relation symbol.

(a) State the compactness theorem for first-order logic and state what it means for an $L$-structure $\mathcal{N}$ to be a substructure of an $L$-structure $\mathcal{M}$.

Define $\mathcal{N}$ to be the $L$-structure with universe $\mathbb{N} = \{0, 1, 2 \ldots\}$, where $c$ is interpreted as 0, $f$ is interpreted as the successor function, and $<$ is interpreted as the usual linear ordering on $\mathbb{N}$. Let $T$ be the set of all $L$-sentences true in $\mathcal{N}$.

(b) Show that there is an $L$-structure $\mathcal{M}$ satisfying $T$ and containing $\mathcal{N}$ as a substructure such that there is an element $a$ of $\mathcal{M}$ with
\[ f^\mathcal{M}(f^\mathcal{M}(\cdots f^\mathcal{M}(c^\mathcal{M})\cdots)) <^\mathcal{M} a \]
for all $k \geq 0$. 8
Solution. (a) Compactness theorem: Let $\Gamma$ be any set of $L$-formulas. If every finite subset $\Gamma_0$ of $\Gamma$ is satisfiable, then $\Gamma$ is satisfiable.

Fix $L$-structures $\mathcal{M}$ and $\mathcal{N}$ with underlying sets $M$ and $N$, respectively. $\mathcal{N}$ is a substructure of $\mathcal{M}$ if the following conditions hold:

1. $N \subseteq M$
2. $c^N = c^M$
3. $a <^N b \iff a <^M b$, for all $a, b \in N$
4. $f^N(a) = f^M(a)$, for all $a \in N$

(b) We wish to satisfy the following set $\Gamma$ of $L$-formulas

$\{ T \cup \{ f(f(\cdots f(c)\cdots)) : 0 \leq k, k \in \mathbb{N} \} \}$

However, any finite subset of $\Gamma$ is contained in a set $\Gamma_0$ of the form

$\{ T \cup \{ f(f(\cdots f(c)\cdots)) : 0 \leq k \leq m \} \}$

for some integer, $m$. But $\Gamma_0$ is satisfiable in $\mathcal{N}$ by choosing any element $l$ in $\mathbb{N}$ to interpret $x$, provided that $m <^N l$.

Thus, $\Gamma$ is satisfied by some structure $\mathcal{M}'$. The induced substructure $\mathcal{N}'$ of $\mathcal{M}'$ with domain

$\{ f^{\mathcal{M}'}(f^{\mathcal{M}'}(\cdots f^{\mathcal{M}'}(c^{\mathcal{M}'})) : k \geq 0 \}$

is isomorphic to $\mathcal{N}'$ by the fact that both $\mathcal{M}, \mathcal{N} \models T$. Thus we may construct an isomorphic copy $\mathcal{M}$ of $\mathcal{M}'$, that contains $\mathcal{N}$ as a substructure. $\mathcal{M}$ is the desired structure.

Topology

T1. Recall that a space $X$ is said to be limit point compact if every infinite subset of $X$ has a limit point. Prove that every compact space is limit point compact.

Solution. Suppose $X$ is a compact space, and let $A$ be a subset of $X$ without a limit point. We will show $A$ is finite. Since $A$ has no limit points, for each $x \in X - A$ there is an open set $U_x$ containing $x$ not intersecting $A$, and for each $x \in A$ there is an open set $U_x$ containing $x$ such that $U_x \cap A = \{x\}$. These $U_x$ form an open cover of $X$, which has a finite subcover $U_{x_1}, \ldots, U_{x_n}$ by compactness of $X$. Note $U_{x_i} \cap A = \{x_i\}$ or $\emptyset$, depending on whether or not $x_i \in A$. Thus $A = \bigcup_{i=1}^{n}(A \cap U_{x_i}) \subset \{x_1, \ldots, x_n\}$, a finite set. So $A$ is finite.

T2.

1. Prove that the unit interval $[0, 1]$ is not homeomorphic to the unit square $S = [0, 1] \times [0, 1]$.
2. Prove that a continuous surjective map from $[0, 1]$ to $S$ (i.e. a “space-filling curve”) cannot be injective.

Solution.
1) Suppose \( f : [0, 1] \to S \) were a homeomorphism. Then we could find an \( x \in (0, 1) \) such that \( f(x) \) is in the interior of \( S \). This would imply that \([0, 1] - \{x\}\) were homeomorphic to \( S - \{f(x)\} \). But \([0, 1] - \{x\}\) is not connected, while \( S - \{f(x)\}\) is connected, a contradiction.

2) Suppose now \( f : [0, 1] \to S \) were a continuous surjection. Suppose \( f \) were injective. If \( F \subset [0, 1] \) is closed then \( F \) is compact, so \( f(F) \) is compact and thus closed. Hence \( f \) takes closed sets to closed sets, implying \( f^{-1} \) is continuous. Thus \( f \) is a homeomorphism, which by part a) can’t happen.

T3. Let \( X \) and \( Y \) be topological spaces and \( p : X \to Y \) a continuous surjective function such that \( p(U) \) is open for every open set \( U \subset X \). Suppose that \( Y \) is connected and that \( p^{-1}(\{y\}) \) is connected for every \( y \in Y \). Prove that \( X \) is connected.

Solution. Suppose \( X \) were not connected, we will obtain a contradiction. Then \( X = U \cup V \), where \( U \) and \( V \) are both open and nonempty. Note that \( p(U) \) and \( p(V) \) are nonempty open sets in \( Y \), and by surjectivity \( p(U) \cup p(V) = Y \). We will show that \( p(U) \) and \( p(V) \) are disjoint, contradicting connectivity of \( Y \). For suppose \( y \) were in \( p(U) \cap p(V) \). Then \( p^{-1}(y) \) intersects both \( U \) and \( V \), and thus \( p^{-1}(y) = (p^{-1}(y) \cap U) \cup (p^{-1}(y) \cap V) \) expresses \( p^{-1}(y) \) as a union of two disjoint relatively open sets, contradicting connectedness of \( p^{-1}(y) \). Hence there cannot be any \( y \in p(U) \cap p(V) \) and we are done.