Full points can be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth 20 points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS. When you have finished, insert all your answer sheets into the envelope provided, then seal it.

Any student whose answers need clarification may be required to submit to an oral examination.

Algebra

A1. Classify groups of order 21 up to isomorphism.

Solution. The 7-Sylow subgroup is $\mathbb{Z}/7\mathbb{Z}$; the 3-Sylow subgroup is $\mathbb{Z}/3\mathbb{Z}$. By Sylow’s theorem, the 7-Sylow subgroup is normal. Hence, the group is a semi-direct product of its 3 and 7-Sylow subgroups. Since $\text{Aut}(\mathbb{Z}/7\mathbb{Z}) = \mathbb{Z}/6\mathbb{Z}$ has a unique subgroup of order 3, there are up to isomorphism exactly two groups of order 21: $\mathbb{Z}/21\mathbb{Z}$ and the group with presentation $\langle x, y | x^3 = y^7 = 1, xyx^{-1} = y^2 \rangle$.

A2. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial with rational coefficients that has exactly one real root. Calculate the Galois group of the splitting field of $f(x)$ over the field of rational numbers $\mathbb{Q}$.

Solution. Since $f$ has one real root, complex conjugation gives a nontrivial automorphism of the splitting field of $f$. Since $f$ is irreducible of degree 3, its Galois group is a transitive subgroup of $S_3$, so it is either $S_3$ or $A_3$. Since it contains an element of order 2, it must be $S_3$.

A3. Let $G$ be a simple group. Let $f : G \to H$ be a non-constant group homomorphism. Prove that $f$ is injective.

Solution. Since $G$ is simple, the kernel of a homomorphism (which is a normal subgroup) is either trivial or all of $G$. Since $f$ is a non-constant homomorphism, its kernel must be trivial so $f$ is injective.
Complex Analysis

C1. Let \( f(x) := \frac{x^2 \cos x}{(x^2+1)(x^2+9)} \) for \( x \in \mathbb{R} \).

(1) Show that \( \int_0^\infty |f(x)| \, dx < \infty \).

(2) Calculate the value of \( \int_0^\infty f(x) \, dx \) using residues. (Justify briefly the steps you are using.)

Solution. Since \( |x^2 \cos x| \leq x^2 < x^2 + 9 \), one has that \( |f(x)| < \frac{1}{x^2+1} < \frac{1}{x^2} \). So by the p-test \( \int_1^\infty |f(x)| < \infty \). Since \( f(x) \) is bounded on \([0, 1]\) one also has \( \int_0^1 |f(x)| < \infty \).

For the second part, \( f(x) = \text{Re} \ g(x) \), where \( g(x) = \frac{x^2 e^{ix}}{(x^2+1)(x^2+9)} \). Since the degree of the denominator is at least one plus the degree of the numerator, \( \int_1^\infty g(x) \, dx \) can be evaluated via the residue theorem. The residue at \( z = i \) is \( \frac{(3i)^2 e^{3i}}{(3i)^3 + 1} = \frac{9e^{-3}}{48i} \). Hence

\[
\int_1^\infty g(x) \, dx = 2\pi i \left( -\frac{e^{-1}}{16i} + \frac{3e^{-3}}{16i} \right)
\]

\[
= \pi \left( -\frac{e^{-1}}{8} + \frac{3e^{-3}}{8} \right)
\]

The original integral is one-half of the real part of this, or

\[
= \pi \left( -\frac{e^{-1}}{16} + \frac{3e^{-3}}{16} \right)
\]

C2. Consider the function \( g(z) = \frac{1}{z \cos z} \).

(1) Show that \( g(z) \) has a Laurent series of the following form on \( 0 < |z| < r \) for some \( r > 0 \):

\[
g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n.
\]

(2) Find \( b_0, b_1, b_2, b_3, b_4 \)

(3) What is the maximum value of \( r \) that the above Laurent series converges?

Solution. \( \frac{1}{\cos(z)} \) is analytic on \( |z| < \frac{\pi}{2} \) and takes the value 1 at \( z = 0 \), so on \( |z| < \frac{\pi}{2} \) one has a Taylor expansion of the form

\[
\frac{1}{\cos(z)} = 1 + \sum_{n=0}^{\infty} b_n z^{n+1}
\]

Dividing by \( z \) gives part (1). The Laurent series will converge up to the radius of the location of the first singularity of \( \frac{1}{z \cos z} \), so the supremum of such \( r \) is exactly \( \frac{\pi}{2} \), giving part (3).
Since \(\cos(z)\) is even, \(\frac{1}{z\cos z}\) is odd, and therefore \(b_0 = b_2 = b_4 = 0\). To find \(b_1\) and \(b_3\), one can use the fact that \(\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)\), and use long division to find the Taylor series coefficients for \(\frac{1}{z\cos z}\). The result is \(b_2 = \frac{1}{2}\) and \(b_4 = \frac{5}{24}\). This gives (2).

**C3.** How many solutions of 
\[e^z - 3z^4 = 0\]
are in the disk \(|z| < 1|)? Explain.

**Solution.** Apply Rouche’s Theorem with \(f(z) = -3z^4\) and \(g(z) = e^z\). On the circle \(|z| = 1|,
|g(z)| = |e^{x+iy}| = e^x < e^1 < 3 = |f(z)|, so |g(z)| < |f(z)| on this circle. Hence \(f(z)\) and \(f(z) - g(z)\) have the same number of zeroes inside the disk (counting multiplicities), namely 4.

**Number Theory**

**N1.**
1. State and prove the Quotient-Remainder Theorem.
2. Define the greatest common divisor of two integers.
3. Use the Euclidean algorithm to calculate \(\gcd(561; 105)\). Show all work.
4. Using the solution to part 3, find an integral solution to the diophantine equation 
\[561x + 105y = 3\]

**N2.**
1. Define the Euler function as a counting function.
2. Prove the product formula for the Euler function.
3. Using part 2, prove that the Euler function is multiplicative.
4. Recalling that the divisor sum of a multiplicative function is a multiplicative function, prove that for any integer \(n \geq 1\), we have \(\sum_{d|n} \phi(d) = n\).
5. Find all integers \(n \geq 1\) such that \(\phi(n) = 6\).

**N3.**
1. Let \(a \in \mathbb{Z}\) and \(p\) an odd prime. Define the Legendre symbol \(\left(\frac{a}{p}\right)\).
2. State the Quadratic Reciprocity Law.
3. Calculate \(\left(\frac{30}{23}\right)\).
4. Let \(p \geq 5\) be a prime. Use Fermat’s Little Theorem to calculate \((3^p + 1) \mod p\) and prove that \(\left(\frac{3^p+1}{p}\right) = 1\).
5. Prove that, for any natural numbers \(n, m\), we have \(167 \nmid 2^n + 3^m\).
In each of the True or False problems: If true, give a proof of the statement. If false, give a counterexample or a proof that the statement is false.

**R1.** (a) True or False: Given any set $X$, there does not exist a surjective mapping from $X$ to $P(X)$ where $P(X)$ denotes the set of all subsets of $X$.

(b) Prove that $41/333 = .123123123\ldots$ by using the definition of limit. Note that you will have to explain what is the meaning of an expression like $123123123\ldots$ as a limit of a certain sequence. (You can use the fact that a monotone increasing sequence with an upper bound has a limit that is equal to the least upper bound of the sequence. If you use this result, you should explain exactly what the terms mean and how you are using it. And you should state the least upper bound axiom for the real numbers.)

**Solution.**

a) This is True. To see this, first note that if $X = \emptyset$ then $P(X) = \{\emptyset\}$ and there is no surjection from a set of size 0 to a set of size 1. Therefore, we can assume that $X$ is nonempty.

Now suppose that $f : X \to P(X)$ is any function. Find a subset $Y$ of $X$ by taking

$$Y = \{x \in X \mid x \not\in f(x)\}.$$  

We claim that $Y$ is not in the image of $f$. If it were, then there would be some $x_0 \in X$ so that $f(x_0) = Y$. We now consider whether or not $x_0 \in Y$. If $x_0 \in Y$ then we have $x_0 \in f(x_0)$ and by the criterion for membership in $Y$ we have $x_0 \not\in Y$. On the other hand, if $x_0 \not\in Y$ then $x_0 \not\in f(x_0)$ and $x_0$ meets the criterion for membership of $Y$, which means that $x_0 \in Y$. Either way, we have a contradiction, which means that there is no such $x_0$, and $f$ is not surjective.

b) The expression $0.123123123\ldots$ means

$$\lim_{n \to \infty} a_n$$

where $a_n$ is defined to be the decimal approximation to $n$ places of this number, namely, we have $a_1 = 0.1$, $a_2 = 0.12$, $a_3 = 0.123$ and so on. This sequence is monotone increasing (meaning that for all $n$ we have $a_{n+1} \geq a_n$) and bounded above by 0.2 (meaning that for all $n$ we have $a_n \leq 0.2$), and so has a limit in the real numbers and this limit $l$ is the least upper bound of the set $\{a_n\}$. This means that for all $n$ we have $a_n \leq l$ and if $y$ is any other number so that $a_n \leq y$ for all $n$ then $l \leq y$.

Now, consider $x = 0.123123123\ldots$ and note that $1000x = 123 + x$. Thus $999x = 123$ and $x = 123/999 = 41/333$. The steps in this derivation are justified since the limit of the series for $x$ exists.
R2. (a) True or False: The intersection of an arbitrary collection of open subsets of the real line is always an open set.

(b) Give a specific example of an empty intersection of an infinite collection of nested open intervals where each individual interval is non-empty.

(c) Recall that the Cantor set is constructed from an open interval $[0,1]$ by removing the open middle third of this interval, then removing the open middle thirds of the two remaining intervals and continuing in this fashion infinitely. One starts by deleting the open middle third $(1/3, 2/3)$ from the interval $[0, 1]$, leaving two line segments: $[0, 1/3] \cup [2/3, 1]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $[0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. This process is continued ad infinitum.

True or False: The Cantor set contains uncountably many irrational real numbers.

Solution.

a) This is False. Consider the sets $U_n = (-1/n, 1/n)$ for $n = 1, 2, 3, ....$ These sets are all open, but their intersection is $\{0\}$ which is not open.

b) For $n \geq 1$ an integer, define $U_n = (0, 1/n)$. Then each $U_n$ is a nonempty open interval and $U_{n+1} \subset U_n$, so the intervals are nested. We claim that the intersection of the $U_n$ is empty. For, suppose that $y \in \cap U_n$. Then, certainly $y > 0$, since every element of $U_1 = (0, 1)$ is positive. However, if $y$ is in each $U_n$, then for all natural numbers $n$ we have $y < 1/n$, which implies that $1/y > n$. However, the set of natural numbers is not bounded above, so this is absurd. Thus $\cap U_n$ is empty.

c) This is True. If we write real numbers using base 3, the elements of the Cantor set are exactly those between 0 and 1 whose base 3 representations contain only the digits 0 and 1, and no occurrences of 2. Since the base 3 expansion of a number which contains no 2’s is unique, there is a bijection between the elements of the Cantor set and the set of all sequences from $\{0, 1\}$. This is in bijection with $P(\mathbb{N})$, where $\mathbb{N}$ is the set of all natural numbers, and so the Cantor set is uncountable. Since there are only countably many rational numbers, the set of irrational numbers in the Cantor set must be uncountable.

R3.

(a) Show from the definition of continuity that $f(x) = x^2$ is continuous at every point in $\mathbb{R}$.

(b) Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = \sin(1/x)$ when $x \neq 0$ and $g(0) = 0$. The function $g$ is continuous at every point in $\mathbb{R}$ except $x = 0$. (You do not have to show this.) Explain why the definition of continuity fails for $g$ at $x = 0$.

(c) True or False: One cannot find a continuous function $f$ defined on the closed interval $[0, 1]$ such that $f(0) < 0$ and $f(1) > 0$ and $f(x) \neq 0$ for all $x \in [0, 1]$.

Solution.
a) Suppose that $c \in \mathbb{R}$ is a number, and let $\epsilon > 0$ be arbitrary. Choose $\delta = \min\{1, \frac{\epsilon}{2|c|+1}\}$. If $0 < |x-c| < \delta$ then we know that $x \in (c-\delta, c+\delta)$, which clearly implies that $|x+c| < 2|c|+1$. Therefore, for such an $x$ we have

$$|f(x) - f(c)| = |x^2 - c^2| = |x+c||x-c| < (2|c|+1)\delta \leq \epsilon.$$  

Since $\epsilon > 0$ was arbitrary, this implies that $f$ is continuous as $c$, as required.

b) If $g$ were continuous at 0 then we must have $\lim_{x \to 0} g(x) = g(0) = 0$.

However, let $\epsilon = \frac{1}{2}$ and let $\delta > 0$ be arbitrary. There is some natural number $n$ so that $\frac{1}{2n\pi+\pi/2} < \delta$. Choose such an $n$ and let $x_n = \frac{1}{2n\pi+\pi/2}$. Note that $g(x_n) = \sin(2n\pi + \pi/2) = 1$.

We therefore have $0 < |x_n - 0| < \delta$ but $|g(x_n) - g(0)| = 1 > \frac{1}{2}$.

This proves that we cannot have $\lim_{x \to 0} g(x) = g(0)$, and so $g$ is not continuous at 0.

c) This is True as the Intermediate Value Theorem rules out such functions.

Logic

L1. Let $T$ be the theory of $(\mathbb{N}, <)$. Show that there is a model $\mathcal{M}$ of $T$, such that there is some $b \in |\mathcal{M}|$ for which $\{a \in |\mathcal{M}| : a < b\}$ is infinite. (Hint: use Compactness)

L2. Suppose that $\{\alpha_1, \alpha_2, \ldots\}$ is a set of formulae in propositional logic such that for all valuations $v$, there is an $n$ for which $v(\alpha_n) = \top$. Prove that there is $m$ for which $\vdash \alpha_1 \lor \cdots \lor \alpha_m$.

Solution. Suppose no such $m$ exists. Then the set $X := \{-\alpha_1, -\alpha_2, \ldots\}$ is finitely satisfiable. By the Compactness Theorem, $X$ is satisfiable, that is, there is a valuation $v$ such that $\overline{v}(\alpha_i) = \bot$ for all $i$; this contradicts the assumption of the problem.

L3.

(a) Prove the Los-Vaught Test: Suppose that $\mathcal{L}$ is a countable language and $T$ is an $\mathcal{L}$-theory that has no finite models and is $\kappa$-categorical for some infinite cardinal $\kappa$. Prove that $T$ is complete. (Hint: Use the Löwenheim-Skolem Theorem.)

(b) Suppose that $\mathcal{L} = \{E\}$, where $E$ is a binary relation symbol. Let $T$ be the $\mathcal{L}$-theory that asserts that the interpretation of $E$ is an equivalence relation with precisely two equivalence classes, both of which are infinite. Prove that $T$ is complete.

Solution.

a) Let $M, N \models T$; we must show that $M \equiv N$. By assumption, $M$ and $N$ are both infinite. Thus, by the Löwenheim-Skolem Theorem, there are $M' \equiv M$ and $N' \equiv N$ such that $|M'| = |N'| = \kappa$. ($M'$ is either an elementary extension of elementary substructure of $M$, depending on whether or not $|M| \leq \kappa$ or $|M| \geq \kappa$; likewise for $N'$). By $\kappa$-categoricity,
\(M' \cong N'.\) Thus, we have
\[M \equiv M' \cong N' \equiv N,\]
whence \(M \equiv N.\)

b) Note that \(T\) has no finite models and is \(\aleph_0\)-categorical: if \((M, E^M), (N, E^N) \models T\) are countable, then \(E^M = E_1 \sqcup E_2\) and \(E^N = E_1' \sqcup E_2'\), where both \(E_1\) and \(E_2\) are countably infinite. Thus, by mapping \(E_1\) bijectively onto \(E_1'\) and \(E_2\) bijectively onto \(E_2'\), we have an isomorphism \((M, E^M) \rightarrow (N, E^N)\). So, by the Los-Vaught test, we have that \(T\) is complete.

**Topology**

**T1.** Let \(f, g : X \rightarrow Y\) continuous maps between topological spaces \(X, Y\), where \(Y\) is Hausdorff.

(a) Prove that the set \(\{x \in X : f(x) = g(x)\}\) is a closed subset of \(X\).
(b) Give an example to show the Hausdorff assumption is necessary.

**Solution.**

(a) Let \(x_0 \in X\) satisfy \(f(x_0) \neq g(x_0)\). Assuming \(Y\) is Hausdorff, there exist disjoint open sets \(U, V \subset Y\) with \(f(x_0) \in U, g(x_0) \in V\) and \(U \cap V = \emptyset\). The sets \(W_1 = f^{-1}(U)\) and \(W_2 = g^{-1}(V)\) are open because \(f, g\) are continuous, and \(x_0 \in W_1, W_2\). Then \(W = W_1 \cap W_2\) is an open set containing \(x_0\) where \(f(x) \neq g(x)\) for every \(x \in W\). This shows that \(\{x \in X : f(x) \neq g(x)\}\) is an open set, hence its complement \(\{x \in X : f(x) = g(x)\}\) is closed.

(b) Take \(Y = \{y_1, y_2\}\) to be a two point set with the trivial topology \(\{\emptyset, Y\}\). Let \(X\) be a topological space with non-discrete topology, then it has a subset \(X_1 \subset X\) which is not closed. Set \(f : X \rightarrow Y\) to be constant \(f(x) = y_1\), and \(g : X \rightarrow Y\) to be \(g(x) = y_1\) for \(x \in X_1\) and \(g(x) = y_2\) on \(X_2 = X \setminus X_1\). Then both functions are continuous, while \(\{x \in X : f(x) = g(x)\} = X_1\) is not closed.

**T2.** Let \(X = \{0, 1\}^\mathbb{N}\) be the infinite countable product of the discrete two point space \(\{0, 1\}\) with itself taken with the product topology. Prove that

(a) \(X\) does not contain isolated points.
(b) For any two points \(x \neq y\) in \(X\) there exists a clopen (closed and open) set \(A\) such that \(x \in A\) and \(y \not\in A\).

**Solution.**

(a) Given any point \(x = (x_i)_{i \in \mathbb{N}}\) and any open set \(U \ni x\), we need to find another point \(y \neq x\) in \(U\). By the definition of product topology, there exists a finite set of indices \(i_1, \ldots, i_n \in \mathbb{N}\) and non-empty open sets \(V_1, \ldots, V_n\) so that the set from the base of the
product topology

\[ V = \{ y \in X \mid y_{i_j} \in V_j, j = 1, \ldots, n \} \]
satisfies \( x \in V \subset U \). Choose index \( k \in \mathbb{N} \setminus \{i_1, \ldots, i_n\} \) and set \( y \in X \) to be \( y_i = x_i \) if \( i \neq k \) and \( y_k \neq x_k \). Then \( y \neq x \) and yet \( y \in V \subset U \).

(b) Since \( x \neq y \) there exists an index \( k \in \mathbb{N} \) where \( x_k \neq y_k \). Let \( A = \{ z \in X \mid z_k = x_k \} \). Then \( A \) is open and closed, \( x \in A \) and \( y \not\in A \).

\[ \textbf{T3.} \text{ Give an example of a continuous map } f : (0, 1) \to \mathbb{R}^2 \text{ that is continuous and 1-to-1 but is not homeomorphic to its image } f((0, 1)). \]

\[ \textbf{Solution.} \]

Imagine drawing digit 6 where the loop does not quite close... More precisely, we consider first the piecewise linear map of the closed interval \( p : [0, 1] \to \mathbb{R}^2 \), where

\[
\begin{align*}
p(0) &= (0, 2), & p(1/5) &= (0, 1), & p(2/5) &= (0, 0) \\
p(3/5) &= (1, 0), & p(4/5) &= (1, 1), & p(1) &= (0, 1)
\end{align*}
\]
and \( p \) is linear between these points: \( p_i(ta + (1-t)b) = tp_i(a) + (1-t)p_i(b) \) for \( i = 1, 2 \), where \( p_1 : [0, 1] \to \mathbb{R} \) and \( p_2 : [0, 1] \to \mathbb{R} \) are coordinate functions.

The function \( p : [0, 1] \to \mathbb{R}^2 \) is continuous, because \( p_1 \) and \( p_2 \) are continuous, being linear on subsets \([0, 1/5], \ldots, [4/5, 1]\) and agreeing on their intersections.

Therefore, the restriction of \( p \) to the subset \((0, 1) \to \mathbb{R}^2 \) is also continuous. It is also clear from the definition, that \( p \) is one-to-one on the open interval \((0, 1)\).

Yet the image \( p((0, 1)) \) is not homeomorphic to the interval \((0, 1)\). Indeed, removing \textit{any} point \( t \) from \((0, 1)\) we get a disconnected set \((0, t) \cup (t, 1)\). At the same time removing \((1, 1) = p(4/5) \) (or any \( p(t) \) with \( 1/5 < t < 1 \)) leaves \( p((0, 1)) \) connected.