The written Master’s Examination
Option Probability and Statistics, Option I
FALL 2003

Full points may be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth the same number of points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS. When you have finished, insert all your answer sheets into the envelope provided, then seal and print your name on it.

Any student whose answers need clarification may be required to submit to an oral examination.
Option Probability and Statistics, Option I – FALL 2003

1. (Math 310)
For any matrix A (may be rectangular) show that \( \text{rank}(A^T A) = \text{rank}(A) \).

2. (Math 413)
For \( x \in (-1,1) \) define \( f(x) = \begin{cases} (1+x)^\frac{1}{2}, & x \neq 0, \\ e, & x = 0. \end{cases} \)

(a) Is this function continuous at zero? Justify your answer.
(b) Is this function differentiable at zero? If so find its derivative.

3. (Stat 401)
Let \((X, Y)\) be uniformly distributed over the semicircle \( \{ (x, y) : x^2 + y^2 \leq 1, y \geq 0 \} \).

(a) Find the conditional expectation \( E \left( 1 + Y^2 \mid X = x \right) \).
(b) Find the probability density function of \( Z = X^2 + Y^2 \).

4. (Stat 411)
Let \( X_1, X_2, \ldots, X_n \) be a random sample from a Poisson distribution with parameter \( \theta \), where \( \theta > 0 \).
Suppose we are interested in the function \( h(\theta) = P(X = 0) = e^{-\theta} \).

(1) Find the maximum likelihood estimator for \( h(\theta) \).
(2) Find the minimum variance unbiased estimator for \( h(\theta) \).
[You have to justify your answers. If you are using a theorem, you have to mention it.]

5. (Stat 461)
The life of brand A bulbs is exponential with mean 1000 hrs. The life of brand B bulbs is also exponential with mean 800 hours. Assume that the two brands are independent. A dark room is lighted with both bulbs simultaneously. Find the average number of hours until the room is turning dark again. (Hint: \( X + Y = \min(X, Y) + \max(X, Y) \).)
Option Probability and Statistics, Option I – FALL 2003

6. (Stat 411)
Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with parameter $\theta > 0$, i.e.,
$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0,$$
zero otherwise.

Using the Neyman Pearson Lemma, derive the uniformly most powerful size $\alpha$ test for $H_0: \theta = 2$ versus $H_1: \theta < 2$.

(1) Show that this test can be based on $\bar{X}$.

(2) Show that this test can be based on a chi-square statistic. What is the degree of freedom of the chi-square?

7. (Stat 416)
Let $X_1, \ldots, X_n$ be an i.i.d. random sample from a continuous type distribution that has density $f$.

Give a definition of the ranks $R_1, \ldots, R_n$ of $X_1, \ldots, X_n$. What is the joint distribution of $R_1, \ldots, R_n$?

Determine the expectation $E(R_i)$, variance $\text{Var}(R_i)$, and covariance $\text{Cov}(R_i, R_j)$. A useful formula hereby is $i^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$. Explain briefly where these quantities are needed.

8. (Stat 431)
We have $n = 15$ data points collected under a fixed size (15) SRS sampling plan without replacement from a population of size $N = 200$. The 15 data points are arbitrarily labeled by $y_1, y_2, \ldots, y_{15}$.

A- Find the mean and the variance of $y_2$.

B- Find the covariance and the correlation coefficient between $y_2$ and $y_5$.

C- Does the covariance between $y_2$ and $y_5$ depend on $n$ or $N$?
9. (Stat 481)
An engineer wishes to compare the strengths of two types of beams. Several beams of each type are selected at random and the deflections (in units of 0.001 inch) are measured when submitted to a force of 3000 pounds. The respective sample characteristics are \( n_1 = 10 \), \( \bar{x} = 82.6 \), \( s_1^2 = 6.52 \), and \( n_2 = 12 \), \( \bar{y} = 78.1 \), \( s_2^2 = 7.02 \). To test the null hypothesis of equal strengths against the two-sided alternative, what are your model assumptions, and which test would you use? Give a formula for the test statistic and calculate its value for the given data. What is the distribution of the test statistic under the null hypothesis?

10. (Stat 481)
In a study of the relationship between the weight \( x \) of an automobile and its fuel consumption \( y \), an experiment was performed with one car that carried different selected loads. The results of the experiment are summarized in the following ANOVA table.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>13.195</td>
<td>1</td>
<td>13.915</td>
<td>165.2</td>
</tr>
<tr>
<td>Error</td>
<td>0.674</td>
<td>8</td>
<td>0.084</td>
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</tr>
<tr>
<td>Total</td>
<td>14.589</td>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is the underlying model, the null hypothesis and alternative, and the test statistic? Explain all entries in the table.
The written Master's Examination

Option Probability and Statistics, Option I

FALL 2003

SOLUTIONS

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For any matrix $A$ (may be rectangular) show that $\text{rank}(A^T A) = \text{rank}(A)$.

Since $A^T A = (A^T)^T A$,

$$\text{rank}(A^T A) \leq \text{rank}(A). \quad (1)$$

Also for a vector $x$,

$$A^T A x = 0$$

$$\Rightarrow x^T A^T A x = 0$$

$$\Rightarrow A x = 0.$$

So $x \in \text{Null}(A) \Rightarrow x \in \text{Null}(A^T A)$

So $\text{nullity}(A^T A) \leq \text{nullity}(A)$

Let $A$ be an $m \times n$ matrix.

Hence $n - \text{rank}(A^T A) \leq n - \text{rank}(A)$

i.e.

$$\text{rank}(A) \leq \text{rank}(A^T A). \quad (2)$$

(1) and (2) gives:

$$\text{rank}(A^T A) = \text{rank}(A).$$
(a) \( \lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \frac{\ln(1+x)}{1} \)

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(1) \( \lim_{x \to 0} \frac{\ln(1+x)}{x} = e \)

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Fall 2 (b) 

\[ \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{\frac{1}{2} \ln(1+x)}{x} = \lim_{x \to 0} \frac{e^{\frac{1}{2} \ln(1+x)} - e^0}{x} \]

\[ \lim_{x \to 0} \frac{\left(\frac{1}{x} \ln(1+x)\right)'(1+x)^{1/2}}{1} = \lim_{x \to 0} \left(\frac{\ln(1+x)}{x}\right)' = e \]

\[ = \lim_{x \to 0} \frac{\ln(1+x)}{x} \cdot e \]

\[ = \lim_{x \to 0} \frac{\ln(1+x) - \ln(1+1)}{x^2} \]

\[ = e \lim_{x \to 0} \frac{x - (1+x) \ln(1+x)}{x^2 (1+x)} \]

\[ = e \lim_{x \to 0} \frac{-\ln(1+x)}{2x + 3x^2} \]

\[ = e \lim_{x \to 0} \frac{1}{2 + 6x} \cdot \frac{1}{x} \]

\[ = -\frac{1}{2} e \]

(5) \[ \frac{1}{(1+x)(2+6x)} = \frac{1}{3+8x+6x^2} \]

Now since (5) is true, we can work backward to obtain all the equalities.
\[ F (\mathbf{b}) = \int_{\mathbf{b} \times \mathbf{b}} f (x, y) \, dx \, dy \]

where

\[ f (x, y) = \begin{cases} \frac{2}{\pi} & (x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1, \, y > 0\} \\ 0 & (x, y) \notin D \end{cases} \]

\[ \mathbf{b} = \left[ 0, \frac{3}{4} \right] \times \left[ 0, 1 \right] \]

\[ \int_{\mathbf{b} \times \mathbf{b}} f (x, y) \, dx \, dy = \int_{0}^{\frac{3}{4}} \int_{0}^{1} \frac{2}{\pi} \, dx \, dy = \frac{3}{4\pi} \]

\[ S_2 (\mathbf{b}) = \int \int_{D \times D} \frac{\partial f}{\partial x} \, dx \, dy \]

\[ \frac{\partial f}{\partial x} (x, y) = \begin{cases} \frac{2}{\pi} & (x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1, \, y > 0\} \\ 0 & (x, y) \notin D \end{cases} \]

\[ \int_{D \times D} \frac{\partial f}{\partial x} \, dx \, dy = \int_{0}^{\frac{3}{4}} \int_{0}^{1} \frac{2}{\pi} \, dx \, dy = \frac{3}{4\pi} \]

\[ S_3 (\mathbf{b}) = \int \int_{D \times D} \frac{\partial f}{\partial y} \, dx \, dy \]

\[ \frac{\partial f}{\partial y} (x, y) = \begin{cases} \frac{2}{\pi} & (x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1, \, y > 0\} \\ 0 & (x, y) \notin D \end{cases} \]

\[ \int_{D \times D} \frac{\partial f}{\partial y} \, dx \, dy = \int_{0}^{\frac{3}{4}} \int_{0}^{1} \frac{2}{\pi} \, dx \, dy = \frac{3}{4\pi} \]
Let \( X_1, X_2, \ldots, X_n \) be a random sample from a Poisson distribution with parameter \( \theta \), where \( \theta > 0 \). Suppose we are interested in the function \( h(\theta) = P(X = 0) = e^{-\theta} \).

1. Find the maximum likelihood estimator for \( h(\theta) \).
2. Find the minimum variance unbiased estimator for \( h(\theta) \).

[You have to justify your answers. If you are using a theorem, you have to mention it.]

\[
(\text{i}) \quad \text{MLE of } \theta: \quad L(\theta) = e^{-n\theta} \frac{\theta^{\sum X_i}}{\prod X_i!} \quad \text{mle } \theta = \bar{X}.
\]

\( h(\theta) \) is a 1-1 function. So \( h(\theta) = \text{mle of } h(\theta) \) = \( h(\bar{X}) = e^{-\bar{X}} \).

\[
(\text{ii}) \quad \text{Let } Y = \begin{cases} 1 & \text{if } X_i = 0 \\ 0 & \text{if } X_i > 0 \end{cases} \quad E(Y) = P(X_i = 0) = e^{-\theta} = h(\theta),
\]

\[
\tilde{X} = \frac{\sum X_i}{n} \quad \text{is complete sufficient for } \theta.
\]

So MVUE is

\[
E(Y | \tilde{X} = z) = \frac{P(X_i = 0 | \tilde{X} = z)}{P(\tilde{X} = z)} = \frac{P(X_i = 0, \frac{\sum X_i}{n} = z)}{P(\tilde{X} = z)}
\]

Since \( \sum X_i \sim \text{Poisson} \),

\[
= \frac{e^{-\theta} \cdot e^{-(n-1)\theta} \frac{(\theta(n-1))^z}{z!}}{e^{-n\theta} \frac{(n\theta)^z}{z!}} = \left( \frac{n-1}{n} \right)^z , \quad \text{where } z = \frac{\sum X_i}{n}.
\]
5. (Stat 461)

Solution: Let \( X, Y \) denote the life of the two bulbs. Since \( X, Y \) are independent random variables with exponential distribution, the random variable \( Z = \min(X, Y) \) is also exponentially distributed. Its expectation is simply \( \frac{1}{\mu} \) where \( \mu = \lambda_1 + \lambda_2 \) where \( \frac{1}{\lambda_1} \) is the mean of \( X \) and \( \frac{1}{\lambda_2} \) is the mean of \( Y \). In our case, \( \frac{1}{\lambda_1} = 1000 \), and \( \frac{1}{\lambda_2} = 800 \).

Thus \( E(Z) = \frac{1}{\frac{1}{1000} + \frac{1}{800}} \). Now \( X + Y = \min(X, Y) + \max(X, Y) \). Thus \( E(X + Y) = E(X) + E(Y) = 1000 + 800 = E(\min(X, Y) + \max(X, Y)) = E(\min(X, Y)) + E(\max(X, Y)) \).

Thus \( E(\max(X, Y)) = 1800 - \frac{1}{\frac{1}{1000} + \frac{1}{800}} = 1355.56 \) hrs. This is the expected time the room will be lighted.
Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with parameter $\theta > 0$, i.e.,

$$f(x; \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0,$$

zero otherwise.

Using the Neyman Pearson Lemma, derive the uniformly most powerful size $\alpha$ test for

$H_0: \theta = 2$ versus $H_1: \theta < 2$.

1. Show that this test can be based on $\bar{X}$.
2. Show that this test can be based on a chi-square statistic. What is the degree of freedom of the chi-square?

\[ L(2) = \frac{\theta^n e^{-\frac{1}{\theta} \sum x_i}}{\theta^n e^{-\frac{1}{\theta} \sum x_i}} \leq \kappa \] i.e., \[ \sum x_i \left( \frac{1}{\theta} - \frac{1}{2} \right) \leq k_1 \]

i.e., \[ \sum x_i \leq k_1 \text{ since } \theta_1 < 2. \]

$\kappa_1$ is determined from $\bar{X} \leq k_4$. $\kappa_4$ determined from

\[ P(\bar{X} \leq k_4 | H_0) = \alpha. \]

This test does not depend on $\theta_1$. So it is UMP size $\alpha$ for $H_0: \theta = 2$, $H_1: \theta < 2$.

Under $H_0$ $f(x; 2) = \frac{1}{2} e^{-\frac{x}{2}} = \frac{1}{2} e^{-\frac{x}{2} \left( \frac{x}{2} \right)^2}$

So $X \sim \chi^2(2)$, $\sum x_i \sim \chi^2(2n)$

Critical region: \[ \sum x_i \leq \eta k_4 \]

so this is a $\chi^2$ test of $df = 2$. 

\[ nk_4 = \chi^2(2n, 1 - \alpha). \]
Let $X_1, \ldots, X_n$ be an i.i.d. random sample from a continuous type distribution that has density $f$.

Give a definition of the ranks $R_1, \ldots, R_n$ of $X_1, \ldots, X_n$. What is the joint distribution of $R_1, \ldots, R_n$?

Determine the expectation $E(R_i)$, variance $\text{Var}(R_i)$, and covariance $\text{Cov}(R_i, R_j)$. A useful formula hereby is $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$. Explain briefly where these quantities are needed.

$$P \left\{ (R_1, \ldots, R_n) = (r_1, \ldots, r_n) \right\} = \frac{1}{n!}, \text{ for every permutation } \pi.$$

$$E(R_i) = \sum_{i=1}^{n} i^2 \frac{1}{n} = \frac{n+1}{2}$$

$$E(R_i^2) = \sum_{i=1}^{n} i^2 \frac{1}{n} = \frac{(n+1)(2n+1)}{6}$$

$$\text{Var}(R_i) = E(R_i^2) - E(R_i)^2 = \frac{n^2 - 1}{12}$$

$$\text{Cov}(R_i, R_j) = E(R_i R_j) - E(R_i) E(R_j)$$

$$E(R_i R_j) = \sum_{i,j} i^2 j \frac{1}{n(n-1)}$$

$$= \sum_{i,j} i^2 j \frac{1}{n(n-1)} - \sum_{i} i^2 \frac{1}{n(n-1)}$$

$$= (\sum_{i} i^2)^2 \frac{1}{n(n-1)} - \sum_{i} i^2 \frac{1}{n(n-1)}$$

$$= \frac{1}{12} (n+1)(3n+2)$$

$$\text{Cov}(R_i, R_j) = -\frac{1}{12} (n+1)$$

To find the asymptotic distribution of a linear rank statistic $T = \sum_{i=1}^{n} c_i R_i$, we need $E(T)$ and $\text{Var}(T)$, both of which can be determined with the above quantities.
8. (Stat 431)
We have \( n = 15 \) data points collected under a fixed size (15) SRS sampling plan without replacement from a population of size \( N = 200 \). The 15 data points are arbitrarily labeled by \( y_1, y_2, \ldots, y_{15} \).

A- Find the mean and the variance of \( y_2 \).

B- Find the covariance and the correlation coefficient between \( y_2 \) and \( y_3 \).

C- Does the covariance between \( y_2 \) and \( y_3 \) depend on \( n \) or \( N \)?
9. (Stat 481)

An engineer wishes to compare the strengths of two type of beams. Several beams of each type are selected at random and the deflections (in units of 0.001 inch) are measured when submitted to a force of 3000 pounds. The respective sample characteristics are \( n_1 = 10, \bar{x} = 82.6, s_1^2 = 6.52 \), and \( n_2 = 12, \bar{y} = 78.1, s_2^2 = 7.02 \). To test the null hypothesis of equal strengths against the two-sided alternative, what are your model assumptions, and which test would you use? Give a formula for the test statistic and calculate its value for the given data. What is the distribution of the test statistic under the null hypothesis?

Model:
\[
X_1, \ldots, X_{n_1} \text{ i.i.d. } N(\mu_1, \sigma^2) \quad \text{equal, but unknown variance}
\]
\[
Y_1, \ldots, Y_{n_2} \text{ i.i.d. } N(\mu_2, \sigma^2) \quad \text{Independence of all observations}
\]

\[
H_0: \mu_1 = \mu_2 \quad H_1: \mu_1 \neq \mu_2
\]

Two-sample t-test is used. The test statistic is
\[
T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1 + n_2 - 2} \left( \frac{1}{n_1} + \frac{1}{n_2} \right)}}
\]
\[
= 4.03
\]

Under \( H_0 \), \( T \) has a \( t \)-distribution with \( n_1 + n_2 - 2 = 20 \) degrees of freedom.

Reject \( H_0 \) if \(|T| > t_{20, 0.02}\).
10. (Stat 481)
In a study of the relationship between the weight \( x \) of an automobile and its fuel consumption \( Y \), an experiment was performed with one car that carried different selected loads. The results of the experiment are summarized in the following ANOVA table.

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>df</th>
<th>MS</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>13.195</td>
<td>1</td>
<td>13.915</td>
<td>165.2</td>
</tr>
<tr>
<td>Error</td>
<td>0.674</td>
<td>8</td>
<td>0.084</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>14.589</td>
<td>9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is the underlying model, the null hypothesis and alternative, and the test statistic? Explain all entries in the table.

**Model:** Simple Linear Regression,

\[
y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i = 1, \ldots, n
\]

- \( y_i \) = observed values
- \( \beta_0 \) = constant
- \( \beta_1 \) = selected parameter
- \( \epsilon_i \) = error terms, i.i.d., \( N(0, \sigma^2) \), \( \sigma^2 \) unknown

**Hypotheses:**

\[ H_0 : \beta_1 = 0 \quad H_1 : \beta_1 \neq 0 \]

- Reject \( H_0 \) if the test statistic \( F \geq F_{1, 8, \alpha} \)

\[ F = \frac{MS_{Regression}}{MS_{Error}} \]

Explanations of all entries in the table are given in the book by Hogg & Ledolter on p. 356-358.