Full points may be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth the same number of points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. **DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS.** When you have finished, insert all your answer sheets into the envelope provided, then seal and print your name on it.

Any student whose answers need clarification may be required to submit to an oral examination.
Option Probability and Statistics, Option II – FALL 2005

1. (Math 310)
For a matrix $A$ let $R(A)$ denote the range (column space) and $N(A)$ the null space. Suppose $A$ is a (real) square matrix such that $R(A) \cap N(A) = \{0\}$, the vector space consisting of the null vector only. Show that then $\text{Rank } (A^2) = \text{Rank } (A)$.

2. (Math 413)
Let $f$ be a continuous function on the line such that $f(x)/x^2$ vanishes at infinity, i.e.,
$$\lim_{x \to \pm \infty} \left( \frac{f(x)}{x^2} \right) = 0.$$ Show that $x^2 + f(x)$ achieves its minimum on the line, i.e. that for some $x_0 \in (-\infty, +\infty)$, $x_0^2 + f(x_0) \leq x^2 + f(x)$ for all $x \in (-\infty, +\infty)$.

3. (Stat 401)
Suppose that the radius $R$ of a circle is uniformly distributed over the interval $(0,1)$. Determine the joint probability distribution function of the area $X$ of the circle and the perimeter $Y$ of the circle, and compute the correlation coefficient $\rho_{xy}$ of $X$ and $Y$.

4. (Stat 411)
Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with p.d.f.
$$f(x; \theta) = \frac{1}{\theta^2} x \ e^{-x/\theta}, x > 0, \text{ zero otherwise.}$$
Derive the likelihood ratio test for $H_0: \theta = 1$, against $H_1: \theta \neq 1$, with a given level of significance $\alpha$, and show that it can be expressed as a two-tailed test based on a chi-square statistic. What are the degrees of freedom of the chi-square?

5. (Stat 461)
Three out of every four trucks on the road are followed by a car and only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?
6. (Stat 462)
Let \( \{X(t) : t \geq 0\} \) be a Yule process that is observed at a random time \( U \), where \( U \) is uniformly distributed over the interval \([0, 1]\). Find \( \operatorname{P}[X(U) = k], k = 1, 2, \ldots \)
[Recall that for a Yule process \( P_n(t) = e^{-\beta t} (1 - e^{-\beta t})^{n-1}, n = 1, 2, \ldots \).]

7. (Stat 462)
Operations 1, 2, and 3 are to be performed in succession on a major piece of equipment. Operation \( k \) takes a random duration \( S_k \) that is exponentially distributed with parameter \( \lambda_k \) for \( k = 1, 2, 3 \), and all operation times are independent. Let \( X(t) \) denote the operation performed at time \( t \), with time \( t = 0 \) marking the start of the first operation. Suppose that \( \lambda_1 = 5, \lambda_2 = 3 \) and \( \lambda_3 = 13 \). Determine \( \operatorname{P}[X(t) = k], k = 1, 2, 3 \).

8. (Stat 471)
Consider the problem of maximizing
\[
4x_1 + 5x_2 + 5x_3 + 2x_4
\]
subject to
\[
33x_1 + 49x_2 + 51x_3 + 22x_4 \leq 120,
\]
where \( x_1, x_2, x_3, x_4 \) are nonnegative integers.

Use the branch and bound method to initialize at a feasible solution and show how the method works for the next iteration.

9. (Stat 473)
Players Anne and Bob are very intelligent and they play the following game. From a basket with 15 oranges, players alternately remove at least 1 but at most 4 oranges. The game ends once the basket is empty. The winner is the one who removed the last orange from the basket that was non-empty.

(a) If Anne starts the game who will win the game?
(b) Describe verbally a winning strategy for the winner.
10. (Stat 477)
Consider a coherent system consisting of four components whose lifetimes are independent, exponentially distributed with mean $1/\lambda$. Suppose the cut sets of the system are \{1\},\{2,3\},\{2,4\},\{3,4\}.

(a) Compute the reliability importance of component 1 at $t=1$.

(b) Find the distribution of the life time of the system.

(c) Find the probability that the failure of the system coincides with the failure of component 1.
Math 310, Fall 05

For a matrix $A$ let $\text{R}(A)$ denote the range (column space) and $\text{N}(A)$ the null space. Suppose $A$ is a square matrix such that $\text{R}(A) \cap \text{N}(A) = \{0\}$, the vector space consisting of the null vector only. Then show that,

$$\text{Rank } (A^2) = \text{Rank } (A)$$

Solution

Let $A$ be $n \times n$

Since $Ax = 0 \Rightarrow A^2x = 0$, $\text{N}(A) \subset \text{N}(A^2)$.

Now suppose $x \in \mathbb{R}^n$ be such $A^2x = 0$.

Then $y = Ax \in \text{R}(A) \cap \text{N}(A)$.

Hence $y = 0$, i.e. $Ax = 0$

So $\text{N}(A^2) = \text{N}(A)$

So $n - \text{rank } (A^2) = n - \text{rank } (A)$

So $\text{rank } (A^2) = \text{rank } (A)$
Fall 2005

Let \( f \) be a continuous function on the line such that \( f(x)/x^2 \) vanishes at infinity, i.e., \( \lim_{x \to \pm \infty} \frac{f(x)}{x^2} = 0 \). Show that \( x^2 + f(x) \) achieves its minimum on the line, i.e., for some \( x_0 \in (-\infty, +\infty) \), \( x_0^2 + f(x_0) \leq x^2 + f(x) \) for all \( x \in (-\infty, +\infty) \).

Proof. Let \( g(x) = x^2 + f(x) \) and note that

\[
\lim_{x \to \pm \infty} g(x) = x^2(1 + \frac{f(x)}{x^2}) = +\infty.
\]

Find \( y_0 \) such that \( g(y_0) > 0 \). Next find \( M, N \) both positive such that \( x > M \implies g(x) > g(y_0) \) and \( x < -M \implies g(x) > g(y_0) \). Consequently for all \( x \) such that \( |x| > b := \max(M, N) \), \( g(x) > g(y_0) \). Now since \( g \) is continuous on \([-b, b] \), by a fundamental theorem for continuous functions there is \( x_0 \) such that \( g(x_0) \leq g(x) \) for all \( x \in [-b, b] \). As a result we can take \( x_0 = y_0 \) if \( g(y_0) \leq g(x_0) \) and \( x_0 = y_0 \) otherwise.
Suppose that the radius $R$ of a circle is uniformly distributed over the interval $(0, 1)$. Determine the joint probability distribution function of the area of the circle $X$ and its perimeter $Y$ and compute the correlation coefficient, $\rho_{XY}$, of $X$ and $Y$.

Solution. $X = \pi R^2$ and $Y = 2\pi R$, thus

$$F(x, y) := P\{X \leq x, Y \leq y\} = P\{\pi R^2 \leq x, 2\pi R \leq y\} = 0 \quad x \leq 0, \text{ or } y \leq 0$$

$$= P\{R \leq \min(\sqrt{\frac{x}{\pi}}, \frac{y}{2\pi})\} = \begin{cases} \sqrt{\frac{x}{\pi}} \quad \sqrt{\frac{x}{\pi}} < \frac{y}{2\pi}, \sqrt{\frac{x}{\pi}} < 1 \\ \frac{y}{2\pi} \quad \frac{y}{2\pi} \leq \sqrt{\frac{x}{\pi}}, \frac{y}{2\pi} < 1 \\ 1 \quad \sqrt{\frac{x}{\pi}} \geq 1, \frac{y}{2\pi} \geq 1. \end{cases}$$

$$= \begin{cases} 0 \quad x \leq 0, \text{ or } y \leq 0 \\ \sqrt{\frac{x}{\pi}} \quad 4\pi x < y^2, x < \pi \\ \frac{y}{2\pi} \quad 4\pi x \geq y^2, y < 2\pi \\ 1 \quad x \geq \pi, y \geq 2\pi \end{cases}$$

$$F_X(x) = \begin{cases} \sqrt{\frac{x}{\pi}} \quad 0 < x < \pi \\ 1 \quad x \geq \pi \end{cases}$$

$$F_Y(y) = \begin{cases} \frac{y}{2\pi} \quad 0 < y < 2\pi \\ 1 \quad y \geq 2\pi \end{cases}$$

$$f_X(x) = \frac{1}{2\sqrt{\pi x}} \quad 0 < x < \pi \text{ and } 0 \text{ otherwise}$$

$$f_Y(y) = \frac{1}{2\pi} \quad 0 < y < 2\pi \text{ and } 0 \text{ otherwise}$$

Next,

$$EX = \pi ER^2 = \pi \int_0^1 r^2 \, dr = \frac{\pi}{3}, \quad EX^2 = \pi^2 ER^4 = \pi^2 \int_0^1 r^4 \, dr = \frac{\pi^2}{5}, \quad Var(X) = \frac{4\pi^2}{45}.$$  

$$EY = 2\pi ER = 2\pi \int_0^1 r \, dr = \pi, \quad EY^2 = 4\pi^2 ER^2 = 4\pi^2 \int_0^1 r^2 \, dr = \frac{4\pi^2}{3}, \quad Var(Y) = \frac{\pi^2}{3}.$$  

$$EXY = 2\pi^2 ER^3 = 2\pi^2 \int_0^1 r^3 \, dr = \frac{\pi^2}{2}, \quad Cov(X, Y) = EXY - EXEY = \frac{\pi^2}{6}.$$  

$$\rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\sqrt{15}}{4} \approx 0.968.$$
Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf,

$$f(x; \theta) = \frac{1}{\theta^2} x e^{-x/\theta}, \quad x > 0, \text{ zero otherwise.}$$

Derive the likelihood ratio test for $H_0: \theta = 1$, against $H_1: \theta \neq 1$, with a given level of significance $\alpha$, and show that it can be expressed as a two-tailed test based on a chi-square statistic. What is the degree of freedom of the chi-square?

**Likelihood**

$$L(\theta) = \frac{1}{\theta^{2n}} e^{-\sum x_i / \theta}$$

$$L(\theta) = \ln L(\theta) = -2n \ln \theta - \frac{\sum x_i}{\theta} + \text{const}.$$  

$$L'(\theta) = -\frac{2n}{\theta} + \frac{\sum x_i}{\theta^2} = 0 \implies \hat{\theta} = \frac{\sum x_i}{2n}.$$

Check $L''(\theta) < 0$ to show $\hat{\theta}$ is MLE.

**Likelihood Ratio**

$$\Lambda = \frac{e^{-\sum x_i}}{(2n)^{2n} e^{-2n}} \leq \infty$$

$$\implies W^{2n} e^{-W} \leq c_1.$$  

Let $g(W) = e^{-W} W^{2n}$

Where $W = \sum x_i$.

$$g'(W) > 0 \quad \text{if} \quad W < 2n$$

$$= 0 \quad \text{if} \quad W = n$$

$$< 0 \quad \text{if} \quad W > n.$$  

So $\Lambda \leq W \leq A$ or $W \leq B$.

Note since $X \sim Gamma$ with $\alpha = 2, \beta = 1$,

$$-\frac{2x}{\theta} \sim \chi^2(4).$$

So under $H_0$, $W \sim \chi^2(4n)$

Thus $W_0 = 2 \sum x_i$.

So $\Lambda < 1$ reject if $W_0 < \chi^2(4n, 1 - \frac{\alpha}{2})$ or $W_0 > \chi^2(4n, \frac{\alpha}{2}).$

**Degree of freedom** $= 4n$. 

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Let \( X_n = 0 \) if the \( n \)-th vehicle is a car and let \( X_n = 1 \) if the \( n \)-th vehicle is a truck. Thus we have a stationary Markov chain with states 0, 1 with transition matrix given by \( p_{10} = \frac{3}{4}, p_{11} = \frac{1}{4}, p_{01} = \frac{1}{5}, p_{00} = \frac{4}{5}. \) Thus the fraction of cars and trucks on the road is given by the stationary distribution \( r = [r_0, r_1] \) where \( rP = r. \) Here

\[
P = \begin{bmatrix}
p_{00} & p_{01} \\
p_{10} & p_{11}
\end{bmatrix}
\]

Solving for

\[
r_0 = \frac{4}{5} r_0 + \frac{3}{4} r_1 \quad r_1 = \frac{1}{5} r_0 + \frac{1}{4} r_1
\]

gives \( r_0 = \frac{15}{19}, \quad r_1 = \frac{4}{19}. \)
Solution:

\[ P[X(U) = k] = \int_0^1 P[X(U) = k | U = u] \, du \]

\[ = \int_0^1 P[X(u) = k] \, du \]

\[ = \int_0^1 \frac{\beta^k}{k!} e^{-\beta u} \, du \]

\[ = \frac{1}{k \beta} \left[ 1 - e^{-\beta} \right]^k \]
\[ P(X(t) = 1) = P(S_1 > t) = e^{-5t} \]

\[ P(X(t) = 3) = P(S_1 + S_2) = \int_0^t P(S_2 > t-u) \cdot 5e^{-5u} \, du \]

\[ = \int_0^t e^{-3(t-u)} \cdot 5e^{-5u} \, du = 5e^{-3t} \left[ \int_0^t e^{-2u} \, du \right] \]

\[ = 5e^{-3t} \left[ \frac{1 - e^{-2t}}{2} \right] \]

\[ P(X(t) = 3) = P(S_1 + S_2 < t < S_1 + S_2 + S_3) \]

\[ = \int_0^t e^{-13(t-u)} \frac{f(u)}{S_1 + S_2} \, du \]

\[ = \int_0^t e^{-13(t-u)} \frac{15}{2} \left[ e^{-3u} - e^{-5u} \right] \, du \]

\[ = \frac{15e^{-13t}}{2} \int_0^t \left[ e^{10u} - e^{8u} \right] \, du \]

\[ = \frac{15e^{-13t}}{2} \left[ \frac{(e^{10t} - 1)}{10} - \frac{(e^{8t} - 1)}{8} \right] \]
Solution to Problem stat 471, Masters exam Spring 2005

Since the coefficient ratios $\frac{4}{33} > \frac{5}{49} > \frac{5}{51} > \frac{2}{22}$ we give the highest priority to $x_1$, followed by $x_2$, then by $x_3$ and finally by $x_4$. Thus we initiate at $x_1 = 3, x_2 = x_3 = x_4 = 0$. The value of the objective function is $(4)(3) + 5(0) + 5(0) + 2(0) = 12$. The branch and bound procedure reduces the first non-zero variable by one unit moving backwards from $x_4$ to $x_3$ to $x_2$ to $x_1$. This releases $120-33(2)=54$ to be shared by $x_2, x_3, x_4$ satisfying $49x_2+51x_3+22x_4 \leq 54$ and $x_2, x_3, x_4$ nonnegative integers. However its highest contribution to the original objective function when $x_1 = 2$ is $4(2) + \frac{5}{49}(54) = 13.51$. Thus there is potential chance to strictly increase the value of the objective function. Thus the next in priority is $x_2 = 1, x_3 = 0, x_4 = 0$. Thus we get an improvement at $(x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 0)$ an improvement from 12 to 13.
Solution to Problem Stat 473, Spring 2005 If Ann starts the game, Bob will win the game by the following strategy: If Ann removes $k$ oranges, Bob removes $5 - k$ oranges. Clearly since $1 \leq k \leq 4$, Bob removes at least one and at most 4. This way finally 5 will be left at some stage and after Ann’s move Bob will make his move and win the game.
STAT 477

(a) \[ h(1, 3, 1, 3, 1) = p_1 \left[ 2b_3 + 2b_4 + b_3 b_4 - 2b_2 b_3 b_4 \right] \]

\[ I(1)^h = \frac{\partial h}{\partial b_1} = \left[ 2b_3 + 2b_4 + b_3 b_4 - 2b_2 b_3 b_4 \right] \]

at \( t = 1 \), \( b_2 = b_3 = b_4 = e^{-\lambda} \)

\[ I(1)^h = 3e^{-2\lambda} - 2e^{-3\lambda} \]

(b) Let \( \bar{F}(t) \) be the survival distribution of the system.

\[ \bar{F}(t) = e^{-\lambda t} \left[ 3e^{-2\lambda t} - 2e^{-3\lambda t} \right] \]

\[ = 3e^{-3\lambda t} - 2e^{-4\lambda t} \]

(c) Let \( T_1 \) be the lifetime of component 1, and let \( T^* \) be the lifetime of the subsystem consisting of components 2, 3, 4, (which is a 2-out-of-3 system). So required probability is

\[ P(T_1 \leq T^*) = \int_0^\infty P(T^* > t) \lambda e^{-\lambda t} dt \]

\[ = \int_0^\infty \left[ 3e^{-2\lambda t} - 2e^{-3\lambda t} \right] \lambda e^{-\lambda t} dt \]

\[ = \int_0^\infty 3\lambda e^{-3\lambda t} dt - \int_0^\infty 2\lambda e^{-4\lambda t} dt \]

\[ = 1 - \frac{1}{2} = \frac{1}{2} \]