The written Master’s Examination

Option Probability and Statistics, Option II     SPRING 2004

Full points may be obtained for correct answers to 8 questions. Each numbered question (which may have several parts) is worth the same number of points. All answers will be graded, but the score for the examination will be the sum of the scores of your best 8 solutions.

Use separate answer sheets for each question. DO NOT PUT YOUR NAME ON YOUR ANSWER SHEETS. When you have finished, insert all your answer sheets into the envelope provided, then seal and print your name on it.

Any student whose answers need clarification may be required to submit to an oral examination.
1. (Math 310)

Given the matrix

\[
A = \begin{bmatrix}
1 & 0 & 5 & 9 & 1 & 0 & 0 \\
2 & 5 & 6 & 9 & 0 & 1 & 0 \\
3 & 2 & -5 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Let \(B = [C_1, C_6, C_7]\) be the 3x3 matrix formed by the columns 1, 6 and 7 of the matrix A. Write down the 3x7 matrix \(B^{-1}A\).

2. (Math 413)

Let \(s_1 = 0\). For \(n \geq 1\), define \(s_n\) by

\[
s_{2n} = \frac{s_{2n-1} + s_{2n+1}}{2}, s_{2n+1} = 1 + s_{2n}.
\]

Find \(\liminf_{n \to \infty} s_n\) and \(\limsup_{n \to \infty} s_n\).

3. (Stat 401)

Let \(X_1, X_2, X_3\) be independent identically distributed random variables each being exponential with mean 1. Let \(Y_1 = \min\{X_1, X_2, X_3\}\) and \(Y_2 = \max\{X_1, X_2, X_3\}\). Find the conditional expectation \(E(Y_1 | Y_2)\).

4. (Stat 411)

Let \(X_1, X_2, \ldots, X_n\) be a random sample of size \(n\) from \(N(\mu, \sigma^2)\).

(i) If the constant \(b\) is defined by \(P(X \leq b) = .975\), find the m.l.e. of \(b\).

(ii) If \(c\) is a given constant, find the m.l.e. of \(P(X \leq c)\).

5. (Stat 461)

A fair coin is tossed repeatedly until either two successive heads appear or two successive tails appear. Suppose that the first toss results in a head, find

(i) The probability that the game ends with two successive tails.

(ii) The expected number of additional tosses until the game ends.

6. (Stat 462)

Let \(\xi_n, n = 0, 1, \ldots\) be a two state Markov chain with transition probabilities

\[
P_{00} = 1 - P_{01} = 0, P_{10} = 1 - P_{11} = 1 - \alpha.
\]

Let \(\{N(t), t \geq 0\}\) be a Poisson process with parameter \(\lambda\).
Let \( X(t) = \xi_{B(t)}, t \geq 0 \). Argue that \( \{X(t), t \geq 0\} \) is a two state birth and death process and determine its parameters \( \lambda_0 \) and \( \mu_i \) in terms of \( \alpha \) and \( \lambda \).

7. (Stat 462)

Let \( X \) be the lifetime of an item. An age replacement policy calls for replacing the item upon its failure or upon its reaching age \( T \), whichever occurs first. A new and identical item is then installed. Suppose each replacement, whether planned or not, costs 1 dollar and each failure costs an additional penalty of 4 dollars. The lifetimes of all items used are independent and identically distributed. Find \( T \) that minimizes the long run expected cost per unit time for each of the following cases:

(i) \( X \) is uniform on \([0,1]\).

(ii) \( X \) is exponential.

8. (Stat 471)

A lumber mill gets planks of length 17 feet from their wholesale supplier. The lumber mill has the following orders:

<table>
<thead>
<tr>
<th>Length</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 feet</td>
<td>200</td>
</tr>
<tr>
<td>5 feet</td>
<td>75</td>
</tr>
<tr>
<td>3 feet</td>
<td>100</td>
</tr>
</tbody>
</table>

Suppose each column of the matrix \( A \) below represents possible cutting patterns.

\[
A = \begin{pmatrix}
3 \text{feet} & 5 & 4 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\
5 \text{feet} & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 3 & 2 & 0 \\
7 \text{feet} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 2
\end{pmatrix}
\]

(1) Choose some 3 cutting patterns to meet all demands.

(2) What is the waste in your case?

(3) Use the revised simplex method to find a better cutting pattern.

9. (Stat 473)

Consider the bimatrix game \((A,B) = (a_{ij},b_{ij})\) given by

\[
\begin{pmatrix}
(3,5) & (4,4) & (5,2) \\
(6,1) & (1,8) & (3,5) \\
(5,4) & (3,6) & (7,3)
\end{pmatrix}
\]

Prove that \( x^* = \left( \frac{2}{3}, 0, \frac{1}{3} \right)^T \) and \( y^* = \left( \frac{1}{3}, \frac{2}{3}, 0 \right)^T \) is a Nash equilibrium pair in mixed strategies.
10. (Stat 477)

Consider a 3-out-of-4 system consisting of 4 components labeled 1, 2, 3, 4. Assume that the lifetime of component \( i \) is exponential with mean \( 1/i \), \( i = 1, 2, 3, 4 \). The components act independently.

(i) List all the minimal path sets.

(ii) List all the minimal cut sets.

(iii) Find the structural importance of component 1.

(iv) Find the Birnbaum reliability importance of component 1 at \( t = 1 \).
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option II

Solutions to problems 6-10

(Problems 1-5 same as option I)
Problem 1. (Linear Algebra): Since $B^{-1}A$ in the matrix will result in the unit vector at columns $(e_1, e_6, e_7) = B$ in the matrix, the matrix $B^{-1}A$ corresponds to pivoting the matrix

$$A = \begin{bmatrix} 2 & 5 & 6 & 9 & 0 & 1 & 0 \\ 3 & 2 & -5 & 1 & 0 & 0 & 1 \end{bmatrix}$$

by elementary row operations. Subtract 2 times row 1 from row 2 & 3 times row 1 from row 3 we get

$$B^{-1}A = \begin{bmatrix} 1 & 0 & 5 & 9 & 1 & 0 & 0 \\ 0 & 5 & -4 & -9 & -2 & 1 & 0 \\ 0 & 2 & -20 & -20 & -30 & 0 & 1 \end{bmatrix}$$
(2) Let $a_m = A_{2m-1} > m \geq 0 \quad (\alpha_0 = \beta_0 = 0)$

$$a_m = A_{2m-1} = \frac{1}{2} + A_{2m} = \frac{1}{2} + \frac{1}{2} A_{2m-2} = \frac{1}{2} + \frac{1}{2} a_{m-1}$$

$$= \frac{1}{2} + \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{2} a_{m-2} \right] = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} a_{m-2}$$

$$= \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + \frac{1}{2} a_{m-2}$$

**induction step**

$$= \frac{1}{2} + \ldots + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} a_{m-k-1} \right)$$

$$= \frac{1}{2} + \ldots + \frac{1}{2} \left( \frac{1}{2} a_{m-k} \right)$$

$$= \ldots = \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} a_0$$

$$= \frac{1}{2} + \ldots + \frac{1}{2} a_0 = 1 - \frac{1}{2^m} \quad \text{as} \quad m \to \infty$$

$$A_{2m} = \frac{1}{2} A_{2m-1} = \frac{1}{2} a_{m-1} \to \frac{1}{2}$$

$$\lim_{n \to \infty} A_n = \frac{1}{2}, \quad \lim_{n \to \infty} \sup A_n = 1$$
$$F \left( y_1, y_2, x \right) = P \left( y_1 \leq y_2, x \right)$$

$$= P \left( y_2 \leq y_1 \right) - P \left( y_2 \leq y_1, X \right)$$

$$= P \left( x \leq y_1, x_2 \leq y_1, x_3 \leq y_2 \right)$$

$$- P \left( x \leq y_1, x_2 \leq y_1, x_3 \leq y_2 \right)$$

$$= (F(y_1)^3 - (F(y_2) - F(y_1)))^{\frac{3}{2}}$$

where

$$F(x) = \int_{0}^{x} e^{-t} dt = 1 - e^{-x}, \quad x > 0, \quad (a < x < 0)$$

$$F \left( y_1, y_2 \right) = \begin{cases} 
(1 - e^{-y_1})^{3} - (e^{-y_2} - e^{-y_1})^{3} & 0 < y_1 < y_2 < \infty \\
0 & \text{otherwise}
\end{cases}$$

$$S \left( y_1, y_2 \right) = \frac{\partial F \left( y_1, y_2 \right)}{\partial y_1 \partial y_2} = \frac{3}{2} \left( -3 (e^{-y_1} - e^{-y_2}) \right)^2$$

$$= 3 \left( e^{-y_1} - e^{-y_2} \right) \left( -e^{-y_2} \right)$$

$$= 3 \left( e^{-y_1} - e^{-y_2} \right) \left( -e^{-y_2} \right)$$

$$= \begin{cases} 
6 \left( e^{-y_1} - e^{-y_2} \right) & 0 < y_1 < y_2 < \infty \\
0 & \text{otherwise}
\end{cases}$$

$$S \left( y_1, y_2 \right) = F \left( y_1 \right) = P \left( y_1 \leq y_2 \right) = P \left( x_1 \leq x_2, x_2 \leq y_1, x_3 \leq y_2 \right)$$

$$= (F(y_1)^3)^{\prime} = (1 - e^{-y_1})^{3} = 3 e^{-y_1} (1 - e^{-y_1})^2$$

$$0 < y_2 < \infty$$

For $$y_2 > 0$$

$$F \left( y_1, y_2 \right) = \frac{S \left( y_1, y_2 \right)}{y_2}$$

$$= \frac{6 \left( e^{-y_1} - e^{-y_2} \right) \left( e^{-y_2} \right)}{y_2}$$

$$0 < y_1 < y_2$$

$$= 0$$

$$0 < y_1 < y_2$$

$$= 0$$

$$0$$
\[
\begin{align*}
\text{E}(Y, 1 | Y_2) &= \left( 2Y_2 - 1 \right) e^{2Y_2} - 2e^{Y_2} + 5) / 2(1 - e^{-Y_2})^2
\end{align*}
\]
(i) $P[X \leq b] = P[Z \leq \frac{b - \mu}{\sigma}]$. Let $Z^*$ be a real number such that $P[Z \leq Z^*] = 0.975$. \( \therefore \frac{b - \mu}{\sigma} = Z^* \), so \( b = \sigma Z^* + \mu \). Hence \( \hat{b} = S Z^* + \bar{X} \), where $\bar{X}$, $S$ are the sample mean and sample standard deviation respectively.

(ii) $P[X \leq c] = P[Z \leq \frac{c - \mu}{\sigma}] = \Phi \left( \frac{c - \mu}{\sigma} \right)$, where $\Phi$ is the c.d.f. of the standard normal r.v. Hence $\hat{P}[X \leq c] = \Phi \left( \frac{c - \bar{X}}{S} \right)$.
Let $X_n$ be the results of the last two tosses. This is a Markov chain with the following transition matrix:

$$
\begin{pmatrix}
HH & HT & TH & TT \\
HH & 1 & 0 & 0 & 0 \\
HT & 0 & 1/2 & 1/2 & 0 \\
TH & 1/2 & 0 & 0 & 0 \\
TT & 0 & 0 & 0 & 1
\end{pmatrix}
$$

where $HH$, $TT$ are absorbing states. Let $N$ be the time until absorption. Let $P[X_n = TT | X_0 = TH] = x$, $P[X_n = TT | X_0 = HT] = y$.

By the 1st step analysis:

$$
\begin{align*}
x &= \frac{1}{2} x + \frac{1}{2} y \quad (1) \\
y &= \frac{1}{2} x + \frac{1}{2} y \quad (2)
\end{align*}
$$

$$
\therefore \quad y = \frac{2}{3}, \quad x = \frac{1}{3}
$$

(i) So the prob. that game ends in $TT$ given that the 1st toss is head = $x = \frac{1}{3}$.

(ii) Let $E[N | X_0 = HT] = V_1$, $E[N | X_0 = TH] = V_2$

$$
\begin{align*}
V_1 &= 1 + \frac{1}{2} V_2 \\
V_2 &= 1 + \frac{1}{2} V_1 \\
\therefore \quad V_1 &= V_2 = 2
\end{align*}
$$

So expected no. of additional tosses is $V_2 = 2$. 

1) The Markovian property of \( (X_t), t \geq 0 \) follows from the independence of the increments for the Poisson process.

\[
P[X(t+h) = 1 | X(t) = 0] \equiv P_{01}(h) = P\left\{ X_{(t+h)} = 1 \left| X_{(t)} = 0 \right. \right\}
\]

\[
= \lambda h + o(h), \quad \{ \text{since} \quad P[\text{N}(t+h) - \text{N}(t) \geq 2] = o(h), \quad P[\text{N}(t+h) - \text{N}(t) = 1] = \lambda h + o(h) \}
\]

Also

\[
P[X(t+h) = 1 | X(t) = 1] = P_{11}(h) = \lambda(1-\lambda) h + o(h)
\]

\[
= \lambda_0 = \lambda_1, \quad \lambda_0 = \lambda(1-\lambda)
\]
2) (i) \( E(\text{length of cycle}) = E[m^2(X,T)] = \int_x xds + \int Tdx \)
\[ = \frac{T}{2} + TC(1-T) = T - \frac{T^2}{2} \]
\[ E(\text{cost per cycle}) = 1 + 4P[X \leq T] = 1 + 4T \]
\[ \text{long run expected cost per unit time} = \frac{1 + 4T}{T - \frac{T^2}{2}} = g(T) \]
\[ g'(T) = 0 \Rightarrow 4T - 2T^2 - 1 - 3T + 4T^2 = 0 \]
\[ \text{i.e. } 2T^2 + T - 1 = 0 \Rightarrow (2T-1)(T+1) = 0 \Rightarrow T = \frac{1}{2} \]

(ii) By similar analysis, or by invoking the memoryless property of the exponential distribution \( T = \infty \), i.e. replace only upon failure.
Problem 5 (linear & Non-linear programming) Cutting pattern problem.
The planks can be initially used to cut into 2 x 3 type, like 3 feet planks out of 17 feet. This corresponds to the pattern

\[
\begin{bmatrix}
5 \\
5 \\
0
\end{bmatrix}
\text{3 feet planks}
\begin{bmatrix}
0 \\
5 \\
7
\end{bmatrix}
\text{"}
\]

Similarly

\[
\begin{bmatrix}
3 \\
5 \\
0
\end{bmatrix}
\text{3 feet planks}
\begin{bmatrix}
0 \\
7 \\
\end{bmatrix}
\text{"}
\]

A third

\[
\begin{bmatrix}
0 \\
5 \\
2
\end{bmatrix}
\text{3 feet planks}
\begin{bmatrix}
0 \\
7 \\
\end{bmatrix}
\text{"}
\]
If we need 100 3 foot planks, 75 5 foot planks, and 200 7 foot planks:

\[
Waste = \frac{17(x_1 + \ldots + x_{12})}{17(x_1 + x_{10} + x_{12})} - \left[ \frac{7(200) + 5(75) + 3(100)}{7(200) + 5(75) + 3(100)} \right]
\]

If we used \(x_j\) time pattern \(j\), \(j = 1, 2, 12\).

In our case:

\[
\begin{align*}
\text{20 of pattern 1} & = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\text{25 of pattern 12} & = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\text{100 of pattern 12} & = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

\(x_1 = 20, x_{10} = 25, x_{12} = 100\) and the rest are 0.

\[
Waste = 17(20 + 25 + 100) - 7(200) - 5(75) - 3(100)
\]

\(= 390\) feet of wood is wasted.

Given \(B = \begin{bmatrix} 3 \text{ feet} \\ 5 \text{ feet} \\ 0 \text{ feet} \\ 0 \text{ feet} \\ 0 \text{ feet} \\ 2 \text{ feet} \end{bmatrix}\)

We can use revised simplex to decide which column is worth entering.

Solve \(x^T B = (1, 1, 1, 1, 1, 1)\) in the last row, we get:

\(x = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}\)

Now compare \((x^T C_2) - 1\) for each column. We do:

Column 3 gives \((x^T C_3) - 1 = \left(\begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\frac{1}{2} \\frac{1}{2} \end{bmatrix} \right) - 1
\]

\(= \frac{3}{2} + \frac{1}{3} - 1 < 0\)

We are looking for \(\max \frac{1}{2} a_1 + \frac{1}{4} a_2 + \frac{1}{4} a_3\)

subject to \(3 a_1 + 5 a_2 + 7 a_3 \leq 17\)
Choose that for which \( 6a_1 + 10a_2 + 15a_3 = 30 \), if there is one.

\[
\begin{align*}
a_1 &= 5 \text{ gives } 30 \quad (5, 0, 0) \text{ gives just } 30 \\
a_1 &= 4 \quad (4, 1, 0) \text{ gives } 24 + 10 = 34
\end{align*}
\]

Thus \( \max \frac{1}{5} a_1 + \frac{1}{3} a_2 + \frac{1}{2} a_3 > 1 \) and so

\[
a_1 = 4 \quad a_2 = 1 \quad a_3 = 0. \\
\text{We need to drop one of the existing patterns and replace}
\]

by

\[
\begin{bmatrix}
4 \\
1 \\
0
\end{bmatrix}
\]

Solving \( B = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \), we get \( d = \begin{bmatrix} 4.5 \\ 1.5 \\ 0 \end{bmatrix} \)

Replace \( \begin{bmatrix} x_1^k \\ x_{10}^k \\ x_{12}^k \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \\ 100 \end{bmatrix} \) by \( \begin{bmatrix} 20 \\ 25 \\ 100 \end{bmatrix} - t \begin{bmatrix} 4.5 \\ 1.5 \\ 0 \end{bmatrix} \) for one complete iteration, where \( t = 25 \).

Thus pattern 1 leaves giving new pattern its position.

\[
\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \rightarrow [25 - \frac{25}{3} + 1] = 18 \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix} \rightarrow 100 \times \begin{bmatrix} 0 \\ 3 \end{bmatrix}
\]

This pattern has wastage = \( 143(17) - \text{demanded total fuel} \)
Problem 2: (Game Theory)

Given \((A, B) = \begin{bmatrix} 3 & 5^* & 4 & 3^* \\ 6 & 1 & 6 \\ 5 & 4 & 3 \\ 7 & 3 \end{bmatrix}\)

and \(x^* = \begin{bmatrix} 2, 0, 1/3 \end{bmatrix}^T, y^* = \begin{bmatrix} 1/3, 2/3, 0 \end{bmatrix}\)

\((x^* A y^*) = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 1 & 3 \\ 5 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix} = \frac{11}{3}\)

Further, \(A y^* = \begin{bmatrix} 11/3 \\ 8/3 \\ 11/3 \end{bmatrix}\)

Since \((x^* A y^*) \leq \max(A y^*_i) = \max(\frac{11}{3}, \frac{8}{3}, \frac{11}{3}) = \frac{11}{3}\)

we have \((x^* A y^*) \leq (x^* A y^*) = \frac{11}{3}\)

Similarly, \(x^* B = \begin{bmatrix} 2/3 & 0 & 1/3 \\ 5 & 4 & 2 \\ 4 & 6 & 3 \end{bmatrix}\)

But \((x^* B, y^*) = (x^* B y) \leq \max(x^* B) = \max(\frac{14}{3}, \frac{14}{3}, \frac{7}{3}) = \frac{14}{3}\)

But \((x^* B y^*) = \frac{14}{3}. \) Thus

\((x^* B y^*) \geq (x^* B y) \forall y \text{ mixed}\)

\((x^* A y^*) \geq (x^* A y^*) \forall x \text{ mixed}\)

Hence \((x^*, y^*)\) is a Nash Equilibrium.
1. \( \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \)

(ii) \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \)

(iii) \( I_\phi(1) = \frac{1}{8} \left[ (\phi(1, 1, 1) - \phi(0, 1, 1)) + (\phi(1, 1, 0) - \phi(0, 1, 0)) \right. \\
+ (\phi(1, 0, 0) - \phi(0, 0, 0)) \]

\( = \frac{1}{8} \cdot 3 = \frac{3}{8} \)

(iv) \[ h_k(h_1, h_2, h_3, h_4) = k_1 h_2 h_3 (1 - h_1) + k_2 (1 - h_2) h_4 + k_3 h_4 + (1 - h_1) k_1 h_3 h_4 + k_2 h_2 h_4 \\
+ k_3 h_3 h_4 \\
= k_1 h_2 h_3 + k_1 h_2 h_4 + k_1 h_3 h_4 + k_2 h_2 h_4 - 3 h_2 h_3 + h_4 \\
\frac{\partial h}{\partial h_1} = h_2 h_3 + h_2 h_4 + h_3 h_4 - 3 h_2 h_3 h_4 \\
\]

At \( t = 1, \) \( h_2 = e^{-2}, h_3 = e^{-3}, h_4 = e^{-4} \)

\[ I_\phi(1) = \left. \frac{\partial h}{\partial h_1} \right|_{t=1} = e^{-2} + e^{-3} + e^{-4} - 2 e^{-5} - 2 e^{-6} - 2 e^{-7} - 2 e^{-8} - 2 e^{-9} \]